Trapped surface formation: physical foundations

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SUSTech

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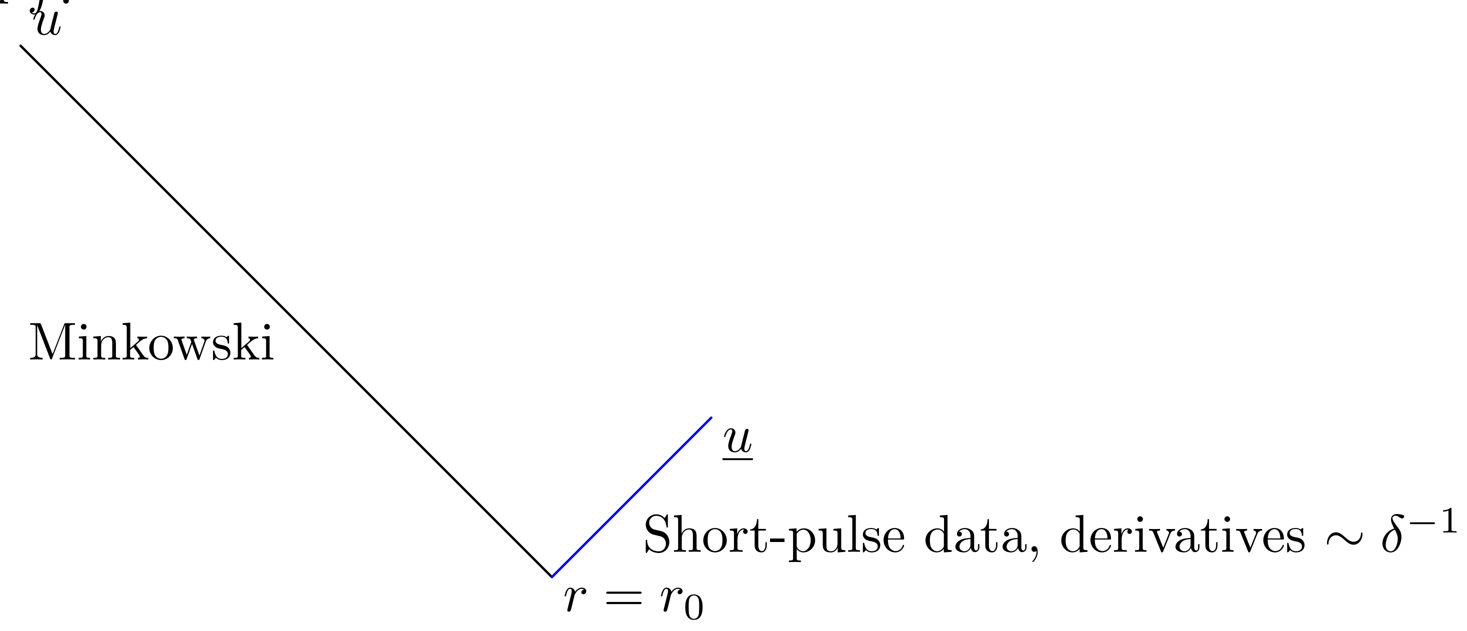

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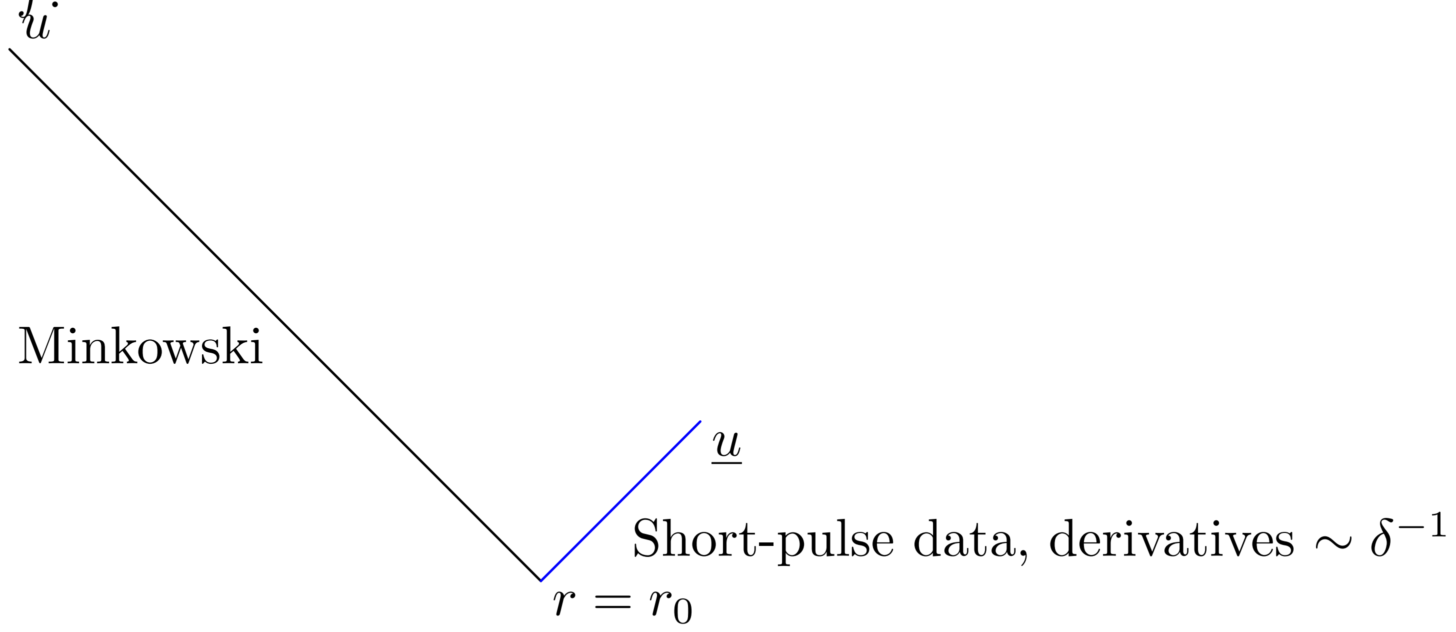

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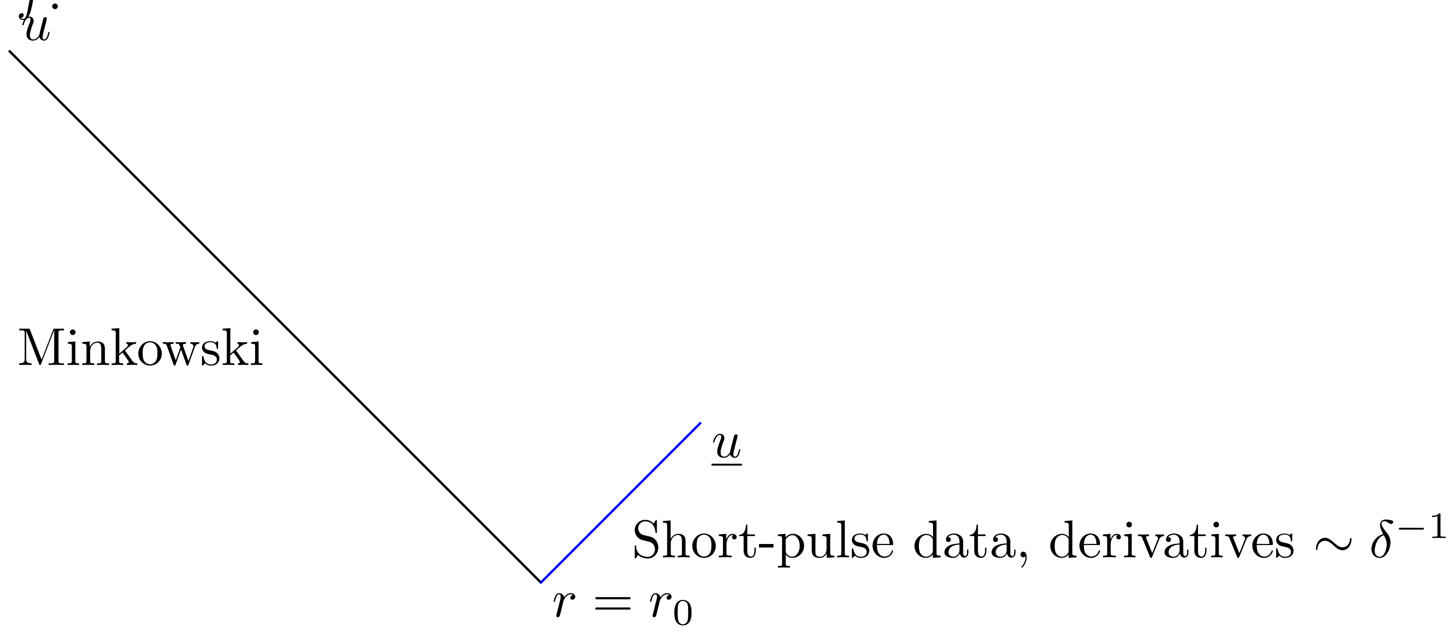

- Conventional wisdom is that such initial data requires a *large-data* existence theorem for the nonlinear, hyperbolic vacuum Einstein equations.
- We will argue that this conventional wisdom is wrong.

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- Define polar coordinates with the aximuthal angle running from $-\pi/2$ to $\pi/2$ and perform the coordinate transformation
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$$u' = u,\ \underline{u}' = \delta^{-1}\underline{u},\ (\theta', \phi') = \delta^{-1/2}(\theta, \phi).$$
- If we perform this coordinate scaling together with an overall conformal scaling by δ^{-1}, then the Minkowski metric is transformed to
$$\eta' = -2du'd\underline{u}' + [r_0 + 2^{-1/2}(\delta \cdot \underline{u}' - u')^2](d\theta'^2 + \cos^2(\delta^{1/2}\theta')d\phi'^2).$$

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- Note that in the limit $\delta \to 0$ η' converges to

$$\eta' = -2du'd\underline{u}' + [r_0 - 2^{-1/2}u'^2](d\theta'^2 + d\phi'^2),$$

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- This suggests that we should be able to obtain trapped surface formation using only the classical local, small-data existence result of Rendall.

Toy model, I

- In order to gain some intuition into all of this, consider the following spacetime diagram (where $t' = \frac{1}{\sqrt{2}}(\underline{u}' + u')$, $\delta r' = \frac{1}{\sqrt{2}}(\underline{u}' - u')$, etc.).

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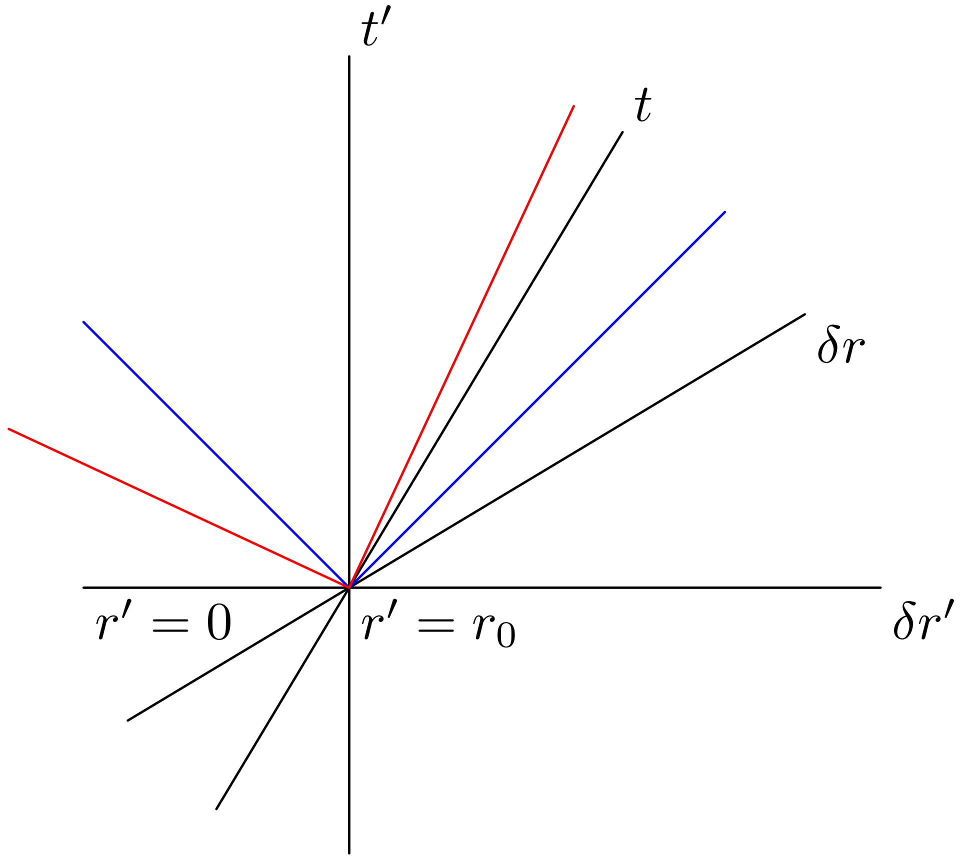

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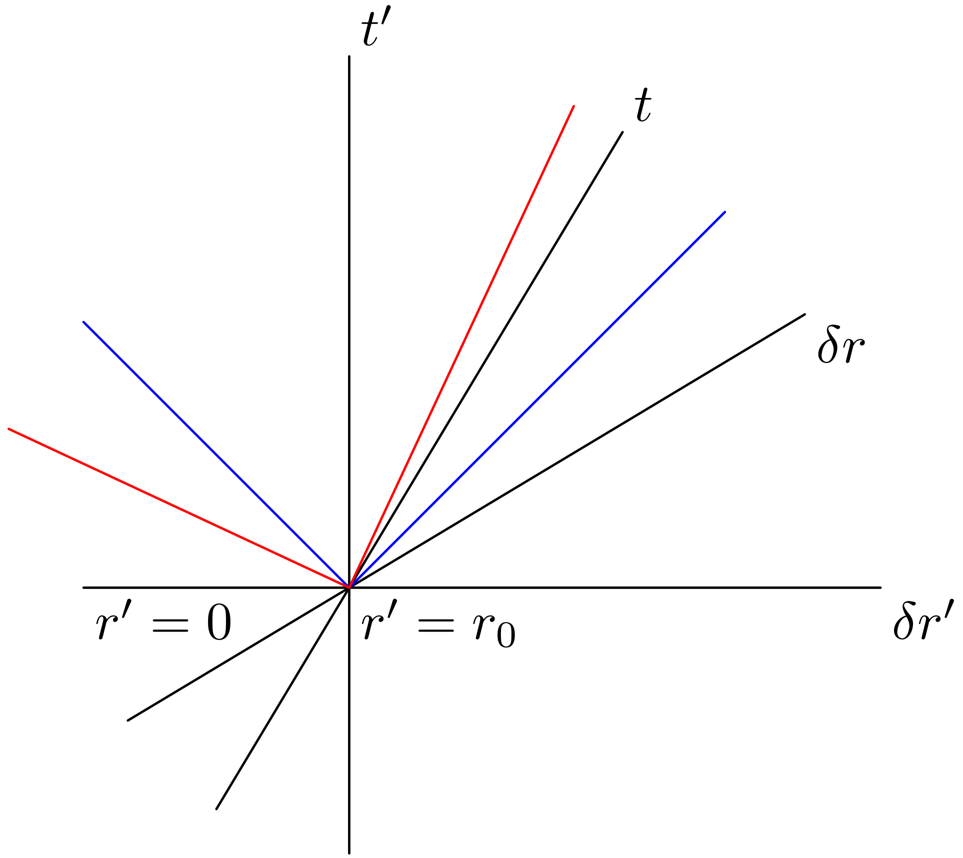

- This suggests that if we were to *rotate* the lightcone by an amount $\sim \delta$, we would obtain trapped surfaces.

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- Let $\Delta = |\det \eta''|$. The incoming and outgoing null expansions of the metric at a point $(u'', \underline{u}'')$ are (up to numerical factors)

$$\partial_{u''}\log\Delta = \Delta^{-1}(-\delta\sin\psi - \cos\psi),\ \partial_{\underline{u}''}\log\Delta = \Delta^{-1}(\delta\cos\psi - \sin\psi).$$

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- We thus get trapped spheres for $\tan\psi > \delta$!

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    - The *outgoing* null expansion along the (scaled) Minkowski data is of size δ.
- While one must take care of some technical points – relating to various uniformity properties – the above result is sufficient to produce solutions to the Einstein vacuum equations which develop trapped surfaces.

Trapped surface formation, I

- We are familiar with the *Raychaudhuri equation*; in our setting, we have schematically

$$\partial_{\underline{u}''} \operatorname{tr} \chi = -\frac{1}{2} |\operatorname{tr} \chi|^2 - |\hat{\chi}|^2.$$

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- This is how, again, we manage to obtain trapped surface formation with *small* incoming data.

Trapped surface formation, II

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\nabla_4 \eta &= -\chi \cdot (\eta - \underline{\eta}) - \beta, \\
\operatorname{div} \hat{\chi} &= \frac{1}{2} \nabla \operatorname{tr} \chi - \frac{1}{2} (\eta - \underline{\eta}) \cdot (\hat{\chi} - \frac{1}{2} \operatorname{tr} \chi) - \beta.
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- We thus expect that $\hat{\chi}$ obeys a *linear wave equation* of the form

$$\nabla_4\nabla_3\hat{\chi} + \frac{1}{2}\mathrm{tr}\,\underline{\chi}\hat{\chi} = -\nabla\hat{\otimes}\mathrm{div}\,\hat{\chi}.$$

Trapped surface formation, III

- Pulling everything together, we find that $\hat{\chi}$ satisfies a wave equation of the form

$$-2\partial_{u'}\partial_{\underline{u}'}\left[(r_0 - 2^{-1/2}u')\hat{\chi}\right] + \frac{1}{(r_0 - 2^{-1/2}u')^2}[\partial_{\theta'}^2 + \partial_{\phi'}^2]\left[(r_0 - 2^{-1/2}u')\hat{\chi}\right] = O(\delta).$$

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- Specifically, if we assume that the *spatial derivatives* $\partial_{\theta'}\hat{\chi}$, $\partial_{\phi'}\hat{\chi}$ have an additional smallness $\sim \eta$, then we obtain that up to small errors, for $\hat{\chi}^0$ the initial data along $u' = 0$,

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- This is sufficient to establish trapped surface formation.

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- The outgoing shear satisfies, up to small errors, a *linear wave equation*.