1. Suppose that I tell you that a certain function has a Taylor series about $x = 0$ given by

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2}.$$

(a) Find the radius of convergence of the series.

As usual, we let $C_n = \frac{1}{n^2}$ and find

$$\lim_{n\to\infty} \frac{C_{n+1}}{C_n} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^2} = 1,$$

which says that the radius of convergence is 1.

(b) What is the interval of convergence of the series?

Since the radius of convergence is 1, we know that the interval of convergence must at least contain $(-1, 1)$, and at most can contain this interval plus one or both of its endpoints; thus we only need to check the points $x = \pm 1$. Note that since we used the ratio test to determine the radius of convergence, and the endpoints of the interval of convergence are exactly where the ratio test fails, we cannot use the ratio test to investigate $x = \pm 1$; instead we must use some other test. In this case, if $x = 1$, the series becomes

$$\sum_{n=1}^{\infty} \frac{1}{n^2},$$

which is a convergent p-series. If $x = -1$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2};$$

now the series of the absolute values of the terms of this series is just

$$\sum_{n=1}^{\infty} \frac{1}{n^2},$$

which is just the convergent p-series we found at $x = 1$. Thus the series at $x = -1$ is absolutely convergent and, hence, convergent.[1] Thus the interval of convergence is exactly $[-1, 1]$.

(c) What is $f'(0)$?

Since the function f is represented by the above power series, the above power series must be the Taylor series for f about $x = 0$; in other words, if we let again $C_n = \frac{1}{n^2}$, then we must have

$$\frac{1}{n!} f^{(n)}(0) = C_n = \frac{1}{n^2},$$

so that

$$f^{(n)}(0) = \frac{n!}{n^2} = \frac{(n-1)!}{n}.$$

[1]One of the reasons absolute convergence is important is as follows. Chapter 9 of the textbook discussed monotone sequences, and in particular pointed out that a monotone increasing sequence which has an upper bound must converge. Now in general it is much easier to show that a sequence has an upper bound than it is to show that a sequence converges; thus monotone increasing sequences are particularly nice to work with. Now if we have a series of nonnegative (positive or zero) terms, then the corresponding sequence of partial sums will be monotone increasing (since at every step we are adding something which is positive or zero); thus if we can show that this sequence is bounded above we will know that it must converge – but that would mean that the original series would converge. In other words, a series of positive terms will converge if it is bounded above. This means that if we can show that the partial sums of the absolute values of the terms of a series is bounded above, then the original series must be absolutely convergent and hence convergent.

Thus $f'(0) = f^{(1)}(0) = \frac{0!}{1} = \frac{1}{1} = 1$ (recall that $0! = 1$ by definition).
(d) What is $f^{(5)}(0)$?
We have only to apply our general formula:

$$f^{(5)}(0) = \frac{4!}{5} = \frac{24}{5} = 4.8.$$

2. Let us find the Taylor series for $\sin x$ about $x = \frac{\pi}{2}$. (Before starting, it may be helpful to sketch the graph of $\sin x$ on $[0, \pi]$.) Let us set $f(x) = \sin x$, $a = \frac{\pi}{2}$.
(a) What is $f(a)$?
Clearly, $f(a) = \sin \frac{\pi}{2} = 1$.
(b) What is $f'(a)$?
Since $f'(x) = \cos x$, we have $f'(a) = \cos \frac{\pi}{2} = 0$.
(c) What is $f''(a)$?
Since $f''(x) = -\sin x$, we have $f''(a) = -\sin \frac{\pi}{2} = -1$.
(d) What is $f'''(a)$?
Since $f'''(x) = -\cos x$, we have $f'''(a) = -\cos \frac{\pi}{2} = 0$.
(e) What is $f^{(4)}(a)$?
Since $f^{(4)}(x) = \sin x$, we have $f^{(4)}(a) = \sin \frac{\pi}{2} = 1$.
(f) Given your answers to (a) – (e), write out the general term in the Taylor series. This formula should remind you of another Taylor series we have already seen. Which of the functions below has a Taylor series about $x = 0$ whose coefficients are the same as those in the current Taylor series?
From our work in (a) – (e), we observe that for $k \geq 0$

$$\begin{aligned} f^{(4k)}(a) &= 1, \\ f^{(4k+1)}(a) &= 0, \\ f^{(4k+2)}(a) &= -1, \\ f^{(4k+3)}(a) &= 0. \end{aligned}$$

Thus only the even-order terms in the Taylor series have nonzero coefficients, and among these terms the coefficients oscillate between 1 and -1. Thus the coefficient of the $2n+1$th order term will always be zero, while the coefficient of the $2n$th order term will be $(-1)^n \frac{1}{(2n)!}$. Thus the Taylor series for $\sin x$ about $x = \frac{\pi}{2}$ is

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(x - \frac{\pi}{2}\right)^{2n}.$$

The coefficients on this series are exactly the same as those on the Taylor series for $\cos x$ centred around $x = 0$; thus the answer is C. (This makes sense since from trigonometric identities we know that $\cos\left(x - \frac{\pi}{2}\right) = \sin x$, and if we substitute $x - \frac{\pi}{2}$ in the Taylor series for $\cos x$ around $x = 0$, we get exactly the series above.)