Research Statement

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1. Introduction

In Section 2 I give a summary of results obtained so far and explain their relation to past work of other researchers. In Section 3 I give three examples of projects which develop the ideas described in Section 2 and which I would be interested in pursuing as part of my postdoctoral research. These are not meant to be exclusive (a brief perusal of my CV should be enough to convince anyone that my research interests are anything but narrow!) and I am of course open to working on other or additional projects in support of the research aims of the Department.

2. Current results

In my thesis [3], I obtained general existence theorems for the Einstein vacuum equations

$$\mathrm{Ric}(g) = 0, \tag{1}$$

with characteristic initial data, for metrics g which are polarised translationally symmetric* and have (reduced) initial data which is large in H^2. Here $\mathrm{Ric}(g)$ denotes the Ricci tensor and g is a Lorentzian metric on a 4-dimensional spacetime M; polarised translational symmetry means that g can be written in the form

$$g = \sum_{i,j=0}^{2} \overline{h}_{ij} dx^i\, dx^j + e^{2\gamma}\left(dx^3\right)^2,$$

where (x_0, x_1, x_2, x_3) give coordinates on M, $\overline{h}_{ij}$ is a $(2+1)$ Lorentzian metric and γ is a scalar function, and $\overline{h}_{ij}$ and γ are independent of x^3. Initial data is given (essentially) by specifying $\overline{h}_{ij}$ and γ along a pair of intersecting null hyperplanes in the reduced $2+1$-dimensional spacetime coordinatised by (x_0, x_1, x_2). I peformed the bulk of the work in the reduced $2+1$-dimensional spacetime, and proved finite-time existence for a certain (broad) class of initial data arbitrarily large in H^2. More concretely, I gave examples of a particular (though still broad) class of families of initial data, depending on a parameter k, for which γ on the initial hyperplanes has H^2 norm of size $k^{3/4}$, but which give rise to solutions which exist up to a time T independent of k (for k sufficiently large). In these latter families of initial data, γ is required to be supported on a rectangle of (coordinate) size $k^{-1/2} \times k^{-1}$ on one of the initial hypersurfaces; I refer to such initial data as a 'blip'.

I then applied these general existence theorems to give examples of Gaussian beam solutions to the Einstein equations which are concentrated along 2-dimensional null hyperplanes in the $3+1$-dimensional spacetime. Specifically, I showed that for a certain choice of initial data, the theory of Gaussian beams (see, e.g., [11] for a general introduction) allows me to show that the *solution* remains concentrated (in an H^1 norm) in a region of (coordinate) size $k^{-1/2} \times k^{-1}$ around a null hyperplane, up to a time T which does not depend on k (although, unlike the general situation, the time T may depend on the degree of concentration). Here k is the parameter introduced in the preceding paragraph.

These results are novel in at least two ways: (a) that the initial data for γ can be large in H^2, without affecting the existence time, indicates that the solutions we have here are beyond those attainable by the L^2 curvature theorem (see [7]); (b) I am not aware of any previous work obtaining Gaussian beam solutions to the Einstein vacuum equations. As I discuss in Section 3 below, these Gaussian beam-type solutions, suitably extended, could give precise examples of so-called *geons*, postulated solutions to the Einstein equations which have so far only been studied in various approximations. (See Section 3 below for details.)

In detail, the results given above are obtained through the following sequence of steps. The work in some of these steps is very standard; those marked with a $*$ involve innovations which I discuss below.

	Reduction of (1) to $2+1$ Einstein equations coupled to free scalar field	(a)
$*$	Introduction of a null geodesic gauge	(b)
	Derivation of Riccati equations from the system in (a), obtaining evolution equations which are a free wave equation coupled to a system of Riccati equations. (This extends work of Alexakis and Fournodavlos [1], where this particular reduction appeared for the first time.)	(c)

* The choice of translational instead of azimuthal symmetry in [3] was made for technical reasons. I anticipate that results similar to those described here hold also in the case of polarised azimuthal symmetry.

- * Derivation of null constraint equations (as ordinary differential equations) in the gauge in (b) from $2+1$ Ricci tensor in (a), and proof that they are preserved by evolution (d)
- * Introduction of a parabolic coordinate scaling, and derivation of above results in the scaled coordinates (e)
- L^∞ estimates on solutions to the constraint equations from (d) (f)
- * Choosing an energy adapted to the Riccati equations in (c) (g)
- * Derivation of energy estimates, using a novel normed algebra to keep track of the norms of the nonlinear terms, and applying the detailed algebraic structure of the system in (c) (h)
- Use of Lipschitz estimates to prove convergence of an iterated sequence arising from the system in (c) (i)
- * Construction of Gaussian beam initial data satisfying uniform lower bounds, in the presence of the constraint equations. (j)

Null geodesic gauge. The null geodesic gauge I used gives only one foliation of the spacetime by null hypersurfaces and is distinct from the usual double-null foliation (used, for example, in [6]). In my null geodesic gauge the (reduced) metric h appearing in step (a) (and which is obtained from $\overline{h}$ by a conformal transformation) takes the form

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & a & b \\ -1 & b & c \end{pmatrix}. \tag{2}$$

In particular, the x^0 coordinate is null and the x^1 coordinate spacelike. The gauge choice forces $c|_{x^0=0} = 0$, so that on $x^0 = 0$ the coordinate x^2 is also null.

That the 02 component of the metric is a constant leads to some simplifications as compared to the double-null gauge (though it should be added that which gauge gives a simpler proof overall can only be determined once my results have been rewritten entirely in double-null, which I haven't done yet).

Constraint equations. The exact form of the constraint equations depends on the choice of gauge. By explicitly calculating all components of the $(2+1)$ Ricci tensor obtained in step (a) and comparing to the Riccati equations obtained in step (c), I am able to determine constraint equations which are preserved under the evolution. More explicitly, I am able to show that if the equations obtained in (a) for the three Ricci components

$$R_{11}, \qquad R_{12}, \qquad R_{22} \tag{3}$$

are satisfied on the null hypersurface $x^0 = 0$, and the Riccati and wave equations from step (c) hold for x^0 in some interval $[0, T']$, then the equations corresponding to (3) hold for all $x^0 \in [0, T']$. These equations, when written out in detail, give a coupled, nonlinear system of partial differential equations on $x^0 = 0$ which, when solved in the correct order, reduce to linear *ordinary* differential (more precisely, transport) equations in x^2.

Parabolic scaling. The structure of the equations from (c) and the constraint equations is such as to admit solutions of the form (here a, b and c are the metric components in (2))

$$a = \left(1 + k^{-1}\overline{\delta\ell}(x^0, k^{1/2}x^1, kx^2)\right)^2, \qquad b = k^{-1/2}\overline{b}(x^0, k^{1/2}x^1, kx^2), \qquad c = \overline{c}(x^0, k^{1/2}x^1, kx^2),$$
$$\gamma = k^{-\iota}\overline{\gamma}(x^0, k^{1/2}x^1, kx^2), \tag{4}$$

$\iota \geq 1/2$, where the functions $\overline{\delta\ell}$, $\overline{b}$, $\overline{c}$, and $\overline{\gamma}$, together with all of their deriviatives up to any desired (finite) order, have L^∞ bounds independent of the parameter k. (Note that the underlying coordinate transformation is simply a Lorentz transformation

$$(x^0, x^1, x^2) \mapsto (k^{-1/2}x^0, x^1, k^{1/2}x^2)$$

followed by an isotropic scaling by $k^{1/2}$, which preserves the Minkowski metric up to multiplication by k.) More specifically, it can be shown that when $\overline{\gamma}|_{x_0=0}$ has support which does not depend on k (for example, on the unit square $[0,1] \times [0,1]$) and is located at either (or both) ends of the $x^0 = 0$ null hypersurface (i.e., near $x^2 = 0$ and/or $x^2 = T$), then the constraint equations from (d) admit solutions of the form (4) on $x^0 = 0$, and the evolution given by the equations in (c) preserves this form. That the constraint equations admit solutions of the form (4) follows by standard ODE theory. That the evolution preserves this form is a key part of the main existence proof, and requires careful work with energy bounds, as I discuss in more detail below.

The importance of these results can be seen by a comparison to the literature. Note first that for solutions of the form (4), the quantities

$$\|k^{m/2+n}\partial_0^\ell \partial_1^m \partial_2^n f\|_\infty, \tag{5}$$

where ℓ, m, and n are positive integers and f is any of $k(a-1)$, $k^{1/2}b$, c, $k^\iota\gamma$, have bounds independent of k, and that if the support of γ is restricted as just indicated, then bounds of this form are propagated by the evolution. The existence and propagation of bounds on norms like that in (5) is a central part of the so-called *short-pulse ansatz* introduced by Christodoulou in [4] and further developed by Klainerman and Rodnianski [6] and Klainerman, Luk and Rodnianski [5]. In particular, Christodoulou [4] uses an ansatz which corresponds in our setting to the simpler assumption (see (2.46))

$$\gamma|_{x^0=0} = k^{-1/2}\overline{\gamma}(x^1, kx^2), \tag{6}$$

i.e., he scales only in the null coordinate x^2 and not in the spatial coordinate x^1. Klainerman and Rodnianski [6] take this a step further by suggesting an ansatz (see [6], (1.20)) for the trace-free part of the second fundamental form, $\hat{\chi}_0$. In our setting, $\hat{\chi}_0$ is dominated by $\partial_2\gamma$, and in terms of this quantity the ansatz in [6] is essentially

$$\partial_2\gamma|_{x^0=0} = k^{1/2}\gamma(k^{1/2}x^1, kx^2, k^{1/2}x^3). \tag{7}$$

In the setting in [6] (where the spatial sections are spheres rather than planes) this ansatz cannot be taken literally without introducing additional support assumptions on f which would potentially cause trouble for the main goal of [6], obtaining trapped surfaces. The work in [6] proceeds instead by using a modification of a corresponding ansatz on norms.†

In both [4] and [6], propagation of the norm ansätze are a central part of the proof. This is what is accomplished in [3] by means of the scaling ansatz (4).

Choice of energy. The structure of the equations in (c), as I will discuss next, is such that it is convenient to close with respect to an energy defined with an extra x^0 derivative. While being important for the derivation of energy estimates, this also means that one obtains bounds on one more x^0 derivative than is typical.

Derivation of energy estimates. The framework I use for the derivation of energy estimates for the wave equation from (c) is entirely standard and even elementary (for example, we make use of only a single time-like multiplier). Showing that this standard framework gives the desired results, however, gives rise to several innovations. One interesting technical innovation is some systematisation of the necessary product-inequality estimates for the nonlinear terms in the differentiated wave equation by making use of a novel normed algebra of what are essentially nonlinear functionals on the solution spaces of the Riccati and wave equations from (c), considered separately or in sequence.

For the estimates to close at all careful use must be made of the detailed structure of the principal part of the wave equation and of the Riccati equations, and the nature of the coupling. As an example: the quantity $\partial_1 b$ appears as a coefficient in the wave equation for γ, and to obtain L^2 bounds on n derivatives of γ requires L^2 bounds on $n-1$ derivatives of $\partial_1 b$. Bounds on b and its derivatives are obtained from the Riccati equation for b, which gives $\partial_0^2 b$ in terms of $\partial_1\gamma$, suggesting that $n-1$ derivatives of $\partial_1 b$ require $n+1$ derivatives of γ, which would result in a loss of derivative. More carefully, though, we need only $n-1$ derivatives of $\partial_1^2\gamma$, and the structure of the principal part of the wave equation for γ allows us to write $\partial_1^2\gamma$ in terms of first order derivatives of γ and their x^0 derivatives. The extra x^0 derivative included in the energy thus allows us to bound the necessary derivatives of $\partial_1 b$ in terms of just n derivatives of γ, avoiding the loss of derivative which is suggested by a standard counting argument.

† While condition (7) certainly holds in our case – where γ does not depend on x^3 at all – the translational symmetry makes it impossible to do a direct comparison with the *norm* ansätze ultimately used in [6].

Still more important for proving finite-time existence for solutions with large H^2 initial data is the observation that the scaling in (4) reduces the wave and Riccati equations from (c) to the corresponding equations in flat, Minkowski space, plus corrections which are of order k^{-1}. With some care to track factors of k through volume and surface area elements, this is the key which allows us finally to conclude that the existence time to such scaled solutions has a lower bound independent of k.

Construction of Gaussian beam. I provide solutions to the system from (c) in which γ is highly focussed along a null geodesic by applying the theory of Gaussian beams. As with the derivation of energy estimates, the general framework I use for obtaining these solutions is entirely standard (see, e.g., [11]). A technical innovation I obtain, which is vital in our case, is the obtaining of lower bounds on the *real part* of the formal Gaussian beam; the standard treatment gives lower bounds on the *modulus of the full complex beam*, which is clearly unsuitable for my situation since I am dealing with a nonlinear coupling and can't simply take the real part at the end.

Two other, more involved, problems arise as follows. A Gaussian beam solution for a wave equation requires knowledge of the metric underlying the wave equation. In my situation this is of course itself coupled back to the Gaussian beam, calling the entire procedure into question. I deal with this by means of the simple observation that once initial data for the Gaussian beam is given, the closeness of the resulting exact solution to the approximate beam follows regardless of how the metric behaves off of the initial surface: in other words, while the approximate Gaussian beam does indeed depend on the behaviour of (some part of) the metric throughout the domain of the solution, the exact Gaussian beam only depends explicitly on the initial values for (some part of) the metric. It turns out that the Gaussian beam depends only on the metric component $a = h_{11}$. This gives rise to the second problem: a must satisfy the constraint equations on the initial hypersurface, and these depend on γ. This suggests one should have to solve some sort of nonlinear system just to obtain the formal Gaussian beam, a major complication. This is resolved by observing that the initial data actually depends only on *a at a single point*, and *multiplicatively*, meaning that we can renormalise $\gamma|_{x^0=0}$ to get rid of this dependence altogether.

3. Potential future projects.

There are many interesting directions in which the research just described can be extended. Here I mention a few.

Considering the limit $k \to \infty$. The work of Luk and Rodnianski in [8], [9], and [10] allows for taking a limit corresponding to $k \to \infty$. While my work in [3] only allows for taking k finite (though arbitrarily large), some hints of what might happen in the limit $k \to \infty$ can be gained from looking at the initial data. Simple scaling arguments suggest the following: (a) along a line of constant x^1 passing through the support of γ, the quantity $(\partial_2\gamma)^2$ (recall that $\partial_2\gamma$ dominates the trace-free second fundamental form $\hat{\chi}_0$) approaches a delta function in x^2 – this is similar to what can be obtained from [9]; (b) along a line of constant x^2 passing through the support of γ, $\partial_2\gamma$ approaches a delta function in x^1 – this has no analogue in [9]; (c) the quantity $\partial_2^2\gamma$, which appears in the curvature tensor, approaches a delta function in x^1 and x^2, apparently giving, in other words, a curvature tensor singular on a set of one dimension less than that obtained in [8] and [9]. It would be very interesting to see whether the full solutions have any meaningful interpretation in this limit, and, if so, how these properties on the initial data might show up in the full solutions.

Blip in the middle. As mentioned in passing above, the function γ must be supported near the ends of the initial hypersurface $x^0 = 0$. This is quite in accordance with the type of initial data used in Christodoulou [4]. Luk and Rodnianski [8], on the other hand, have obtained solutions to the $3+1$ Einstein vacuum equations which possess many similarities to ours but with 'interesting' (i.e., close-to-singular) behaviour at the *center* of the initial hypersurface. This cannot at yet be done within our framework since the corresponding solutions to the constraint equations would not satisfy the scaling (4) unless we take ι so large that the H^2 norm of γ has a uniform bound in k (which holds for $\iota \geq 3/2$). This is unsatisfactory as (a) finite-time existence for such solutions could presumably be shown using known results, e.g., the L^2 curvature theorem; (b) the initial data used in Luk and Rodnianski [8] can be viewed as the limit of initial data scaling similarly to (4) with $\iota = 1$, and the existence results in Luk and Rodnianski [9] evidently even allow $\iota = 1/2$ (though with no scaling in x^1: it is worth noting that the obstruction to solving the constraint equations in our case seems to be due *precisely* to our scaling of x^1). While I have made some progress on producing special initial data with a blip in the middle of the initial hypersurface, it is not clear that the procedure can be carried all the way through, and anyway one would like an existence result for general data. It seems plausible that such a result could be achieved by commuting with additional vector fields, or by using energies associated with additional multipliers (rewriting in the double null gauge may also help). This would be one fairly simple extension which nevertheless holds promise for extending [8] by introducing squeezing in an additional spatial direction.

Solutions concentrated near a geodesic. My thesis work, as noted, gives solutions to the $3+1$ Einstein vacuum equations which are concentrated near a null 2-plane. Obviously, one would like to go a step further and ask for solutions which are concentrated along a single geodesic. I see at least two possible ways of attacking this problem:

(a) I expect it is only a matter of technical difficulty to replace translational symmetry with azimuthal symmetry in the foregoing. (One known issue to be resolved is that the metric h used in the $2+1$ picture is obtained from $\overline{h}$ via a conformal transformation involving γ, and for the case of azimuthal symmetry this conformally rescaled metric will not be Minkowskian even in regions where $\overline{h}$ is.) This suggests the possibility of obtaining a solution to the $3+1$ Einstein vacuum equations which is concentrated near a null cylinder; if the cylinder could be taken of small radius, then this would be a solution concentrated near a geodesic. On the other hand this solution would still possess azimuthal symmetry and hence not be generic.

(b) Klainerman and Rodnianski [6] suggest that the work of Christodoulou [4] can be carried out much more easily if his ansatz (see (6)) and resulting energy bounds are replaced with the energy bounds corresponding to their ansatz (see (7)). I do not believe this work has been carried out in detail, but even so the energy bounds under which [6] obtain existence are weaker. This suggests that the existence results in [6], when applied together with Gaussian beam theory as done in my thesis, have the potential to give solutions concentrated near a single geodesic. The main work to be done here would presumably be the decoupling of the constraint equations from the Gaussian beam initial data, as described for my situation just above.

Colliding beam solutions. Following their work in [8], Luk and Rodnianski in [9] also showed that their framework could be used to treat *colliding* impulsive waves. In my thesis I did not make any real use of the freedom to specify initial data on the null initial surface $x^2 = 0$, and it would be an interesting question to study what can be said about colliding waves within my framework. As two particular examples, one could consider initial data taken near $x^2 = 0$ and $\iota = 1/2$, or the more regular case with initial data in the centre of $x^0 = 0$ and $\iota = 1$, and in either case a supplementary beam imposed at $x^2 = 0$. Should Gaussian beam solutions in $3+1$ dimensions be obtainable from the existence results in Klainerman and Rodnianski [6], it would be interesting to study collisions in that setting as well. There are obviously many possibilities for future research here.

Geons, and extension to Einstein-Maxwell. Geons were originally suggested by Wheeler [12] as possible models for elementary particles. His idea was to consider solutions to the Einstein-Maxwell equations in which a concentrated packet of electromagnetic energy was held together by its own gravitational attraction. Subsequently, Brill and Hartle [2] also considered the case of a pure gravitational geon, where the electromagnetic waves in [12] were replaced by gravitational waves. The results in both [12] and [2] are only valid in certain approximations, and I am not aware of a rigorous mathematical treatment of either geon solution. If Gaussian beam solutions in $3+1$ dimensions can be constructed, it would be most interesting to compare them to the approximate descriptions given in [2]. A separate research project would involve working out – in either our framework, i.e., still with translational symmetry, or more long-term in $3+1$ dimensions – results for focussed solutions to the Einstein-Maxwell equations, and comparing them to the approximate descriptions given in [12].

Again, these are just some concrete examples of future research projects, and I am happy also to work on projects which might be suggested by future mentors.

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