Summary of past research experience

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0. Introduction

My ultimate research goal is to develop new mathematics, and apply existing mathematics, to advance understanding of fundamental physical law. In particular, this includes both

(a) the development of new mathematical techniques to study established and speculative formulations of physical laws and phenomena in terms of existing mathematical concepts;

(b) the formulation of new mathematical concepts to better model, quantitatively and qualitatively, established and speculative formulations of physical laws and phenomena.

In the following I will give a description of some of my past research projects (as of August 2026).

1. Past research

In reverse chronological order (newest first), my past research projects may be summarized as follows. In the following, the term "Impact" is to be understood broadly, to include both impact on the field and impact on my own personal development as a researcher.

Area: Concentrated solutions to Einstein vacuum equations

Years: 2018 – 2025

Goals: The original goal of this project, when it was given to me as a thesis problem by Spyros Alexakis, was to find Gaussian beam solutions to the Einstein vacuum equations under the simplifying assumption of $U(1)$ (essentially, one translational) symmetry. Such a solution could have applications in, for example, a study of inverse problems for the Einstein equations. Pursuant to my further investigations, which showed that Gaussian beam scaling did not fit the natural scaling for the problem, the problem was amended to finding highly concentrated solutions, i.e., solutions in which gravitational energy is concentrated to an arbitrarily high (but fixed) degree in an arbitrarily small spatial region in the $2+1$-dimensional spacetime obtained by quotienting by the $U(1)$ symmetry, while the existence time is uniform in the size of the support. Concentration implies that we are in the regime of large-data (in some sense) solutions to the nonlinear hyperbolic Einstein equations.

Results: In my doctoral thesis [1] I constructed solutions to the above problem, but for which the total incoming gravitational energy vanished for vanishingly small regions. In subsequent work reported in [2] (in collaboration with Spyros Alexakis) I introduced a novel Klainerman-Sobolev inequality which allowed us to extend the existence results in [1] to solutions with total incoming gravitational energy possessing a uniform lower bound. In both [1] and [2] I then proceeded to construct solutions for which the concentration property of the initial data was preserved, in some sense, throughout the evolution. The construction in [2] is stronger than that in [1] in various ways.

Representative theorem: *Let $\varpi : \mathbf{R}^2 \to \mathbf{R}^1$ be any fixed C^∞ function supported on $(0,1)\times(0,1)$. Then for any sufficiently small $\delta > 0$ there is a solution to the Einstein vacuum equations in $U(1)$ symmetry which has the form*

$$g_{ij} = \eta_{ij} + \delta^{1/4} \cdot \delta g_{ij} + o(\delta^{1/4}),$$

and where $\delta g_{u\underline{u}}|_{u=0}(\underline{u}, x^1) = \varpi(\delta^{-1}\underline{u}, \delta^{-1/2}x^1)$. Such a metric has an incoming null shear (incoming gravitational energy) of size $O(1)$.

Impact: The solutions in [1] are the first example in the literature of (generic) solutions to the Einstein vacuum equations in which gravitational energy is concentrated in *two* spatial dimensions. Further, while the size of the initial data used in [1] and [2] is comparable to or less than that in other results already in the literature, such results prove existence on a region which, while having finite extent in one (null) direction, has vanishing extent in the transverse (null) direction in the limit of very high concentration. [1] and [2] gave the first solutions in which the existence time in *both* null directions is finite, independent of the degree of concentration.

Technical innovations:

Coordinate scaling. While seemingly simple, I believe this is the most significant innovation. The use of scaling ansätze is prevalent throughout the literature on concentrated solutions to the Einstein vacuum equations, starting with the original results of Christodoulou in [3]. However, in previous work such ansätze were used systematically only at the level of *derivative scalings* (see [4]), e.g., by introducing scaled norms in which derivatives in different directions "cost" different powers of δ^{-1}. To my knowledge, with the possible exception of the very understudied [5], [1] and [2] are the only works to implement these scaling ansätze by working with respect to scaled *coordinates*. In particular, it is shown in [1] and [2] that the coordinate scaling

$$u \mapsto u,\ \underline{u} \mapsto \delta^{-1}\underline{u}, x^1 \mapsto \delta^{-1/2}x^1, \tag{1}$$

together with the conformal rescaling $g \mapsto \delta^{-1}g$, takes the large-data, short-time problem given in the above theorem to a *small-data*, *long-time* problem which is amenable to treatment by standard methods such as Klainerman-Sobolev inequalities.

Rectangular coordinates. Another crucial innovation from [1] is the use of *rectangular*, rather than *spherical*, spatial sections $S_{u,\underline{u}}$. In this case, the scaling in (1) can be understood as a simple Lorentz boost followed by an isotropic scaling. When the spatial sections are spherical, the scaling (1) does not preserve the form of the Minkowski metric, and its physical significance is much less obvious. It is also far less clear how to treat the resulting problem mathematically ([4] even states that an ansatz derived from a scaling analogous to (1) "does not quite make sense" in the context of spherical spatial sections). (I return to this in Section 2 below.)

Klainerman-Sobolev inequality. Standard forms of the Klainerman-Sobolev inequality (see [6]) apply to regions with some form of angular symmetry, such as spheres, annular regions, or half-planes. In [2] I constructed a version of the Klainerman-Sobolev inequality which applies instead to narrow, infinitely-long rectangular strips, necessitated by the use of rectangular spatial sections just noted. The decay one obtains from such an inequality is worse by $r^{1/4}$ (where r is a radial variable) than one would typically obtain. This inequality is crucial to obtaining the improved existence time.

Nonbreakdown of initial null hypersurfaces.. Incoming energy causes null geodesic congruences to converge, including those ruling an initial null hypersurface, and more detailed calculations suggest that initial data of the size considered in [2] should cause caustics in the initial hypersurface well before the desired existence time. This problem was avoided by, essentially, forcing the expansion of the null congruence to be initially very large; more precisely, the relevant constraint equations are solved from the *future* end of the hypersurface, forcing any caustic to appear to the *past*.

Nonlinear geometric optics. Since the transformed problem is a small-data one, the geometric-optics based construction used in [2] to obtain a solution which *remains* concentrated can treat the nonlinear terms in the relevant hyperbolic equation roughly at the same level as the geometric-optics approximation error terms, which may be new.

Area: Alternative definitions of Henstock integrals

Years: 2023-2024

Goals: Obtain a definition of the Henstock integral modeled after the Lebesgue integral which would work on more general measure spaces.

Results: While the original goal of this project is shown by [7] to be essentially impossible, I have obtained some tentative results on a dominated convergence theorem for various generalized set-valued integrals which may be new.

Representative theorem: N/A
Impact: This project taught me a great deal about non-Lebesgue integrals, particularly about the Henstock integral and the subtle interplay between measure and topology it contains. It also brought home to me the importance of doing literature reviews early in a project.
Technical innovations: N/A

Area: Mathematical neuroscience: modeling of seizures by Wilson-Cowan equations
Years: 2017-2018
Goals: The so-called Epileptor [8] is a small-dimensional dynamical system which produces timeseries showing many features characteristic of seizure dynamics. My advisor for this project, Jeremie Lefebvre, set me the problem of obtaining a dynamical system with similar behaviour but based on the biologically-motivated Wilson-Cowan equations [9] instead. Part of the original project also envisaged extending the model by adding a spatial component.
Results: I constructed a detailed bifurcation diagram of a particular Wilson-Cowan system, showing how different parameter values resulted in different types of dynamics. I attempted, though ultimately unsuccessfully, to couple this system to a separate system so as to induce seizure-like transitions between different regions of the bifurcation diagram.
Representative theorem: N/A
Impact: This research was never published. The impact of this experience on me as a researcher included introducing me to mathematical modeling in an area lacking a comprehensive theoretical foundation (as opposed to fundamental physics) and hence constructing phenomenological models, and giving me experience in dynamical systems research.
Technical innovations: I was able to construct analytic expressions for the boundary curves in the bifurcation diagram by a technique which may have been novel.

Area: Planetary science: modeling Europan oceans
Years: 2014-2017
Goals: Construct a mathematical model of the electromagnetic properties of Europa, considering its ice sheet, potential subsurface ocean, and solid layers, and determine whether terrestrial measurement of the refraction of ambient Jovian radio waves can put new constraints on the depth of the water layer.
Results: By extending a quasistatic model from geophysics to the full electrodynamics case (including replacing scalar spherical harmonics by vector spherical harmonics), I was able to construct a model of the refraction of radio waves by a stratified sphere given the electrical properties of its layers. The next logical steps, of building computational models and obtaining experimental data for comparison, are still pending.
Representative theorem: N/A
Impact: I believe this model has the potential to provide additional insights into subsurface structures of extraterrestrial bodies, though additional work is clearly required.
Technical innovations: To the best of my knowledge, the full electrodynamic problem had not been treated by this technique before.

Area: Space science instrument modeling: rheometry
Years: 2013-2017
Goals: By making measurements on a scale model, obtain electrical properties of the antennas on the NASA Parker Solar Probe used for measuring low-frequency electromagnetic waves in space plasmas, particularly the solar wind.
Results: Provided important baseline data for interpretation of electric field measurements by the NASA Parker Solar Probe [10].
Representative theorem: N/A
Impact: Personal: this project taught me the importance of making an explicit statistical model of a measurement. My dissatisfaction with the current state of theoretical understanding of the

rheometry technique also led me to attempt to obtain a complete mathematical formulation of the properties of an electrical antenna in vacuum, though this project was never completed.

Technical innovations: The basic technique, termed *rheometry*, is quite well-established [11]. My main contribution (unpublished, though reflected in the numbers reported in [10]) was (a) recognize that the previous nonlinear least squares model used at my research lab could be replaced by a far simpler *linear* least squares model; (b) develop a more careful statistical model of the experimental measurements, and use an estimator allowing for heteroskedasticity in the sample values (see, e.g., [12] for one such estimator).

Area: Auroral plasma physics
Years: 2013-2014
Goals: Obtain multichannel (particle, electric field, magnetic field, optical) data on auroral plasma phenomena.
Results: Wavelet analysis of electric field timeseries data from the GREECE aurora mission; modeling and removal of apparent Langmuir probe interference on magnetometer readings from the same mission.
Representative theorem: N/A
Impact: Through this project and related work I learned a great deal about timeseries analysis, and particularly wavelet analysis. I also gained direct hands-on experience working with experimental data, which included observing, modeling, and compensating for an unexplained interference between instruments.
Technical innovations: N/A

Area: Quantum gravity: building models of quantum time
Years: 2009-2010
Goals: The group of conformal transformations of a two-dimensional spacetime acts transitively on the set of spacelike/Cauchy hypersurfaces in the spacetime. The problem set for me by my master's advisor, Charles Torre, was to attempt to apply the group-theoretical quantization programme of Christopher Isham [13] to this setting so as to build a model of quantum time in two dimensions.
Results: Working in a spacetime diffeomorphic to $\mathbf{R}^1 \times \mathbf{S}^1$ and globally conformal to the Minkowski metric, and using a compact-open C^∞ (Whitney) topology, I was able to obtain an identification of the conformal group as a "diagonal" quotient of a semidirect product involving covering spaces of $\operatorname{Diff}\mathbf{S}^1$. This corrects statements in the literature identifying this conformal group with $\operatorname{Diff}\mathbf{S}^1 \times \operatorname{Diff}\mathbf{S}^1$. Further, I was able to identify the set of spacelike hypersurfaces of such a spacetime with a quotient space of the conformal group, and hence obtain a very concise proof that the action of the conformal group on the space of spacelike hypersurfaces was transitive and continuous (with respect to a compact-open C^∞ topology). The isotropy of this action is conjugate to a conformal transformation representing time reversal.
Representative theorem: *Define*

$$\operatorname{Diff}_{2\pi\mathbf{Z}}(\mathbf{R}^1) = \{f \in \operatorname{Diff}\mathbf{R}^1 \mid f(x+2\pi) = f(x) + 2\pi \textit{ for all } x \in \mathbf{R}^1\}.$$

Let $\operatorname{Diff}^{\pm}_{2\pi\mathbf{Z}}(\mathbf{R}^1)$ *denote the components of* $\operatorname{Diff}_{2\pi\mathbf{Z}}$ *containing increasing and decreasing diffeomorphisms, respectively. Let* G *be the conformal group of a metric on* $\mathbf{R}^1 \times \mathbf{S}^1$ *globally conformal to the Minkowski metric,* K *be the set of all spacelike embeddings of* $\mathbf{S}^1$ *in the spacetime, and give* G *and* K *a suitable compact-open* C^∞ *topology. Then there is a surjective, continuous, open group homomorphism*

$$\Phi : (\operatorname{Diff}^{+}_{2\pi\mathbf{Z}}(\mathbf{R}^1) \times \operatorname{Diff}^{+}_{2\pi\mathbf{Z}}(\mathbf{R}^1) \cup \operatorname{Diff}^{-}_{2\pi\mathbf{Z}}(\mathbf{R}^1) \times \operatorname{Diff}^{-}_{2\pi\mathbf{Z}}(\mathbf{R}^1)) \rtimes S_2 \to G \tag{2}$$

with kernel $\{(f_1, f_2, 1) \mid \textit{for some } n \in \mathbf{Z} \textit{ and all } x \in \mathbf{R}^1,\ f_1(x) = x + 2\pi n,\ f_2(x) = x - 2\pi n\}$. *Here* S_2 *is the symmetric group on two letters and the action in (2) is* $(f_1, f_2) \mapsto (f_{\sigma(1)}, f_{\sigma(2)})$.

Furthermore, there is a surjective, continuous, open map

$$\Omega : (\mathrm{Diff}^+_{2\pi\mathbf{Z}}(\mathbf{R}^1) \times \mathrm{Diff}^+_{2\pi\mathbf{Z}}(\mathbf{R}^1) \cup \mathrm{Diff}^-_{2\pi\mathbf{Z}}(\mathbf{R}^1) \times \mathrm{Diff}^-_{2\pi\mathbf{Z}}(\mathbf{R}^1)) \rtimes S_2 \to K, \tag{3}$$

each fiber of which is a union of two cosets of $\ker\Phi$*. Finally, letting* $\hat{\Phi}$*,* $\hat{\Omega}$ *denote the maps obtained from (2) and (3) by quotienting the domain by* $\ker\Phi$*, we have for any suitable* ω_1*,* ω_2*,*

$$\hat{\Phi}(\omega_1)\hat{\Omega}(\omega_2) = \hat{\Omega}(\omega_1\omega_2),$$

and the isotropy of this action corresponds to time reversal.

Impact: While unpublished, this research corrects the identification in the literature of the conformal group of a cylinder, showing that it has, in some sense, an additional "discrete" part (in the covering-space sense). I believe it was also the first demonstration in the literature of the continuity of the action of the conformal group on the set of spacelike embeddings.

Technical innovations: This research was, to my knowledge, the first study of the properties of the conformal group and space of spacelike embeddings as infinite-dimensional spaces using a compact-open C^∞ topology (as opposed to, for example, a Sobolev topology [14]).

2. Other projects.

The following are two side projects I have worked on and may spend more time on in the future as the opportunity presents itself.

Description: Tooling for mathematical knowledge management

Goals: Create computer software to help in the production of semantic information tied to mathematics papers.

Background: Mathematical knowledge management has the goal of producing software tools for the construction, manipulation, and use of (formalized) mathematical knowledge [15]. While various standards for the formal representation of mathematical knowledge exist (e.g., MathML, OpenMath [16], [17]), producing such formal representations from a mathematics paper requires a substantial amount of additional semantic information not contained in the paper itself.

Tentative results: I have pursued two different approaches to this problem. The first is inspired by so-called *type inference* in computer science (Hindley-Milner [18]), in which the type of a variable in a computer program can often be inferred by the compiler from the context without requiring explicit annotation by the programmer. In the context of a mathematics paper, the hope would be that a minimal amount of additional type information provided by the author for certain key quantities in the paper – much of which could even come from standard libraries of control sequences with such information already attached – would allow a text processor to infer type information about all remaining quantities. While simple in concept, implementation runs into various difficulties which suggest applying concepts from two-level (or van Wijngaarden) grammars [19]. Another approach, for which I have produced an elementary prototype, involves creating an editing environment which prompts and supports the user in providing and managing such semantic information. The prototype I have produced is a so-called `webxdc` app [20] (a platform based on the standard HTML/CSS/JavaScript stack coupled to a messaging framework) and allows for TEX mathematical expressions to be rendered in-app.

Discussion: With the explosion of mathematical knowledge, efficient ways of discovering, accessing, and manipulating the available knowledge are essential. Adding detailed annotations would put too much of a burden on the author. The hope is that usable tooling can help alleviate this burden.

Description: Error-detecting solid codes for Unicode

Goals: Produce an encoding scheme for Unicode [21] based on alphanumeric characters which has good robustness properties under bitflip noise.

Background: In coding theory, a so-called *solid* code is one in which any correctly-transmitted codeword in a string of codewords subject to arbitrary corruption can still be correctly and unambiguously decoded. Error-detecting codes allow for assurance that a decodable codeword has not been corrupted in transmission.

Tentative results: In a recent preprint [22] I have presented a new method for constructing solid codes closely related to that in [23], and also obtained a result on the error-detecting properties of solid codes, utilizing their spatial integrity features. This theoretical work was undertaken in support of an applied project aiming at constructing encoding schemes for Unicode on a restricted alphabet (such as the collection of alphanumeric characters); this work is ongoing but close to completion.

Discussion: While the family of solid codes in [22] does not seem to be of any particular theoretical interest in and of itself, I believe the results on error-detection properties of solid codes may have some independent interest.

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