1. Consider the power series $\sum_{n=0}^{\infty} n!x^n$. Using the ratio test, determine all possible values of x for which the series converges. Use this to determine the radius of convergence.

To apply the ratio test to this series, we must calculate the limit of the sequence

$$\left| \frac{(n+1)!x^{n+1}}{n!x^n} \right| = |(n+1)x| .$$

We see that this limit is 0 if $x = 0$, while it does not exist otherwise. Thus the series converges only when $x = 0$, which means that its radius of convergence is 0.

2. Suppose I tell you that the radius of convergence of the power series $\sum_{=0}^{\infty} a_n x^n$ is 2. What is the radius of convergence of $\sum_{n=0}^{\infty} \frac{a_n}{2^n} x^n$?

We know that the series $\sum_{n=0}^{\infty} a_n x^n$ converges as long as $|x| < 2$, and diverges whenever $|x| > 2$ (it may converge or diverge when $x = \pm 2$, we don't know). Now the second series can be written

$$\sum_{n=0}^{\infty} \frac{a_n}{2^n} x^n = \sum_{n=0}^{\infty} a_n \frac{x^n}{2^n} = \sum_{n=0}^{\infty} a_n \left(\frac{x}{2}\right)^n ;$$

this is just the original series with x replaced by $\frac{x}{2}$. Since the original series converged whenever $|x| < 2$, this new series must converge whenever $\left|\frac{x}{2}\right| < 2$, i.e., whenever $|x| < 4$. Similarly, it must diverge when $|x| > 4$. Thus the radius of convergence of the new series is exactly 4.

It is worth pointing out here that the formula for the radius of convergence given in the textbook, derived from applying the ratio test to the power series, cannot be done in reverse; in other words, knowing that the radius of convergence of a certain series is R (say) does *not* allow us to conclude that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{R}$, simply because there is no reason for this limit to exist: it could be that infinitely many a_n were zero, which would mean that the sequence would not even be defined for infinitely many n, and hence could not possibly have a limit. Thus we cannot solve this problem by applying the ratio test to the new series.

3. Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(n!)^3}{(3n)!} x^n .$$

Let $C_n = \frac{(n!)^3}{(3n)!}$; then we have

$$\frac{C_{n+1}}{C_n} = \frac{[(n+1)!]^3}{[3(n+1)]!} \frac{(3n)!}{(n!)^3} = \frac{[(n+1)n!]^3}{(3n+3)(3n+2)(3n+1)(3n)!} \frac{(3n)!}{(n!)^3} = \frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)} .$$

If we multiply out the cube in the numerator, and the three binomials in the denominator, we see that this is a quotient of two third-order polynomials; thus we may find its limit as $n \to \infty$ by dividing numerator and denominator by n^3, as follows:

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{C_{n+1}}{C_n} \right| &= \lim_{n\to\infty} \frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)} \\ &= \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^3}{\left(3+\frac{3}{n}\right)\left(3+\frac{2}{n}\right)\left(3+\frac{1}{n}\right)} \\ &= \frac{1}{27} . \end{aligned}$$

This tells us that the radius of convergence is 27.

4. Find the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{2^n+n^2}{n^{10}+2} x^n$.

We proceed as before. Let $C_n = \frac{2^n+n^2}{n^{10}+2}$; then we have

$$\frac{C_{n+1}}{C_n} = \frac{2^{n+1}+(n+1)^2}{(n+1)^{10}+2} \frac{n^{10}+2}{2^n+n^2} = \frac{2^{n+1}+(n+1)^2}{2^n+n^2} \frac{n^{10}+2}{(n+1)^{10}+2} .$$

Now we look for the largest term in the numerators and denominators of the above fractions. Clearly 2^n is larger than n^2 when n is large (in fact, whenever $n > 2$)[1]; multiplying numerator and denominator by 2^{-n}, we obtain for the first factor above

$$\frac{2 + (n+1)^2 \cdot 2^{-n}}{1 + n^2 \cdot 2^{-n}},$$

which goes to 2 as $n \to \infty$. Similarly, for the second term we multiply the numerator and denominator by n^{-10}, obtaining

$$\frac{1 + 2n^{-10}}{\left(1 + \frac{1}{n}\right)^{10} + 2n^{-10}},$$

which goes to 1 as $n \to \infty$. Thus we obtain finally

$$\lim_{n\to\infty} \left| \frac{C_{n+1}}{C_n} \right| = 2 \cdot 1 = 2,$$

so that the radius of convergence is $\frac{1}{2}$.

(This example illustrates a general principle: powers of n generally do not affect the radius of convergence of a power series. Note the word 'generally' – this is not a theorem! but rather a principle which we can keep in mind to get some notion of what our answer ought to be, and to check whether what we obtain makes sense.)

5. Suppose I tell you that a certain power series $\sum_{n=0}^{\infty} a_n x^n$ converges for $x = 1$ and diverges for $x = 2$. Which of the following statements must be true given just this information?

Since the series is centred at 0 and converges for $x = 1$, we know that the radius of convergence must be at least 1. On the other hand, since the series diverges for $x = 2$, the radius of convergence can be at most 2. (It could equal 1 or 2, in the which case one or the other of the points given would be an endpoint of the interval of convergence.) Thus we can say for certain that the series must converge on the interval $(-1, 1]$, and possibly on a larger one, and must diverge at least on $(-\infty, -2) \cup [2, \infty)$, and possibly on a larger set. This means that the series must diverge at $x = -3$, so the correct answer is D. A and B are not right since we do not know anything about the series on the interval $(1, 2)$. C is not right since convergence at one endpoint does not imply convergence at the other (consider the case $a_n = \frac{(-1)^n}{n}$, for example; this would have interval of convergence exactly equal to $(-1, 1]$).

[1] The author cannot pass up this opportunity to present one of his favourite number problems. Can you find distinct positive integers n and m such that $n^m = m^n$? It turns out that there is only one pair! (Up to ordering, of course.)