1. Using the binomial series on p. 525 of the textbook, find the Taylor series for the function $\sqrt{r^2 - x^2}$ about $x = 0$.
(a) What is the order of the first nonvanishing term after the constant term?
The binomial series is

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots,$$

and the radius of convergence of the series is 1. (Note that if p is an integer then the series above is actually a finite series, since the product $p(p-1)(p-2)\cdots$ will eventually be zero; in fact, in that case we get exactly what we get from the binomial expansion theorem we probably learned in high school.) Now the function we have is not exactly of the form $(1+x)^p$, so we have to do something to it first. Since we are interested in expanding about $x = 0$, we take x as our variable and assume that r is constant. If we factor out r, then, we have (assuming $r > 0$)

$$\sqrt{r^2 - x^2} = r\sqrt{1 - \frac{x^2}{r^2}}.$$

Now $\sqrt{1 - \frac{x^2}{r^2}} = \left(1 - \frac{x^2}{r^2}\right)^{\frac{1}{2}}$ is of the form to which the binomial series can be applied; thus we obtain

$$\begin{aligned}\sqrt{r^2 - x^2} &= r\left(1 - \frac{x^2}{r^2}\right)^{\frac{1}{2}} \\ &= r\left(1 - \frac{x^2}{2r^2} - \frac{1}{8}\frac{x^4}{r^4} - \frac{1}{16}\frac{x^6}{r^6} - \cdots\right).\end{aligned}$$

Thus the order of the first nonvanishing term after the constant term is 2. (This is just a fancy way of saying that the linear term vanishes, or that the coefficient on the linear term is zero. Recall from the notes that in series such as the one we have here, it is not the case that there is no linear term (even if we might get sloppy sometimes and say that), but rather that the coefficient on the linear term is zero.)
(b) Compare this series to the Taylor series for $r\cos\frac{x}{r}$. Which is larger when x is small compared to r?
Substituting in to the Taylor series for $\cos x$,

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n},$$

we have

$$r\cos\frac{x}{r} = r\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\frac{x^{2n}}{r^{2n}} = r\left(1 - \frac{x^2}{2r^2} + \frac{x^4}{6r^4} - \cdots\right).$$

Comparing this to the series in (a), we see that the order 0, 1, 2, and 3 terms are all equal (the order 1 and 3 terms vanish in both series), while the order 4 terms differ and that for $r\cos\frac{x}{r}$ is greater than that for $\sqrt{r^2 - x^2}$. Since when x is small compared to r the fraction $\frac{x}{r}$ will be small, as long as x is sufficiently small compared to r the first nonequal terms in the Taylor series will determine which function is the larger; and thus for x sufficiently small compared to r we have $r\cos\frac{x}{r} > \sqrt{r^2 - x^2}$.

2. Use the Taylor series of e^x and $\sin x$ about $x = 0$ to find $\lim_{x\to 0} \frac{\left(e^x-1\right)^{50}}{\sin x^{50}}$.

We have the series

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n, \qquad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1};$$

thus

$$\left(e^x - 1\right)^{50} = \left(\sum_{n=0}^{\infty} \frac{1}{n!}x^n - 1\right)^{50} = \left(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots\right)^{50} = x^{50}\left(1 + \frac{1}{2}x + \frac{1}{6}x^2 + \cdots\right)^{50},$$

$$\sin x^{50} = x^{50} - \frac{1}{6}x^{150} + \frac{1}{120}x^{250} + \cdots = x^{50}\left(1 - \frac{1}{6}x^{100} + \frac{1}{120}x^{200} + \cdots\right).$$

Now[1] the limits of both quantities in parentheses above as $x \to 0$ are 1; thus we may write

$$\lim_{x\to 0} \frac{\left(e^x - 1\right)^{50}}{\sin x^{50}} = \lim_{x\to 0} \frac{x^{50}\left(1 + \frac{1}{2}x + \frac{1}{6}x^2 + \cdots\right)^{50}}{x^{50}\left(1 - \frac{1}{6}x^{100} + \frac{1}{120}x^{200} + \cdots\right)} = \lim_{x\to 0} \frac{\left(1 + \frac{1}{2}x + \frac{1}{6}x^2 + \cdots\right)^{50}}{1 - \frac{1}{6}x^{100} + \frac{1}{120}x^{200} + \cdots} = 1.$$

3. Use the Taylor series for e^x and $\cos x$ to find $\lim_{x\to 0} \frac{\left(e^{-x^2}-1\right)^{100}}{1-\cos x^{100}}$.

We proceed as before: $e^{-x^2} = 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \cdots$, so

$$\left(e^{-x^2} - 1\right)^{100} = \left(-x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \cdots\right)^{100} = x^{200}\left(1 - \frac{1}{2}x^2 + \frac{1}{6}x^4 - \cdots\right)^{100},$$

$$1 - \cos x^{100} = 1 - \left(1 - \frac{1}{2}x^{200} + \frac{1}{24}x^{400} - \frac{1}{720}x^{600} + \cdots\right) = x^{200}\left(\frac{1}{2} - \frac{1}{24}x^{200} + \frac{1}{720}x^{400} - \cdots\right),$$

so

$$\lim_{x\to 0} \frac{\left(e^{-x^2} - 1\right)^{100}}{1 - \cos x^{100}} = \lim_{x\to 0} \frac{\left(1 - \frac{1}{2}x^2 + \frac{1}{6}x^4 - \cdots\right)^{100}}{\frac{1}{2} - \frac{1}{24}x^{200} + \frac{1}{720}x^{400} - \cdots} = 2.$$

4. Suppose that I tell you thata certain function has a Taylor series about $x = 0$ given by $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$. What is $f'(-1)$?

Differentiating term by term, we have

$$f'(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n},$$

so[2]

$$f'(-1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots = \ln 2 \approx 0.693.$$

[1]Here we are assuming that we can interchange the order of two limits, the limit $\lim_{x\to 0}$ and the limit $\lim_{n\to\infty}$ which we are taking implicitly in writing out an infinite sum. In general, interchanging limits requires great care, but in this case everything works out all right.

[2]note that, since -1 is an endpoint of the interval of convergence, differentiating term-by-term and then evaluating at -1 is not entirely justified. Were we at, say, $-\frac{1}{2}$, this process would be fine. It turns out that it works in this case, but it might not in general.

5. Recall that the derivative of $\sin^{-1} x$ is $\frac{1}{\sqrt{1-x^2}}$. Use this and the binomial series to find the first three terms of the Taylor series of $\sin^{-1} x$. Given that $\sin^{-1} \frac{1}{2} = \frac{\pi}{6}$, what approximation to π does the above series give? Answer to three decimal places.

We recall the binomial series:

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots.$$

In this case, we have $\frac{1}{\sqrt{1-x^2}} = \left(1-x^2\right)^{-\frac{1}{2}}$, so we replace x by $-x^2$ and set $p = -\frac{1}{2}$ to obtain

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4.$$

Since $\sin^{-1}(0) = 0$, we have

$$\begin{aligned}\sin^{-1}(x) &= \int_0^x \frac{1}{\sqrt{1-t^2}}\,dt = \int_0^x \left(1 + \frac{1}{2}t^2 + \frac{3}{8}t^4 + \cdots\right)dt \\ &= \left(t + \frac{1}{6}t^3 + \frac{3}{40}t^5 + \cdots\right)\Bigg|_0^x = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \cdots.\end{aligned}$$

Note that since the radius of convergence for the binomial series is 1, we expect the above series to converge to $\sin^{-1} x$ as long as $|x| < 1$. Thus we may set $x = \frac{1}{2}$ in the above to obtain

$$\frac{\pi}{6} = \sin^{-1}\frac{1}{2} \approx \frac{1}{2} + \frac{1}{48} + \frac{3}{1280} \approx 0.5231771,$$

so the approximation for π we get is

$$\pi \approx 3.139.$$

Not terribly good, but not too bad either, and we could clearly get a better approximation by including a couple more terms in the binomial series.