# FOCUSSED SOLUTIONS TO THE EINSTEIN VACUUM EQUATIONS

by

Nathan Thomas Carruth

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University of Toronto

# ABSTRACT

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Nathan Thomas Carruth, Graduate Department of Mathematics, University of Toronto

We construct solutions to the Einstein vacuum equations in polarised translational symmetry in $3+1$ dimensions which have $H^1$ energy concentrated in an arbitrarily small region around a two-dimensional null plane and large $H^2$ initial data. Specifically, there is a parameter $k$ and coordinates $s$, $x$, $v$, $y$ such that the null plane is given by $x = k^{-1/2}/2$, $v = T\sqrt{2} - k^{-1}/2$ for some $T$ independent of $k$, the $H^1$ energy of the solution is concentrated on the region $[0, T'] \times [0, k^{-1/2}] \times [T\sqrt{2} - k^{-1}, T\sqrt{2}] \times \mathbf{R}^1$, and the $H^2$ norm of the initial data is bounded below by a multiple of $k^{3/4}$. The time $T'$ has a lower bound independent of $k$. This result relies heavily on a new existence theorem for the Einstein vacuum equations with characteristic initial data which is large in $H^2$. This result is proved using parabolically scaled coordinates in a null geodesic gauge.

## ACKNOWLEDGMENTS

First of all, I would like to thank my advisor, Spyros Alexakis, for being willing to take a chance on a fourth-year doctoral student who wanted to switch research areas for at least the fourth time, for giving him a clear problem to work on, for putting up with (sometimes extended) periods of inactivity for ill-explained reasons and putting up with obtuse insistence on bizarre notations, and for helping that doctoral student stay focussed enough to finally finish.

I would like to particularly acknowledge Professor Alexakis' help in editing the final manuscript, his encouragement to keep going and obtain better results even when I would rather have stopped, and his explanations to me of how the current project fits in the broader setting of the mathematical study of the Einstein vacuum equations. Without his editorial review, in particular, this document would probably have remained an impenetrable hodgepodge of random results – to use a technical term from computer programming, 'obfuscated'!

I would like to thank my other committee members, Robert Jerrard and Stefanos Aretakis, for being willing to serve on my committee, and for putting up also with long (and unexplained) periods of silence on my part. I would also like to thank the external reviewer, Jonathan Luk, for his careful review and helpful comments. As usual, all remaining errors are my own responsibility.

I would also like to thank my parents and family, as well as Kullervo Hynynen, David Olsen, and others, for their encouragement and support.

A sizeable portion of the work presented below, as well as much of the final writing, were done while the author was under lockdown due to the Covid-19 pandemic. The notation C-naught nu, which appears regularly in various parts of Chapter 6, was chosen without any attention to its typographical appearance, but when this was noticed it was decided to keep it as a memorial to and reminder of the challenges current during the writing of this work.

Finally, I wish to express my thanks to my God, without whom I am nothing.

> "Therefore, in the beginning the Word was, for he was the Word, even the messenger of salvation – the light and the Redeemer of the world; the Spirit of truth, who came into the world, because the world was made by him, and in him was the life of men and the light of men. The worlds were made by him; men were made by him; all things were made by him, and through him, and of him. ... And truth is knowledge of things as they are, and as they were, and as they are to come; ... I am the Spirit of truth, and John bore record of me, saying: He received a fullness of truth, yea, even of all truth; and no man receiveth a fulness unless he keepeth his commandments. He that keepeth his commandments receiveth truth and light, until he is glorified in truth and knoweth all things. Man was also in the beginning with God. Intelligence, or the light of truth, was not created or made, neither indeed can be. All truth is independent in that sphere in which God has placed it, to act for itself, as all intelligence also; otherwise there is no existence. ... The glory of God is intelligence,

or, in other words, light and truth."

"And the light which shineth, which giveth you light, is through him who enlighteneth your eyes, which is the same light that quickeneth your understandings; which light proceedeth forth from the presence of God to fill the immensity of space – the light which is in all things, which giveth life to all things, which is the law by which all things are governed, even the power of God who sitteth upon his throne, who is in the bosom of eternity, who is in the midst of all things. ... He comprehendeth all things, and all things are before him, and all things are round about him; and he is above all things, and in all things, and is through all things, and is round about all things; and all things are by him, and of him, even God, forever and ever. And again, verily I say unto you, he hath given a law unto all things, by which they move in their times and their seasons; ... Behold, all these are kingdoms, and any man who hath seen any or the least of these hath seen God moving in his majesty and power."

(Doctrine and Covenants 93:8 – 10, 24, 26 – 30, 36; 88:11 – 13, 41 – 42, 47.)

To Him in whom alone I have light and life, I express in reverence the praise and gratitude of my heart, and pray that in studying creation I may honour the Creator. *Soli Deo Gloria.*

Nathan Carruth, Toronto, September 2020

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獻我未來的妻子，
願我們父神的恩典與妳同在，
雖未相識，
還讓妳感到我心裡對妳的愛。

獻我的神，
因爲祂至高無上，
慈悲又公正，
一切能力都在祂的手裡，
永無止境，
永無盡期。
願榮耀僅歸祂一身，
永永遠遠。

# 0. INTRODUCTION

## 0.1. Previous work

The question of finite-time existence of solutions to the Einstein vacuum equations has been studied from various angles. These include attempts to prove existence for general data in as low regularity as possible, such as the $L^2$ curvature conjecture (see Klainerman, Rodnianski and Szeftel [8]). This provides a lower bound on the existence time of a solution to the Einstein vacuum equations with general initial data depending on, among other things, the $L^2$ norm of the curvature of the initial data, and is currently the best result known in this direction;* for an earlier classical result, see Fischer and Marsden [4]. From another direction, one may seek more special solutions with even lower initial regularity. The results in this thesis fall into this second category. Other important examples in the recent literature include the results of Christodoulou [3], Klainerman and Rodnianski [6], and Klainerman, Luk and Rodnianski [5] on the formation of trapped surfaces, as well as Luk and Rodnianski's work on impulsive gravitational waves, where the initial data has a delta-function singularity (see [9], [10]). We will now briefly review these results.

The papers [3], [6], [5], and [9] all make use of the same basic geometric setup, namely a double null foliation of the spacetime. This can be described as follows. The spacetime is foliated by null geodesic cones

$$C_u, \underline{C}_{\underline{u}},$$

where $C_u$ is generated by outgoing null geodesics and $\underline{C}_{\underline{u}}$ by incoming null geodesics, and $u$ and $\underline{u}$ are optical functions. $C_u$ and $\underline{C}_{\underline{u}}$ are assumed to intersect in spheres $S_{u,\underline{u}}$; let $\theta^A$ denote coordinates on these spheres (we assume $\theta^A$ to be transported along the geodesics generating $C_u$ and $\underline{C}_{\underline{u}}$). The work in [3], [6] and [5] makes use of initial data on a particular outgoing null cone $C_{u_0}$ which is assumed to be Minkowskian except for a 'blip' on an interval of length (in $\underline{u}$) equal to a (suitably small) number $\delta$.

Christodoulou [3] then specifies initial data for the (conformal class of the) metric on $C_{u_0}$ as follows (see [3], 2.1). This process is slightly involved, but the important part for our purposes is as follows. Using stereographic coordinates, the specification of the conformal class of a metric on the spheres $S_{u_0,\underline{u}}$ is reduced to the problem of finding positive-definite elements of $SL_2(\mathbf{R})$ (2.28), which are then expressed as the exponentials of elements of $sl_2(\mathbf{R})$ (the set of trace-free, symmetric $2 \times 2$ matrices on $\mathbf{R}$). These matrices are then specified using the ansatz ([3], (2.46))

$$\psi_{u_0}(\underline{u}, \theta) = \frac{\delta^{1/2}}{|u_0|} \psi_0 \left( \frac{\underline{u}}{\delta}, \theta \right), \tag{0.1.1}$$

for a fixed function $\psi_0$. $\psi_0$ is related (though not identical) to the function $\overline{\gamma}_0$ introduced below (see equation (0.2.29) and equation (5.1.20)).

* On the other hand, after reducing by a one-dimensional symmetry as is done in this work, better general results become available: for example, the work of Smith and Tataru [14] suggests a general existence result in $H^{7/4+}$. This would still not be strong enough for the purposes of this work. (We thank the external reviewer for bringing this point to our attention.)

Klainerman and Rodnianski [6] re-express condition (0.1.1) in terms of ansätze on the trace-free part of the second fundamental form of $C_{u_0}$, $\hat{\chi}_0$, as follows (see [6], (1.14)):

$$\hat{\chi}_0(\underline{u},\omega) = \delta^{-1/2} f_0(\delta^{-1}\underline{u},\omega), \tag{0.1.2}$$

where $\omega$ are transported coordinates along their $H_0$ (equivalent to $C_{u_0}$ in [3]); note that this is in line with (0.1.1) since $\hat{\chi}_0$ should contain a $\underline{u}$ derivative of the metric. They then observe that a natural alternative would be the parabolic scaling (see [6], (1.16))

$$\hat{\chi}_0(\underline{u},\omega) = \delta^{-1/2} f_0(\delta^{-1}\underline{u},\delta^{-1/2}\omega), \tag{0.1.3}$$

and use this to motivate conditions on the $L^2$ norms of the curvature components and Ricci coefficients, for example the assumption ([6], (2.2))*

$$\delta^{1/2}\|\hat{\chi}_0\|_{L^\infty(0,\underline{u})} + \sum_{k=0}^{2} \delta^{1/2}\|(\delta\nabla_4)^k\hat{\chi}_0\|_{L^2(0,\underline{u})} + \sum_{k=0}^{1}\sum_{m=1}^{4} \delta^{1/2}\|(\delta^{1/2}\nabla)^{m-1}(\delta\nabla_4)^k\nabla\hat{\chi}_0\|_{L^2(0,\underline{u})} < \infty, \tag{0.1.4}$$

where $\nabla_4$ indicates differentiation with respect to $\underline{u}$, and $\nabla$ indicates differentiation with respect to the angular variables.

It should be noted that in [3], as well, the specific ansatz (0.1.1) is less important than the conditions on the norms of the curvature components which it motivates, see [3], p. 20.

Ansätze (0.1.2) and (0.1.3) give, in the left- and right-hand columns respectively,

$$\begin{aligned} \|\hat{\chi}_0\|_{L^\infty} &\sim \delta^{-1/2}, & \|\hat{\chi}_0\|_{L^\infty} &\sim \delta^{-1/2}, \\ \|\partial_{\underline{u}}^k\hat{\chi}_0\|_{L^\infty} &\sim \delta^{-k-1/2}, & \|\partial_{\underline{u}}^k\hat{\chi}_0\|_{L^\infty} &\sim \delta^{-k-1/2}, \\ \|\nabla^m\hat{\chi}_0\|_{L^\infty} &\sim \delta^{-1/2}, & \|\nabla^m\hat{\chi}_0\|_{L^\infty} &\sim \delta^{(-m-1)/2}, \end{aligned} \tag{0.1.5}$$

where the $L^\infty$ norms are taken over the initial outgoing null cone. The results in the left-hand column can be compared to [3], (2.117), (2.69), (2.71); those in the right-hand column can be compared with the condition (0.1.4).

The goal of [3] and [6] is to prove the existence of trapped surfaces, and this guides their assumptions. Luk and Rodnianski [9], [10] proceed in a different direction (though obtaining, inter alia, a trapped surface result, see [10]), by constructing solutions to the Einstein vacuum equations for which the curvature has a delta function singularity across a null hypersurface. This is done by exploiting the structure of the Einstein equations to compensate for the lack of regularity in one null direction by using high regularity in the spatial directions. In [9] they construct initial data possessing such a curvature singularity, and also give a sequence of $C^\infty$ initial data converging to it. Their main work is concerned with the singular initial data and hence does not make use of scaling ansätze like those above (which would be inapplicable in the singular null

* Here by $L^2(0,\underline{u})$ is meant an $L^2$ norm on the sphere $S_{0,\underline{u}}$, and by $L^\infty(0,\underline{u})$ is evidently meant an $L^\infty$ norm on the same sphere. Unlike [3], [6] solve in the region $u \geq 0$ rather than $u \geq u_0$.

direction). Nevertheless, the smooth approximating sequence they construct has properties which can be compared to (0.1.4). Specifically, use is made of the sequence of functions (see [9], section 3.1; we have corrected an addition to a multiplication, as evidently intended)

$$h_n(x) = 1_{\{x \geq 0\}} \cdot \sum_{j=-\infty}^{n} \tilde{h}(2^j x) \tag{0.1.6}$$

where

$$\tilde{h}(x) = \begin{cases} \tilde{h}_0(x) - \tilde{h}_0(2x), & x \geq 0 \\ 0, & x < 0, \end{cases}$$

and $\tilde{h}_0$ is a $C^\infty$ function with support contained in $[-1,1]$ and identically equal to 1 on $[-1/2, 1/2]$. The smooth approximating sequence to the metric on the initial outgoing null cone is then obtained as follows. First set

$$\underline{\hat{\gamma}}_n = \hat{\gamma}_1 + (\underline{u} - \underline{u}_s) h_n(\underline{u} - \underline{u}_s) \hat{\gamma}_2, \tag{0.1.7}$$

where $\hat{\gamma}_1$, $\hat{\gamma}_2$ are positive definite matrices with $\det\hat{\gamma}_1 = 1$, and $\underline{u}_s$ is a parameter chosen sufficiently small that $\underline{\hat{\gamma}}_n$ is still positive definite. The smooth approximating sequence to the metric is then obtained by normalising $\underline{\hat{\gamma}}_n$:

$$\hat{\gamma}_n = \frac{1}{\sqrt{\det\underline{\hat{\gamma}}_n}} \underline{\hat{\gamma}}_n. \tag{0.1.8}$$

Solutions are found for $\underline{u} \in (0, \epsilon)$, where $\epsilon \geq 2\underline{u}_s$ is sufficiently small.

To connect these results to the scaling ansätze in [3] and [6], we make the following observations. From (0.1.6), we obtain for all $n$

$$h_n(x) = h_{n-1}(2x) = h_0(2^n x), \qquad \text{all}\, x,\, n \tag{0.1.9}$$

$$h_n(x) = \begin{cases} 0, & x < 2^{-(n+2)} \\ 1, & x > 2^{-(n+1)} \end{cases}. \tag{0.1.10}$$

Let us fix $n$ and, for the purpose of comparison with [3] and [6], set $\delta = 2^{-n}$. By (0.1.10), we have

$$h_n'(x) = 0, \qquad x \notin (2^{-(n+2)}, 2^{-(n+1)}),$$

so by (0.1.9) and (0.1.10) we have, setting $M = \sup_{x \in \mathbf{R}} h_0'(x)$,

$$\frac{d}{dx}\left(x h_n(x)\right) = h_n(x) + x h_n'(x) = h_0(2^n x) + 2^n x h_0'(2^n x) \leq 1 + \frac{1}{2} M.$$

From this it is evident that there is some constant $C$ independent of $n$ such that

$$\left| \frac{\partial}{\partial \underline{u}} \hat{\gamma}_n \right| \leq C$$

for $\underline{u} \in (0, \underline{u}_s)$. Continuing, it is clear that there is some constant $C_m$ independent of $n$ such that

$$\left| \frac{\partial^m}{\partial \underline{u}^m} \hat{\gamma}_n \right| \leq C_m \cdot 2^{mn} = C_m \delta^{-m} \tag{0.1.11}$$

for $\underline{u} \in (0, \underline{u}_s)$. (Luk and Rodnianski obtain a similar result for the difference $\hat{\gamma}'_n = \hat{\gamma}_n - \hat{\gamma}_{n-1}$, see [9], section 3.1.) Using (0.1.11) to compare $\partial_{\underline{u}}\hat{\gamma}_n$ with the ansätze in (0.1.5), it is clear that the derivatives of the metric, in the direction along the initial outgoing null cone, are smaller in [9] by a factor of $\delta^{1/2}$. This will be discussed more later in the context of our results, see the discussion at the end of the next section.

Note that, like [3], the result in [9] does not include any kind of scaling in the spatial directions tangent to the spheres $S_{u,\underline{u}}$. As we noted above, the ability to exploit high regularity in the spatial directions is important to the work in [9].

### 0.2. Introduction

We have seen that [3] and [6] both make use of scaling assumptions on initial data. In both cases, however, the work is carried out, and the results proved, in the original coordinate system (what we might term the *physical picture*), and powers of the scaling parameter are kept track of through weighted norms (see (0.1.4), for example). In this thesis we make the next logical step and rewrite Einstein's equations in a coordinate system to which a parabolic scaling (like that in (0.1.3)) has been applied.

More specifically, we work with the class of polarised translationally symmetric* metrics, and apply a parabolic scaling to the reduced system. Specifically, we work in coordinates $(s, x, v, y)$, where the metric is translationally symmetric in $y$ and the coordinate system $s\,x\,v$ can be described as follows (see Section 1.2 for the detailed construction). $v$ parameterises null geodesics foliating the hypersurface $s = 0$, $s$ parameterises null geodesics throughout spacetime, and $x$ is a transverse spatial coordinate, constant along the geodesics parameterised by $v$ and $s$. Thus the hypersurfaces of constant $v$ are null, while the hypersurfaces of constant $s$, other than $s = 0$, need not be. In terms of the double null foliation, $s$ corresponds to the coordinate $u$, while on the initial null hypersurface $s = 0$ the coordinate $v$ corresponds to $\underline{u}$. (This correspondence does not hold for $s > 0$ since the other hypersurfaces of $s$ need not be null.) While for us the hypersurfaces $s = 0$ and $v = 0$ are hyperplanes, not cones, and hence there is no real reason to consider one of them 'outgoing' and the other 'incoming', we may, for ease of comparison with the double null case, refer to $s = 0$ as the 'outgoing' initial null hypersurface, and $v = 0$ as the 'incoming' initial null hypersurface.

The Einstein equations for a metric of this symmetry class can be analysed as follows. In the above coordinate system, a metric which is polarised translationally symmetric along $y$ can be written in the form (where indices 0123 correspond respectively to $sxvy$)

$$g_{ij} = \begin{pmatrix} 0 & 0 & -e^{-2\gamma(s,x,v)} & 0 \\ 0 & a(s,x,v)e^{-2\gamma(s,x,v)} & b(s,x,v)e^{-2\gamma(s,x,v)} & 0 \\ -e^{-2\gamma(s,x,v)} & b(s,x,v)e^{-2\gamma(s,x,v)} & c(s,x,v)e^{-2\gamma(s,x,v)} & 0 \\ 0 & 0 & 0 & e^{2\gamma(s,x,v)} \end{pmatrix}, \qquad (0.2.1)$$

* Note that our use of this symmetry class means that our results are not properly contained in those of the papers just discussed, though there are connections; see Section 0.4 for further discussion.

and the Einstein vacuum equations Ric $(g) = 0$ give rise* to Riccati equations

$$\partial_s^2 a = \frac{(\partial_s a)^2}{2a} - 4a\,(\partial_s \gamma)^2 \tag{0.2.2}$$

$$\partial_s^2 b = \frac{1}{2a}\,(\partial_s a)\,(\partial_s b) - 4\partial_s\gamma\,(b\partial_s\gamma + \partial_x\gamma) \tag{0.2.3}$$

$$\partial_s^2 c = \frac{(\partial_s b)^2}{2a} - 2\partial_s\gamma\left(2\partial_v\gamma + \left(\frac{b^2}{a} + c\right)\partial_s\gamma + 2\frac{b}{a}\partial_x\gamma\right) - \frac{2}{a}\,(\partial_x\gamma)^2 \tag{0.2.4}$$

for the quantities $a$, $b$, and $c$, and a free wave equation

$$\left[\left(\frac{b^2}{a} - c\right)\partial_s^2 + 2\frac{b}{a}\partial_s\partial_x - 2\partial_s\partial_v + \frac{1}{a}\partial_x^2 - \frac{1}{2}\left(\left(\frac{b^2}{a} + c\right)\frac{\partial_s a}{a} - 4\frac{b}{a}\partial_s b + \frac{b}{a^2}\partial_x a + 2\partial_s c - \frac{2}{a}\partial_x b + \frac{\partial_v a}{a}\right)\partial_s \right.$$
$$\left. - \frac{1}{2}\left(\frac{b}{a^2}\partial_s a - \frac{2}{a}\partial_s b + \frac{\partial_x a}{a^2}\right)\partial_x - \frac{1}{2}\frac{\partial_s a}{a}\partial_v\right]\gamma = 0 \tag{0.2.5}$$

for $\gamma$. This is in fact simply the wave equation $\Box_h \gamma = 0$ with respect to the metric (where indices 012 correspond to $sxv$)

$$h_{ij} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & a & b \\ -1 & b & c \end{pmatrix}; \tag{0.2.6}$$

this can be compared to the form of the metric with respect to a double null foliation in $3+1$ dimensions, see for example Section 2.2 in [9]. Note that the system (0.2.2 – 0.2.5) is in $2+1$ dimensions because of the translational symmetry in $y$ imposed on $g$. While the wave equation (0.2.5) is linear in $\gamma$, it is coupled in a nonlinear way to the Riccati equations (0.2.2 – 0.2.4), which taken as a system are nonlinear.†

Most of our work will be done in a scaled picture, given as follows. Define the scaled coordinates

$$\overline{s} = s, \qquad \overline{x} = k^{1/2}x, \qquad \overline{v} = kv, \tag{0.2.7}$$

and scaled quantities‡

$$\overline{\delta\ell} = k(a^{1/2} - 1), \qquad \overline{b} = k^{1/2}b, \qquad \overline{c} = c, \qquad \overline{\gamma} = k^{\iota}\gamma, \tag{0.2.8}$$

* As usual, the Einstein vacuum equations also give rise to constraint equations. We shall say more about these shortly.

† Curiously, the Riccati equations can be turned into a system of equations which are linear if solved in the correct sequence: specifically, equation (0.2.2) gives a linear equation for $\ell = a^{1/2}$; given $a$ (or, equivalently, $\ell$), equation (0.2.3) is then a linear equation for $b$; and given $a$ (or $\ell$) and $b$, equation (0.2.4) is then a linear equation for $c$. These linear equations are however still nonlinearly coupled to (0.2.5); also the requirement $a > 0$ – necessary to ensure the metric (0.2.6) stays nonsingular – will for a general solution of (0.2.2) fail to hold after a finite time.

‡ Roughly speaking – see equation (0.2.14) below – ‘barred’ quantities, e.g., $\overline{b}$, $\overline{c}$, $\overline{\gamma}$, will have bounds independent of $k$ – in appropriate spaces to be detailed later, see e.g. equation (0.3.30) – as will derivatives of ‘barred’ quantities with respect to ‘barred’ variables.

where $\iota \geq 1/2$ is an exponent we shall leave unspecified for the moment. In terms of these coordinates and quantities, system (0.2.2 – 0.2.5) can be rewritten as

$$\partial_{\overline{s}}^2 \overline{\delta\ell} = -2(1 + k^{-1}\overline{\delta\ell})k^{1-2\iota}(\partial_{\overline{s}}\overline{\gamma})^2, \tag{0.2.9}$$

$$\partial_{\overline{s}}^2 \overline{b} = \frac{1}{\overline{\ell}} k^{-1}(\partial_{\overline{s}}\overline{\delta\ell})(\partial_{\overline{s}}\overline{b}) - 4k^{1-2\iota}\partial_{\overline{s}}\overline{\gamma}(\partial_{\overline{x}}\overline{\gamma} + k^{-1}\overline{b}\partial_{\overline{s}}\overline{\gamma}), \tag{0.2.10}$$

$$\partial_{\overline{s}}^2 \overline{c} = k^{-1}\frac{(\partial_{\overline{s}}\overline{b})^2}{2\overline{a}} - 2k^{1-2\iota}\partial_{\overline{s}}\overline{\gamma}\left(2\partial_{\overline{v}}\overline{\gamma} + 2k^{-1}\frac{\overline{b}}{\overline{a}}\partial_{\overline{x}}\overline{\gamma} + k^{-1}\left(\overline{c} + k^{-1}\frac{\overline{b}^2}{\overline{a}}\right)\partial_{\overline{s}}\overline{\gamma}\right) - k^{1-2\iota}\frac{2}{\overline{a}}(\partial_{\overline{x}}\overline{\gamma})^2, \tag{0.2.11}$$

while the wave equation (0.2.5) can be rewritten as

$$\Bigg[(-2\partial_{\overline{s}}\partial_{\overline{v}} + \partial_{\overline{x}}^2) + \frac{1}{k}\left(2\frac{\overline{b}}{\overline{a}}\partial_{\overline{s}}\partial_{\overline{x}} - \overline{c}\partial_{\overline{s}}^2 - \overline{\delta^{-1}a}\partial_{\overline{x}}^2 - \left(\partial_{\overline{s}}\overline{c} - \frac{1}{\overline{a}}\partial_{\overline{x}}\overline{b} + \frac{\partial_{\overline{v}}\overline{\delta\ell}}{\overline{\ell}}\right)\partial_{\overline{s}} + \left(\frac{1}{\overline{a}}\partial_{\overline{s}}\overline{b} - \frac{\overline{\ell}\partial_{\overline{x}}\overline{\delta\ell}}{\overline{a}^2}\right)\partial_{\overline{x}} - \frac{\partial_{\overline{s}}\overline{\delta\ell}}{\overline{\ell}}\partial_{\overline{v}}\right)$$
$$+ \frac{1}{k^2}\left(\frac{\overline{b}^2}{\overline{a}}\partial_{\overline{s}}^2 - \left(\overline{c}\frac{\partial_{\overline{s}}\overline{\delta\ell}}{\overline{\ell}} - 2\frac{\overline{b}}{\overline{a}}\partial_{\overline{s}}\overline{b}\right)\partial_{\overline{s}} - \frac{\overline{b\ell}\partial_{\overline{s}}\overline{\delta\ell}}{\overline{a}^2}\partial_{\overline{x}}\right) - \frac{1}{k^3}\frac{\overline{b}^2\overline{\ell}\partial_{\overline{s}}\overline{\delta\ell}}{\overline{a}^2}\partial_{\overline{s}}\Bigg]\overline{\gamma} = 0, \tag{0.2.12}$$

which is the wave equation $\Box_{\overline{h}}\gamma = 0$ corresponding to the metric $\overline{h} = kh$ represented in the $\overline{s}\,\overline{x}\,\overline{v}$ coordinate system by

$$\overline{h}_{ij} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} + \frac{1}{k}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \overline{\delta\ell}(1 + k^{-1}\overline{\delta\ell}) & \overline{b} \\ 0 & \overline{b} & \overline{c} \end{pmatrix}. \tag{0.2.13}$$

Note that, for $\iota \in [1/2, 1)$, the terms of leading order in $k$ on the right-hand sides of (0.2.9 – 0.2.11) are forcing terms quadratic in $\overline{\gamma}$ and (at least if we expand out $1/\overline{a} = 1/(1 + k^{-1}\overline{\delta\ell})^2$) independent of $\overline{\delta\ell}$, $\overline{b}$, and $\overline{c}$, and that for any $\iota \geq 1/2$ all other terms decay at least as fast as $k^{-1}$. Similarly, the wave equation (0.2.12) is clearly the Minkowski wave equation $(-2\partial_{\overline{s}}\partial_{\overline{v}} + \partial_{\overline{x}}^2)\overline{\gamma} = 0$ with a correction of order $k^{-1}$. Thus we have the preliminary rough ansatz for the behaviour of the full solution on the bulk $\Gamma$. It will be sharpened and extended considerably momentarily in terms of the $L^2$-based energies defined in equation (0.3.14) and equation (0.3.30) below. Our ansatz is that there is some $C > 0$, depending, among other things, on the size of the initial data, but independent of $k$, such that

$$\left\|\partial_{\overline{s}}^j\overline{\delta\ell}\right\|_{L^\infty(\Gamma)}, \left\|\partial_{\overline{s}}^j\partial_{\overline{x}}\overline{\delta\ell}\right\|_{L^\infty(\Gamma)}, \left\|\partial_{\overline{s}}^j\partial_{\overline{v}}\overline{\delta\ell}\right\|_{L^\infty(\Gamma)}, \left\|\partial_{\overline{s}}^j\overline{b}\right\|_{L^\infty(\Gamma)}, \left\|\partial_{\overline{s}}^j\partial_{\overline{x}}\overline{b}\right\|_{L^\infty(\Gamma)}, \left\|\partial_{\overline{s}}^j\overline{c}\right\|_{L^\infty(\Gamma)}, \left\|\partial_{\overline{s}}^j\partial_i^m\overline{\gamma}\right\|_{L^\infty(\Gamma)} < C \tag{0.2.14}$$

where $\Gamma$ is the region (see the figure)

$$\Gamma = \{(\overline{s}, \overline{x}, \overline{v}) \in \mathbf{R}^3 \mid \overline{s} \in [0, T'],\ \overline{v} \in [0, kT\sqrt{2}],\ \frac{1}{\sqrt{2}}(\overline{s} + \overline{v}) \leq kT\} \tag{0.2.15}$$

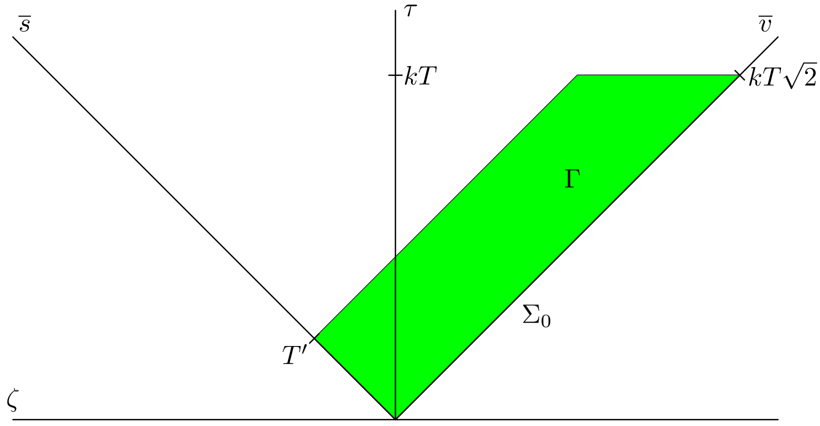


Figure 0.2.1

for some $T > 0$, $T' \geq 1$,* $j, m = 0, 1$, and $\partial_i$ denotes one of $\partial_{\overline{s}}$, $\partial_{\overline{x}}$ and $\partial_{\overline{v}}$. Equivalently, in terms of the unscaled coordinates and variables,

$$\left\| k\partial_s^j (a^{1/2} - 1) \right\|_{L^\infty({}^0\Gamma)}, \left\| k^{1/2}\partial_s^j \partial_x (a^{1/2} - 1) \right\|_{L^\infty({}^0\Gamma)}, \left\| \partial_s^j \partial_v (a^{1/2} - 1) \right\|_{L^\infty({}^0\Gamma)}, \left\| k^{1/2}\partial_s^j b \right\|_{L^\infty({}^0\Gamma)},$$
$$\left\| \partial_s^j \partial_x b \right\|_{L^\infty({}^0\Gamma)}, \left\| \partial_s^j c \right\|_{L^\infty({}^0\Gamma)}, \left\| k^\iota \partial_s^m \gamma \right\|_{L^\infty({}^0\Gamma)}, \left\| k^{\iota - 1/2}\partial_s^j \partial_x \gamma \right\|_{L^\infty({}^0\Gamma)}, \left\| k^{\iota - 1}\partial_s^j \partial_v \gamma \right\|_{L^\infty({}^0\Gamma)} < C, \tag{0.2.16}$$

where here

$${}^0\Gamma = \{(s, x, v) \in \mathbf{R}^3 \mid s \in [0, T'],\ v \in [0, T\sqrt{2}],\ \frac{1}{\sqrt{2}}(k^{-1}s + v) \leq T\} \tag{0.2.17}$$

$j = 0, 1$, and $m = 0, 1, 2$. As noted, we shall sharpen this considerably after we introduce energies for the quantities $a$, $b$, $c$, and $\gamma$, below; in particular, we shall in addition require that derivatives with respect to $\overline{x}$ and $\overline{v}$ do not change the order in $k$ of any of the above quantities, up to the degree of regularity at which we close our estimates (see Chapter 6 for the details). Note that this implies, effectively, that derivatives with respect to $x$ cost an extra factor of $k^{1/2}$ while derivatives with respect to $v$ cost an extra factor of $k$, exactly as in [6], (1.23) (see also equation (0.1.4) above).

Our choice of gauge gives, on $s = 0$, the conditions (see Proposition 1.2.1)

$$b = c = \frac{\partial c}{\partial s} = 0. \tag{0.2.18}$$

Given this, it can be shown that the Einstein vacuum equations are equivalent to the system (0.2.2 – 0.2.5) together with the constraint equations on $s = 0$ (note the similarity between the first of these and (0.2.9))

$$\frac{\partial^2 a}{\partial v^2} = \frac{(\partial_v a)^2}{2a} - 4a\,(\partial_v \gamma)^2 \tag{0.2.19}$$

* The restriction $T' \geq 1$ is made for technical reasons related to the need to take a cutoff in the $\overline{s}$ direction (see the discussion after Theorem 0.3.3 at the end of Section 0.3 below); specifically, we do not wish the cutoff to increase norms of $\overline{s}$ derivatives. We shall occasionally ignore it below, but we may always impose it when needed.

$$\frac{\partial}{\partial v}\left(a^{1/2}\frac{\partial b}{\partial s}\right) = 4a^{1/2}\partial_x\gamma\partial_v\gamma \tag{0.2.20}$$

$$2a\frac{\partial^2 a}{\partial v\partial s} = 2a\frac{\partial^2 b}{\partial x\partial s} - \frac{\partial b}{\partial s}\frac{\partial a}{\partial x} + \frac{\partial a}{\partial v}\frac{\partial a}{\partial s} + a\left(\frac{\partial b}{\partial s}\right)^2 + 4a\left(\partial_x\gamma\right)^2 \qquad (0.2.21);$$

in other words, we have the following lemma.

0.2.1. Lemma. If the system (0.2.2 – 0.2.5) holds on the set ${}^0\Gamma$ defined in (0.2.17), while the constraint equations (0.2.19 – 0.2.21) hold on

$${}^0\Sigma_0 = \{(s,x,v) \in {}^0\Gamma \,|\, s = 0\}, \tag{0.2.22}$$

then the Einstein vacuum equations

$$\mathrm{Ric}\,(g) = 0 \tag{0.2.23}$$

for the metric given by (0.2.1) hold on ${}^0\Gamma \times \mathbf{R}^1$.

The proof is given in Chapter 2.*

In the scaled picture, this result implies that the Einstein vacuum equations (0.2.23) are equivalent to requiring that (0.2.9 – 0.2.12) hold on $\Gamma$ (as defined in (0.2.15)), and that the constraint equations

$$\frac{\partial^2\overline{\delta\ell}}{\partial\overline{v}^2} = -2(1+k^{-1}\overline{\delta\ell})k^{1-2\iota}\left(\partial_{\overline{v}}\overline{\gamma}\right)^2 \tag{0.2.24}$$

$$\partial_{\overline{v}}\left([1+k^{-1}\overline{\delta\ell}]\,\partial_{\overline{s}}\overline{b}\right) = 4(1+k^{-1}\overline{\delta\ell})k^{1-2\iota}\partial_{\overline{v}}\overline{\gamma}\partial_{\overline{x}}\overline{\gamma} \tag{0.2.25}$$

$$2(1+k^{-1}\overline{\delta\ell})\cdot\frac{\partial^2\overline{\delta\ell}}{\partial\overline{v}\partial\overline{s}} = (1+k^{-1}\overline{\delta\ell})\partial_{\overline{x}}\left([1+k^{-1}\overline{\delta\ell}]^{-1}\partial_{\overline{s}}\overline{b}\right) + \frac{1}{2k}\left(\partial_{\overline{s}}\overline{b}\right)^2 + 2k^{1-2\iota}\left(\partial_{\overline{x}}\overline{\gamma}\right)^2 \tag{0.2.26}$$

hold on the set

$$\Sigma_0 = \{(\overline{s},\overline{x},\overline{v}) \in \Gamma \,|\, \overline{s} = 0\}.$$

It is principally in the scaled form (0.2.24 – 0.2.26) that we shall study the constraint equations (see Chapter 5).

As initial data for the system (0.2.2 – 0.2.5), we must specify

$$a|_{s=0}, \qquad \partial_s a|_{s=0}, \qquad \partial_s b|_{s=0}, \qquad \gamma|_{s=0}, \qquad \gamma|_{v=0}$$

satisfying (0.2.19 – 0.2.21). By (0.2.19 – 0.2.21), specifying $\gamma|_{s=0}$ and

$$a|_{s=0,\,v=v_0}, \qquad \partial_v a|_{s=0,\,v=v_0}, \qquad \partial_s a|_{s=0,\,v=v_0}, \qquad \partial_s b|_{s=0,\,v=v_0}, \tag{0.2.27}$$

for some $v_0 \in [0, T\sqrt{2}]$ will uniquely determine $a$, $\partial_s a$ and $\partial_s b$ on some neighbourhood of the line $\{v = v_0\}$ in ${}^0\Sigma_0$. To extend these solutions to $[0, T\sqrt{2}]$, with a uniform lower bound for $a$, requires additional conditions

* Note that – unlike the treatment in [11] – here there are no separate constraint equations along the incoming null hypersurface $v = 0$, essentially because the Riccati equations for $a$, $b$ and $c$ play the role of the constraint equations there.

on $\gamma|_{s=0}$ as well as the quantities specified in (0.2.27). In particular, we have the following result (see Section 5.1). Note the similarity of equation (0.2.29) to the scalings in (0.1.1), (0.1.2), (0.1.3).

0.2.2. LEMMA. Let $\delta\overline{v}_1$, $\delta\overline{v}_2 \in (0,1)$ be two fixed numbers,* independent of $k$, and assume that $k$ is large enough that $kT/\sqrt{2} \in (\delta\overline{v}_1, kT\sqrt{2} - \delta\overline{v}_2)$. Let $\varpi_1$, $\varpi_2$ be $C^\infty$ functions on $\mathbf{R}^2$ with support contained in

$$[0,1] \times [0, \delta\overline{v}_1], \qquad [0,1] \times [0, \delta\overline{v}_2],$$

respectively, and which, together with all of their derivatives, have $L^\infty$ bounds on $\mathbf{R}^2$ which are independent of $k$. Define $\varpi_0(\overline{x}, \overline{v})$ on $\Sigma_0$ by

$$\varpi_0(\overline{x},\overline{v}) = \begin{cases} \varpi_1(\overline{x},\overline{v}), & \overline{v} \in [0, \delta\overline{v}_1] \\ 0, & \overline{v} \in [\delta\overline{v}_1, kT\sqrt{2} - \delta\overline{v}_2] \\ \varpi_2(\overline{x}, \overline{v} - (kT\sqrt{2} - \delta\overline{v}_2)), & \overline{v} \in [kT\sqrt{2} - \delta\overline{v}_2, kT\sqrt{2}] \end{cases}. \tag{0.2.28}$$

If we specify

$$\gamma|_{s=0} = k^{-\iota} o \cdot \varpi_0(k^{1/2}x, kv), \tag{0.2.29}$$

where $o \leq 1$ is a scaling parameter independent of $k$, and

$$a|_{s=0,\, v=T/\sqrt{2}} = 1, \qquad \partial_v a|_{s=0,\, v=T/\sqrt{2}} = 0, \qquad \partial_s a|_{s=0,\, v=T/\sqrt{2}} = 0, \qquad \partial_s b|_{s=0,\, v=T/\sqrt{2}} = 0, \tag{0.2.30}$$

and furthermore assume that $\iota \geq 1/2$, then for $o$ sufficiently small (depending only on $\varpi_0$) the equations (0.2.19 – 0.2.21) have a unique solution on $[0, T\sqrt{2}]$, and $a$ has a uniform lower bound on that interval.

We shall show in Chapter 2 that under the conditions of Lemma 0.2.2, if we specify $\gamma|_{v=0}$ appropriately (in a way which satisfies the necessary consistency conditions, see equation (1.3.2)), the transverse derivatives

$$\partial_s^\ell \gamma|_{s=0},\ \partial_s^\ell a|_{s=0},\ \partial_s^\ell b|_{s=0}, \partial_s^\ell c|_{s=0},$$
$$\partial_v^\ell \gamma|_{v=0},\ \partial_v^\ell a|_{v=0},\ \partial_v^\ell b|_{v=0}, \partial_v^\ell c|_{v=0},$$

for suitable values of $\ell$, will also be uniquely determined in a way consistent with the system (0.2.2 – 0.2.5), and will have $L^\infty$ bounds on $s = 0$ and $v = 0$ respectively which are independent of $k$. See Proposition 5.4.1 for the details.

The treatment of the constraints in the scaled picture sheds light on the requirements on $\gamma|_{s=0}$ in (0.2.29). As initial data for the system (0.2.9 – 0.2.12), we must specify

$$\overline{\delta\ell}|_{\overline{s}=0}, \qquad \partial_{\overline{s}}\overline{\delta\ell}|_{\overline{s}=0}, \qquad \partial_{\overline{s}}\overline{b}|_{\overline{s}=0}, \qquad \overline{\gamma}|_{\overline{s}=0}, \qquad \overline{\gamma}|_{\overline{v}=0}$$

satisfying (0.2.24 – 0.2.25). As before, specifying $\overline{\gamma}|_{\overline{s}=0}$ and

$$\overline{\delta\ell}|_{\overline{s}=0,\,\overline{v}=\overline{v}_0}, \qquad \partial_{\overline{v}}\overline{\delta\ell}|_{\overline{s}=0,\,\overline{v}=\overline{v}_0}, \qquad \partial_{\overline{s}}\overline{\delta\ell}|_{\overline{s}=0,\,\overline{v}=\overline{v}_0}, \qquad \partial_{\overline{s}}\overline{b}|_{\overline{s}=0,\,\overline{v}=\overline{v}_0}, \tag{0.2.31}$$

* The restriction $\delta\overline{v}_1$, $\delta\overline{v}_2 < 1$ is purely for technical convenience.

for some $\overline{v}_0 \in [0, kT\sqrt{2}]$, will uniquely determine $\overline{\delta\ell}$, $\partial_{\overline{s}}\overline{\delta\ell}$ and $\partial_{\overline{s}}\overline{b}$ on $\overline{s} = 0$. For the ansatz (0.2.14) to hold with $\Gamma$ replaced by $\Sigma_0$, the quantities $\overline{\delta\ell}$, $\partial_{\overline{s}}\overline{\delta\ell}$ and $\partial_{\overline{s}}\overline{b}$ must have uniform bounds, independent of $k$, on all of $\Sigma_0$. For general $\overline{\gamma}|_{\overline{s}=0}$, this is nontrivial since (0.2.24 – 0.2.26) are transport equations in $\overline{v}$, and on $\Sigma_0$, $\overline{v}$ ranges from 0 to $kT\sqrt{2}$. To avoid this difficulty we require first that $\overline{\gamma}|_{\overline{s}=0}$ be supported in a region whose size is independent of $k$.* A more careful study of (0.2.24) shows that for $\iota < 1$, $\overline{\gamma}|_{\overline{s}=0}$ can only have support near $\overline{v} = 0$ and $\overline{v} = kT\sqrt{2}$: in other words, it cannot be supported in the middle of $\Sigma_0$. In the unscaled picture, this means that $\gamma|_{s=0}$ must be supported in a double strip $\{0 < v < C_1/k\} \cup \{T\sqrt{2} - C_2/k < v < T\sqrt{2}\}$, as in (0.2.29). For technical reasons, we also require the metric $\overline{h}$ to equal the Minkowski metric on $\Sigma_0 \backslash \operatorname{supp} \overline{\gamma}$; to obtain this condition, it suffices to require the quantities in (0.2.31) to vanish for $\overline{v}_0 \in [C_1, kT\sqrt{2} - C_2]$, explaining (0.2.30).

Note that taking $\gamma|_{s=0}$ supported on $\{T\sqrt{2} - C/k < v < T\sqrt{2}\}$ (only) is similar to what is done in [3], [5], and [6], inasmuch as it allows for the initial data to be Minkowskian followed by a short pulse whose extent (in the null coordinate) is of size $1/k \sim \delta$.

For $\iota = 1$, the ansatz (0.2.14) with $\Gamma$ replaced by $\Sigma_0$ will hold for $\overline{\delta\ell}$, $\overline{b}$ and $\overline{c}$ as long as $\overline{\gamma}|_{\overline{s}=0}$ is compactly supported, regardless of where its support lies.† Note that in this case the ansatz (0.2.14) implies that when $\iota = 1$, $\gamma$ is smaller by a factor of $k^{-1/2}$ than when $\iota = 1/2$. This is exactly in line with our observation, when comparing equation (0.1.5) with equation (0.1.11) in Section 0.2, that the derivatives of the initial metric in Luk and Rodnianski [9] – where the initial data is large in the middle of the initial null hypersurface – are smaller by a factor $\delta^{1/2} \sim k^{-1/2}$ than the corresponding derivatives in Christodoulou [3] and Klainerman and Rodnianski [6], where the initial data is large at the end of the initial null hypersurface.

The detailed construction of, and resulting $L^\infty$ bounds on, the initial data are given in Chapter 5.

## 0.3. Main results

Our main goal in this thesis is to prove finite-time existence of solutions to the Einstein vacuum equations which are highly localised in $H^1$ near a two-dimensional null plane. We achieve this in Chapter 7. Our results there rely heavily on a more general existence theorem, for initial data satisfying the requirements laid out at the end of Section 0.2.

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* Presumably it would be sufficient to require instead that $\overline{\gamma}|_{\overline{s}=0}$ fall off sufficiently rapidly in $\overline{v}$, but we do not investigate that possibility in this thesis.

† There is, however, a further obstruction in our situation to obtaining initial data satisfying the ansatz (0.2.14) and the corresponding ansätze for higher derivatives: in order to obtain higher $\overline{s}$ derivatives of $\overline{\gamma}|_{\overline{s}=0}$ we must differentiate the wave equation (0.2.12) with respect to $\overline{s}$ and then integrate with respect to $\overline{v}$ – again, *over an interval whose length is of order* $k$. Thus in general it appears that higher $\overline{s}$ derivatives of $\overline{\gamma}$ will pick up extra factors of $k$, violating the ansatz for the higher derivatives. We believe that there are ways of producing solutions in this case – either by circumventing the problem using other multipliers, or by specialising still further to functions $\varpi$ which are constructed in such a way that the higher-order (in $k$) terms ultimately all cancel – but leave the treatment of the matter for another time.

Our main result may be stated in words as follows:

For every $\epsilon \in (0,1)$ and every $k$ sufficiently large, there is a solution to the Einstein vacuum equations of the form (0.2.1) such that the fraction $1-\epsilon$ of the $H^1$ energy of $\gamma$, as a function of $s\,x\,v$, is contained within a rectangular tube of size $k^{-1/2} \times k^{-1}$ centred on a null geodesic. The existence time of this solution does not depend on $k$.

By way of comparison with [6], note that their ansatz (0.1.3) gives an isotropic scaling in the spatial variables $\omega$ which amounts to taking initial data concentrated in a region equally small in *both* spatial directions, whereas (because of the translational symmetry of the metric $g$ in (0.2.1)) our result involves initial data which is concentrated in a region small in only *one* direction ($x$), with both the data and the solution constant in the other ($y$). Further, our solution remains concentrated off of the initial null surface.

More precisely, we have the following theorem (see Corollary 7.3.1).

0.3.1. THEOREM. Let $\epsilon \in (0,1)$ be given, and let ${}^0\Gamma$ be as defined in (0.2.17),

$$ {}^0\Gamma = \{(s,x,v) \in \mathbf{R}^3 \mid s \in [0,T'],\ v \in [0, T\sqrt{2}],\ \frac{1}{\sqrt{2}}(s+kv) \le kT\}. $$

Then there are constants $C, C' > 0$, independent of $\epsilon$ and $k$, such that for $T'$, $T$ sufficiently small, independent of $k$, and all $k$ sufficiently large there is a function $\gamma_{GB,o}$, supported on the prism

$$ \{(s,x,v) \in {}^0\Gamma \mid v \in [T\sqrt{2} - k^{-1}, T\sqrt{2}],\ x \in [0, k^{-1/2}]\} $$

and satisfying

$$ \|\gamma_{GB,o}\|_{H^\ell({}^0\Sigma_0)} \ge C' k^{\ell - 5/4}, \tag{0.3.1} $$

$\ell \ge 0$, and a solution to the Einstein vacuum equations of the form (0.2.1) such that

$$ \gamma|_{{}^0\Sigma_0} = \gamma_{GB,o}|_{{}^0\Sigma_0}, \qquad \frac{\|\gamma - \gamma_{GB,o}\|_{H^1({}^0\Gamma)}}{\|\gamma\|_{H^1({}^0\Gamma)}} \le C\epsilon. \tag{0.3.2} $$

Here $\gamma_{GB,o}$ will be a formal Gaussian beam for the *solution* metric (0.2.1), of whose initial data it is a part. Thus the construction of $\gamma_{GB,o}$ is more delicate than in the standard case, where the metric is known from the outset. Also, for fixed $T'$, $T$, the $H^1$ norm of $\gamma_{GB,o}$ on ${}^0\Sigma_0$ will go to zero as a (in general, high) power of $\epsilon$. See Chapter 7 for the details.

The hard part in establishing this result is to show that the existence times $T'$ and $T$ do not depend on $k$. This is nontrivial, since by (0.3.1) and (0.3.2) the initial data we use for $\gamma$ have $H^2$ norm on ${}^0\Gamma$ of size $k^{3/4}$, which is too large to apply general results such as the $L^2$ curvature conjecture [8].

Thus the main work in this thesis is to prove a finite-time existence result of solutions to the Einstein vacuum equations for initial data in a class sufficiently broad to encompass that needed in Theorem 0.3.1. We shall describe the result we obtain first in the unscaled picture, and then in the scaled picture, though most of our work in the actual proof is done in the scaled picture.

To begin, we define coordinates

$$t = \frac{1}{\sqrt{2}}(s+v), \qquad z = \frac{1}{\sqrt{2}}(s-v)$$
$$\tau = \frac{1}{\sqrt{2}}(s+kv), \qquad \zeta = \frac{1}{\sqrt{2}}(s-kv); \tag{0.3.3}$$

regions (see the figure)*

$$\begin{aligned}
{}^0\Gamma &= \{(s,x,v) \in \mathbf{R}^3 \mid s \in [0,T'],\ v \in [0, T\sqrt{2}],\ \frac{1}{\sqrt{2}}(k^{-1}s+v) \le T\} \\
{}^0\Sigma_0 &= \{(s,x,v) \in {}^0\Gamma \mid s = 0\} \\
{}^0\Sigma'_\sigma &= \{(0,x,v) \in {}^0\Sigma_0 \mid v \in [\sigma - k^{-1}T'\sqrt{2}, \sigma]\} \\
{}^0U_0 &= \{(s,x,v) \in {}^0\Gamma \mid v = 0\} \\
{}^0A_\sigma &= \{(s,x,v) \in {}^0\Gamma \mid \frac{1}{\sqrt{2}}(k^{-1}s+v) = \sigma\} \\
\underline{\partial}{}^0A_\sigma &= {}^0A_\sigma \cap {}^0\Sigma_0
\end{aligned} \tag{0.3.4}$$

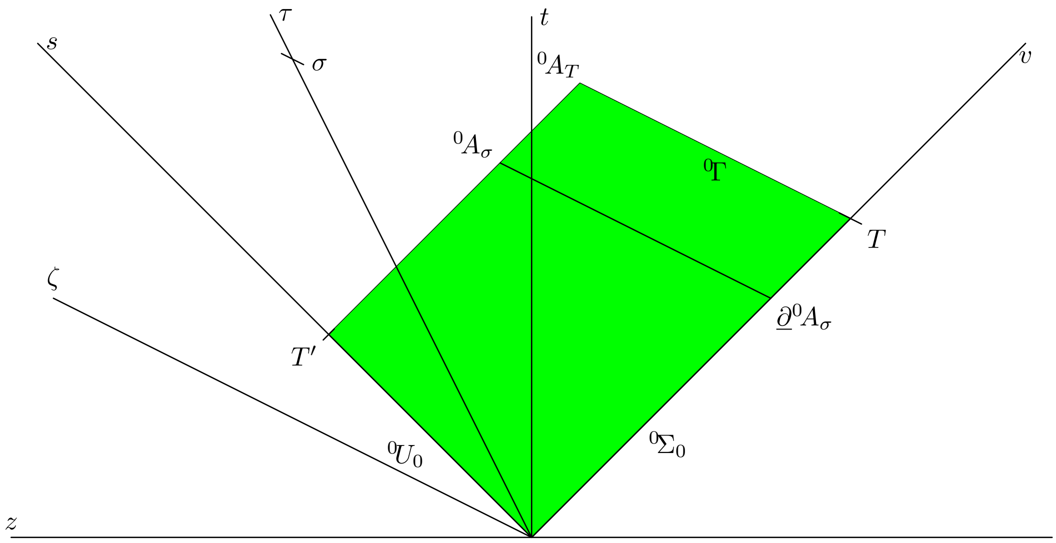


Figure 0.3.2

(we take $x$ and $\zeta$ as coordinates on ${}^0A_\sigma$); auxiliary quantities†

On ${}^0\Gamma$:

$${}^0\overline{\mu} = k\left[\left(\frac{\left(1+\frac{c}{2k}\right)a - \frac{b^2}{2k}}{1-\frac{c}{2k}}\right)^{1/2} - 1\right]; \tag{0.3.5}$$

On ${}^0\Gamma$, indices $cd$ correspond to $t\,x\,z$:

---

* Regions labelled with variables containing a leading 0 superscript are the unscaled (physical space) equivalent of the regions in the scaled picture labelled by the corresponding unmarked variables; e.g., ${}^0\Gamma$ corresponds to $\Gamma$, ${}^0\Sigma_0$ to $\Sigma_0$, etc.

† As with regions, quantities marked with a leading 0 superscript are the unscaled (physical space) equivalents of the corresponding unmarked quantities in the scaled picture.

$$
{}^0\Delta^{cd}_{{}^0A}(t,x,z) =
$$

$$
\left(1+\frac{1}{k}{}^0\overline{\mu}\right)\left\{-\frac{c}{4}\begin{pmatrix}-1 & 0 & 0\\ 0 & k^{-1} & 0\\ 0 & 0 & k^{-2}\end{pmatrix}+\frac{1}{2}\left(1-\frac{c}{2k}\right)\begin{pmatrix}\frac{b^2}{2a}-\frac{c}{2} & \frac{b}{a\sqrt{2}} & k^{-1}\left(\frac{b^2}{2a}-\frac{c}{2}\right)\\ \frac{b}{a\sqrt{2}} & a^{-1}-1 & \frac{k^{-1}b}{a\sqrt{2}}\\ k^{-1}\left(\frac{b^2}{2a}-\frac{c}{2}\right) & \frac{k^{-1}b}{a\sqrt{2}} & \frac{b^2}{2a}-\frac{c}{2}\end{pmatrix}\right\}
$$

$$
+\frac{1}{2}{}^0\overline{\mu}\begin{pmatrix}1 & 0 & 0\\ 0 & k^{-1} & 0\\ 0 & 0 & k^{-2}\end{pmatrix}; \tag{0.3.6}
$$

On ${}^0\Sigma_0$, indices $cd$ correspond to $x\,v$:

$$
{}^0\Delta^{cd}_{{}^0\Sigma} = \frac{1}{\sqrt{2}}\left(a^{1/2}-1\right)\begin{pmatrix}1 & 0\\ 0 & \frac{1}{2a}k^{-1}\end{pmatrix}+\frac{1}{2\sqrt{2}}\left(a^{1/2}-a^{-1/2}\right)\begin{pmatrix}0 & 0\\ 0 & k^{-1}\end{pmatrix}; \tag{0.3.7}
$$

On ${}^0U_0$, indices $cd$ correspond to $s\,x$:

$$
{}^0\Delta^{cd}_{{}^0U} = \frac{1}{\sqrt{2}}\left(a^{1/2}-1\right)\begin{pmatrix}k & 0\\ 0 & \frac{1}{2}\end{pmatrix}+\frac{1}{2\sqrt{2}}a^{1/2}\begin{pmatrix}\frac{b^2}{a}-c & \frac{b}{a}\\ \frac{b}{a} & \left(a^{-1}-1\right)\end{pmatrix}, \tag{0.3.8}
$$

where the factors of $k$ come from the $k$ in the definition of ${}^0A_\sigma$ (see (0.3.4)) and the ansatz (0.2.16); sets of dependent variables ($\delta\ell = \left(a^{1/2}-1\right)$)

$$
\Omega_0 = \{\delta\ell, b, c\}, \qquad \Omega = \{\delta\ell, b, c, \partial_x\delta\ell, \partial_v\delta\ell, \partial_x b\}; \tag{0.3.9}
$$

norms on the initial data (here $I = (i_1, i_2)$ denotes a multiindex, $|I| = i_1 + i_2$, $\partial^I = \partial_x^{i_1}\partial_z^{i_2}$, we define a function $\eta$ by*

$$
\eta(I) = i_2 + i_1/2,
$$

$X$ denotes one of ${}^0\Sigma_0$ and ${}^0U_0$, and the $L^2$ norms are with respect to the coordinates $x\,v$ (${}^0\Sigma_0, {}^0\Sigma'_\sigma$) or $s\,x$ (${}^0U_0$))†

$$
\begin{aligned}
{}^0\iota_{n,\ell}[h](\sigma) &= \sqrt{2}\sum_{|I|\le n-1}\sum_{\omega\in\Omega} k^{-2\eta(I)+3/2}\left\|\partial^I\partial_s^\ell\omega\right\|^2_{L^2({}^0\Sigma'_\sigma)},\\
{}^0\iota_n[h](\sigma) &= \sum_{\ell=0}^{1}{}^0\iota_{n,\ell}[h](\sigma),\\
{}^0\overline{I}_{{}^0\Sigma_0}[f] &= k^{3/2}\int_{\Sigma_0}\frac{1}{\sqrt{2}}\left[\frac{1}{2}k^{-1}(\partial_x f)^2 + k^{-2}(\partial_v f)^2\right] + k^{-1}\cdot{}^0\Delta^{cd}_{{}^0\Sigma}\partial_c f\partial_d f\,dv\,dx\\
{}^0\overline{I}_{{}^0U_0}[f] &= k^{1/2}\int_{U_0}\frac{1}{\sqrt{2}}\left[(\partial_s f)^2 + \frac{1}{2}k^{-1}(\partial_x f)^2\right] + k^{-1}\cdot{}^0\Delta^{cd}_{{}^0U}\partial_c f\partial_d f\,ds\,dx\\
{}^0\overline{\iota}_{X,n,\ell}[\gamma] &= \sum_{|I|\le n-1} k^{-2\eta(I)}\overline{I}_X[\partial^I\partial_s^\ell\gamma]\\
{}^0\overline{\iota}_{X,n}[\gamma] &= \sum_{\ell=0}^{1}\overline{\iota}_{X,n,\ell}[\gamma],
\end{aligned} \tag{0.3.10}
$$

---

* Compare Klainerman and Rodnianski's concept of signature and scaling in [6].

† The explicit factors of $k$ arise because we are working in the *unscaled* picture; compare the use of weighted norms in Christodoulou [3] (e.g., (2.41), (12.125 – 12.140)) or explicit factors of the scaling parameter in Klainerman and Rodnianski [6] (e.g., (1.22), (1.23)). One of the innovations of the current work is the observation that explicitly scaling the coordinates allows one to work with energies which have no positive powers of $k$, see equation (0.3.26), equation (0.3.27), and equation (0.3.30) below.

squares of norms of the initial data on lines* (here the $L^2$ norm on $\underline{\partial} A_\sigma$ is with respect to $x$)

$$
\begin{aligned}
{}^0\underline{I}^\circ{}_m[f](\sigma) &= \sum_{|I|\le m} k^{-2\eta(I)+1/2} \left\|\partial^I f\right\|^2_{L^2(\underline{\partial} A_\sigma)}, \\
{}^0\underline{I}^1{}_m[f](\sigma) &= \sum_{|I|\le m} k^{-2\eta(I)+1/2} \left[\left\|\partial^I f\right\|^2_{L^2(\underline{\partial} A_\sigma)} + k^{-1}\left\|\partial^I \partial_x f\right\|^2_{L^2(\underline{\partial} A_\sigma)}\right], \\
{}^0\underline{\iota}[h](\sigma) &= \sum_{\ell=0}^{1}\sum_{\omega\in\Omega} \underline{I}^\circ_{n-1}[\partial_s^\ell \omega](\sigma), \qquad {}^0\underline{\iota}^1[h](\sigma) = \sum_{\ell=0}^{1}\sum_{\omega\in\Omega} \underline{I}^1_{n-1}[\partial_s^\ell \omega](\sigma), \\
{}^0\underline{I}[\gamma](\sigma) &= \sum_{\ell_1,\ell_2=0}^{1} \left\{{}^0\underline{I}^\circ{}_{n-1}[\partial_s^{\ell_1+\ell_2}\gamma](\sigma) + k^{-\ell_2/2}\,{}^0\underline{I}^\circ{}_{n-1}[\partial_s^{\ell_1}\partial_x^{\ell_2}\gamma](\sigma) + k^{-\ell_2}\,{}^0\underline{I}^\circ{}_{n-1}[\partial_s^{\ell_1}\partial_v^{\ell_2}\gamma](\sigma)\right\}, \\
{}^0\underline{I}^1[\gamma](\sigma) &= \sum_{\ell_1,\ell_2=0}^{1} \left\{{}^0\underline{I}^1{}_{n-1}[\partial_s^{\ell_1+\ell_2}\gamma](\sigma) + k^{-\ell_2/2}\,{}^0\underline{I}^1{}_{n-1}[\partial_s^{\ell_1}\partial_x^{\ell_2}\gamma](\sigma) + k^{-\ell_2}\,{}^0\underline{I}^1{}_{n-1}[\partial_s^{\ell_1}\partial_v^{\ell_2}\gamma](\sigma)\right\}
\end{aligned}
\tag{0.3.11}
$$

(note that

$$
{}^0\underline{\iota}[h](\sigma) \le {}^0\underline{\iota}^1[h](\sigma), \qquad {}^0\underline{I}[h](\sigma) \le {}^0\underline{I}^1[h](\sigma)), \tag{0.3.12}
$$

set

$$
{}^0\epsilon[f](\sigma) = k^{1/2}\int_{{}^0A_\sigma} \frac{1}{2}\left[(\partial_s f)^2 + k^{-1}(\partial_x f)^2 + k^{-2}(\partial_v f)^2\right] + k^{-1}\cdot{}^0\Delta^{cd}_{{}^0A}\partial_c f\partial_d f\,dx\,d\zeta \tag{0.3.13}
$$

and finally define energies (the $L^2$ norm on ${}^0A_\sigma$ is with respect to $x$ and $\zeta$)

$$
\begin{aligned}
{}^0\overline{E}_n[\gamma](\sigma) &= \sum_{|I|\le n-1}\sum_{\ell=0}^{1} k^{-2\eta(I)}\,{}^0\epsilon[\partial^I\partial_s^\ell\gamma](\sigma), \\
{}^0E_{n,\ell}[h](\sigma) &= \sum_{|I|\le n-1}\sum_{\omega\in\Omega} k^{-2\eta(I)+1/2}\left|\partial^I\partial_s^\ell\omega\right|^2_{L^2({}^0A_\sigma)}, \\
{}^0E_n[h](\sigma) &= \sum_{\ell=0}^{1} {}^0E_{n,\ell}[h](\sigma).
\end{aligned}
\tag{0.3.14}
$$

These give squares of (semi-) norms of $C^\infty$ functions on ${}^0A_\sigma$, and it is effectively with respect to these norms that we shall close the energy estimates for our system. Note the extra $s$ derivative (appearing as the explicit $\partial_s^\ell$) in (0.3.10), (0.3.11) and (0.3.14).

We fix $\nu\in(0,1)$,† and assume that the initial data satisfy

$$
\sup_{\sigma\le T} {}^0\iota_n[h](\sigma) \le \frac{1}{32}\nu^2, \qquad {}^0\overline{\iota}_{{}^0\Sigma_0,n}[\gamma] + {}^0\overline{\iota}_{{}^0U_0,n}[\gamma] \le \frac{1}{12}\nu^2 \tag{0.3.15}
$$

$$
\sup_{\sigma\le T} {}^0\underline{\iota}^1[h](\sigma) \le \frac{1}{1+\sqrt{2}(n+2)C_S^2C_\chi^2}\nu^2 \tag{0.3.16}
$$

$$
\sup_{\sigma\le T} {}^0\underline{I}^1[\gamma](\sigma) \le \frac{1}{1+\sqrt{2}(n+2)C_S^2C_\chi^2}\nu^2, \tag{0.3.17}
$$

---

* Both here and in the scaled picture – see equation (0.3.27) – we use an underline to indicate squares of norms defined along lines (in this case, $\underline{\partial}{}^0A_\sigma$).

† The restriction $\nu<1$ is mostly for convenience, in that it allows us to bound $\nu^p$ by $\nu^q$ when $q<p$ and hence gives the inclusions $\widehat{X}^{m,p}\subset\widehat{X}^{m,q}$ and $\widehat{X}_0^{m,p}\subset\widehat{X}_0^{m,q}$, see the discussion after Definition 6.3.1 below. The methods in Chapter 6 would probably allow us to treat the case $\nu>1$ if more careful track were kept of the exponents of the admissible nonlinearities used.

where $C_S$ is the Sobolev embedding constant on $\mathbf{R}^1$ and $C_\chi$ is another constant introduced for technical reasons (see equation (0.3.37) below). Existence of initial data satisfying the constraints (0.2.19 – 0.2.21) and the bounds (0.3.15 – 0.3.17) will be shown (by working in the scaled picture) in Chapter 5; see especially Corollary 5.4.1.

Given the above definitions, we may prove the following theorem.

0.3.2. THEOREM. Let $n \geq 4$, $T' > 1$, $T > 0$. There is a constant $C > 0$ and a positive integer $N$, both independent of $k$, such that if $\nu \in (0,1)$ satisfies

$$\nu \leq C \min \left\{ T'^{-N}, \frac{T'^N}{T} \right\},$$

then for all initial data satisfying (0.2.19 – 0.2.21) and (0.3.15 – 0.3.17), there exist functions $\delta\ell$, $b$ and $c$ on $\Gamma$, having the given initial data on ${}^0\Sigma_0$, and a function $\gamma$ on ${}^0\Gamma$ having the given initial data on ${}^0\Sigma_0 \cup {}^0U_0$, satisfying the Riccati equations (0.2.2 – 0.2.4) and the wave equation (0.2.5), and such that the bounds

$$\begin{aligned} {}^0E_n[h](\sigma) &\leq \nu^2, \\ {}^0\overline{E}_n[\gamma](\sigma) &\leq \nu^2 \end{aligned} \tag{0.3.18}$$

hold for $\sigma \in [0, T]$. The metric (0.2.1), with $a = (1 + \delta\ell)^2$, will satisfy the Einstein vacuum equations on the region ${}^0\Gamma$.

In the scaled coordinates, the foregoing can be written out as follows. We define for convenience the quantities

$$\overline{\delta a} = k(\overline{a} - 1), \qquad \overline{\delta^{-1} a} = k(\overline{a}^{-1} - 1).$$

We define coordinates

$$\tau = \frac{1}{\sqrt{2}}(\overline{s} + \overline{v}), \qquad \xi = \overline{x}, \qquad \zeta = \frac{1}{\sqrt{2}}(\overline{s} - \overline{v}), \tag{0.3.19}$$

and note that $\tau$ and $\zeta$ will be timelike and spacelike, respectively, for $k$ sufficiently large. We define regions (see the figure)

$$\begin{aligned} \Gamma &= \{(\overline{s}, \overline{x}, \overline{v}) \in \mathbf{R}^3 \,|\, \overline{s} \in [0, T'], \, \overline{v} \in [0, kT\sqrt{2}], \, \frac{1}{\sqrt{2}}(\overline{s} + \overline{v}) \leq kT\} \\ \Sigma_0 &= \{(\overline{s}, \overline{x}, \overline{v}) \in \Gamma \,|\, \overline{s} = 0\} \\ \Sigma'_\sigma &= \{(0, \overline{x}, \upsilon) \in \Sigma_0 \,|\, \upsilon \in [\sigma - T'\sqrt{2}, \sigma]\} \\ U_0 &= \{(\overline{s}, \overline{x}, \overline{v}) \in \Gamma \,|\, \overline{v} = 0\} \\ A_\sigma &= \{(\overline{s}, \overline{x}, \overline{v}) \in \Gamma \,|\, \frac{1}{\sqrt{2}}(\overline{s} + \overline{v}) = \sigma\} \\ \underline{\partial} A_\sigma &= A_\sigma \cap \Sigma_0, \end{aligned} \tag{0.3.20}$$

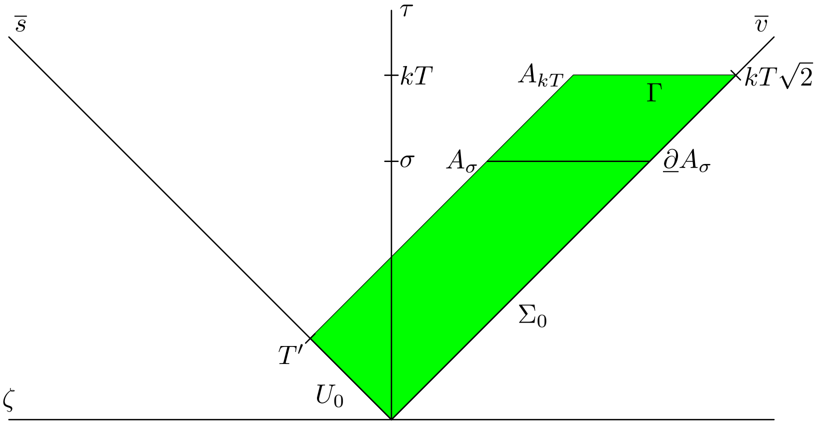


Figure 0.3.3

auxiliary quantities

On $\Gamma$:

$$\overline{\mu} = k\left[\frac{\sqrt{1+\frac{1}{k}(\overline{\delta a}+\frac{\overline{c}}{2})+\frac{1}{2k^2}(\overline{c}\,\overline{\delta a}-\overline{b}^2)}}{\sqrt{1-\frac{\overline{c}}{2k}}}-1\right]; \tag{0.3.21}$$

On $\Gamma$, indices $cd$ correspond to $\tau\,\xi\,\zeta$:

$$\Delta_A^{cd}(\tau,\xi,\zeta) = \left(1+\frac{1}{k}\overline{\mu}\right)\left\{-\frac{\overline{c}}{4}\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}+\frac{1}{2}\left(1-\frac{\overline{c}}{2k}\right)\begin{pmatrix}\frac{\overline{b}^2}{2k\overline{a}}-\frac{\overline{c}}{2} & \frac{\overline{b}}{\overline{a}\sqrt{2}} & \frac{\overline{b}^2}{2k\overline{a}}-\frac{\overline{c}}{2}\\ \frac{\overline{b}}{\overline{a}\sqrt{2}} & \overline{\delta^{-1}a} & \frac{\overline{b}}{\overline{a}\sqrt{2}}\\ \frac{\overline{b}^2}{2k\overline{a}}-\frac{\overline{c}}{2} & \frac{\overline{b}}{\overline{a}\sqrt{2}} & \frac{\overline{b}^2}{2k\overline{a}}-\frac{\overline{c}}{2}\end{pmatrix}\right\} + \frac{1}{2}\overline{\mu}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}; \tag{0.3.22}$$

On $\Sigma_0$ and $\Sigma_1$, indices $cd$ correspond to $\overline{x}\,\overline{v}$:

$$\Delta_\Sigma^{cd} = \frac{\overline{\delta\ell}}{\sqrt{2}}\begin{pmatrix}1&0\\0&\frac{1}{2\overline{a}}\end{pmatrix}+\frac{1}{2\sqrt{2}}\left(1+k^{-1}\overline{\delta\ell}\right)\overline{\delta^{-1}a}\begin{pmatrix}0&0\\0&1\end{pmatrix}; \tag{0.3.23}$$

On $U_0$, indices $cd$ correspond to $\overline{s}\,\overline{x}\,\overline{v}$:

$$\Delta_U^{cd} = \overline{\delta\ell}\begin{pmatrix}\frac{1}{\sqrt{2}}&0&0\\0&\frac{1}{2\sqrt{2}}&0\\0&0&0\end{pmatrix}+\frac{1}{2\sqrt{2}}\left(1+k^{-1}\overline{\delta\ell}\right)\begin{pmatrix}\frac{\overline{b}^2}{2k\overline{a}}-\frac{\overline{c}}{2} & \frac{\overline{b}}{\overline{a}\sqrt{2}} & \frac{\overline{b}^2}{2k\overline{a}}-\frac{\overline{c}}{2}\\ \frac{\overline{b}}{\overline{a}\sqrt{2}} & \overline{\delta^{-1}a} & \frac{\overline{b}}{\overline{a}\sqrt{2}}\\ \frac{\overline{b}^2}{2k\overline{a}}-\frac{\overline{c}}{2} & \frac{\overline{b}}{\overline{a}\sqrt{2}} & \frac{\overline{b}^2}{2k\overline{a}}-\frac{\overline{c}}{2}\end{pmatrix} \tag{0.3.24}$$

sets of dependent variables

$$\overline{\Omega}_0 = \{\overline{\delta\ell},\overline{b},\overline{c}\}, \qquad \overline{\Omega} = \{\overline{\delta\ell},\overline{b},\overline{c},\partial_{\overline{x}}\overline{\delta\ell},\partial_{\overline{v}}\overline{\delta\ell},\partial_{\overline{x}}\overline{b}\}, \tag{0.3.25}$$

norms on the initial data (here $I=(i_1,i_2)$ denotes a multiindex, $|I|=i_1+i_2$, $\partial^I=\partial_\xi^{i_1}\partial_\zeta^{i_2}$, $X$ denotes one of $\Sigma_0$ and $U_0$, and the $L^2$ norms are with respect to the scaled coordinates $\overline{s}$, $\overline{x}$, $\overline{v}$, $\tau$, $\xi$, $\zeta$, as appropriate)

$$\begin{aligned}
\iota_{n,\ell}[h](\sigma) &= \sqrt{2} \sum_{|I|\le n-1} \sum_{\overline{\omega}\in\overline{\Omega}} \left\| \partial^I \partial_{\overline{s}}^{\ell} \overline{\omega} \right\|^2_{L^2(\Sigma'_\sigma)}, \\
\iota_n[h](\sigma) &= \sum_{\ell=0}^{1} \iota_{n,\ell}[h](\sigma), \\
\overline{I}_{\Sigma_0}[f] &= \int_{\Sigma_0} \frac{1}{\sqrt{2}} \left[ \frac{1}{2} (\partial_{\overline{x}} f)^2 + (\partial_{\overline{v}} f)^2 \right] + k^{-1} \Delta_{\Sigma}^{cd} \partial_c f \partial_d f \, d\overline{v}\, d\overline{x} \\
\overline{I}_{U_0}[f] &= \int_{U_0} \frac{1}{\sqrt{2}} \left[ (\partial_{\overline{s}} f)^2 + \frac{1}{2} (\partial_{\overline{x}} f)^2 \right] + k^{-1} \Delta_{U}^{cd} \partial_c f \partial_d f \, d\overline{s}\, d\overline{x} \\
\overline{\iota}_{X,n,\ell}[\overline{\gamma}] &= \sum_{|I|\le n-1} \overline{I}_X [\partial^I \partial_{\overline{s}}^{\ell} \overline{\gamma}] \\
\overline{\iota}_{X,n}[\overline{\gamma}] &= \sum_{\ell=0}^{1} \overline{\iota}_{X,n,\ell}[\overline{\gamma}],
\end{aligned} \tag{0.3.26}$$

squares of norms of the initial data on lines (as before, $\partial_i$ represents one of $\partial_{\overline{s}}$, $\partial_{\overline{x}}$, or $\partial_{\overline{v}}$, or equivalently $\partial_\tau$, $\partial_\xi$, $\partial_\zeta$)

$$\begin{aligned}
&\underline{I}^{\circ}_m[f](\sigma) = \sum_{|I|\le m} \left\| \partial^I f \right\|^2_{L^2(\underline{\partial} A_\sigma)}, &&\underline{I}^{1}_m[f](\sigma) = \sum_{|I|\le m} \left\| \partial^I f \right\|^2_{H^1(\underline{\partial} A_\sigma)}, \\
&\underline{\iota}[h](\sigma) = \sum_{\ell=0}^{1} \sum_{\overline{\omega}\in\overline{\Omega}} \underline{I}^{\circ}_{n-1}[\partial_{\overline{s}}^{\ell} \overline{\omega}](\sigma), &&\underline{\iota}^1[h](\sigma) = \sum_{\ell=0}^{1} \sum_{\overline{\omega}\in\overline{\Omega}} \underline{I}^{1}_{n-1}[\partial_{\overline{s}}^{\ell} \overline{\omega}](\sigma), \\
&\underline{I}[\overline{\gamma}](\sigma) = \sum_{\ell_1,\ell_2=0}^{1} \sum_{i=0}^{2} \underline{I}^{\circ}_{n-1}[\partial_{\overline{s}}^{\ell_1} \partial_i^{\ell_2} \overline{\gamma}](\sigma), &&\underline{I}^1[\overline{\gamma}](\sigma) = \sum_{\ell_1,\ell_2=0}^{1} \sum_{i=0}^{2} \underline{I}^{1}_{n-1}[\partial_{\overline{s}}^{\ell_1} \partial_i^{\ell_2} \overline{\gamma}](\sigma)
\end{aligned} \tag{0.3.27}$$

(note that

$$\underline{\iota}[h](\sigma) \le \underline{\iota}^1[h](\sigma), \qquad \underline{I}[h](\sigma) \le \underline{I}^1[h](\sigma)), \tag{0.3.28}$$

set

$$\epsilon[f](\sigma) = \int_{A_\sigma} \frac{1}{2} \left[ (\partial_\tau f)^2 + (\partial_\xi f)^2 + (\partial_\zeta f)^2 \right] + k^{-1} \Delta_A^{cd} \partial_c f \partial_d f \, d\xi\, d\zeta, \tag{0.3.29}$$

and finally define energies

$$\begin{aligned}
\overline{E}_n[\overline{\gamma}](\sigma) &= \sum_{|I|\le n-1} \sum_{\ell=0}^{1} \epsilon[\partial^I \partial_{\overline{s}}^{\ell} \overline{\gamma}](\sigma), \\
E_{n,\ell}[h](\sigma) &= \sum_{|I|\le n-1} \sum_{\overline{\omega}\in\overline{\Omega}} \left| \partial^I \partial_{\overline{s}}^{\ell} \overline{\omega} \right|^2_{L^2(A_\sigma)}, \\
E_n[h](\sigma) &= \sum_{\ell=0}^{1} E_{n,\ell}[h](\sigma).
\end{aligned} \tag{0.3.30}$$

These give squares of norms of $C^\infty$ functions on $A_\sigma$, and it is effectively with respect to these norms that we shall close. Note the extra $\overline{s}$ derivative.

We fix $\nu \in (0,1)$, and assume that the initial data satisfy

$$\sup_{\sigma\le kT} \iota_n[h](\sigma) \le \frac{1}{32}\nu^2, \qquad \overline{\iota}_{\Sigma_0,n}[\overline{\gamma}] + \overline{\iota}_{U_0,n}[\overline{\gamma}] \le \frac{1}{12}\nu^2 \tag{0.3.31}$$

$$\sup_{\sigma\le kT} \underline{\iota}^1[h](\sigma) \le \frac{1}{1+\sqrt{2}(n+2)C_S^2C_\chi^2}\nu^2 \tag{0.3.32}$$

$$\sup_{\sigma\le kT} \underline{I}^1[\overline{\gamma}](\sigma) \le \frac{1}{1+\sqrt{2}(n+2)C_S^2C_\chi^2}\nu^2, \tag{0.3.33}$$

where $C_S$ and $C_\chi$ are as in (0.3.15 – 0.3.17) above. Existence of initial data satisfying the constraints (0.2.24 – 0.2.26) and the bounds (0.3.31 – 0.3.33) will be shown in Chapter 5; see especially Corollary 5.4.1.

Given the above definitions, we prove the following theorem.

0.3.3. THEOREM. Let $n \ge 4$, $T' > 1$, $T > 0$. There is a constant $C > 0$, independent of $k$, such that if $\nu \in (0,1)$ satisfies

$$\nu \le C \min\left\{T'^{-N}, \frac{T'^N}{T}\right\},$$

then for all initial data satisfying (0.2.24 – 0.2.26) and (0.3.31 – 0.3.33), there exist functions $\overline{\delta\ell}$, $\overline{b}$ and $\overline{c}$ on $\Gamma$, having the given initial data on $\Sigma_0$, and a function $\overline{\gamma}$ on $\Gamma$ having the given initial data on $\Sigma_0 \cup U_0$, satisfying the Riccati equations (0.2.9 – 0.2.11) and the wave equation (0.2.12), and such that the bounds

$$\begin{aligned} E_n[h](\sigma) &\le \nu^2, \\ \overline{E}_n[\overline{\gamma}](\sigma) &\le \nu^2 \end{aligned} \tag{0.3.34}$$

hold for $\sigma \in [0, kT]$. The metric (0.2.1), with

$$a = [1 + k^{-1}\overline{\delta\ell}]^2, \qquad b = k^{-1/2}\overline{b}, \qquad c = \overline{c}, \qquad \gamma = k^{-\iota}\overline{\gamma}, \tag{0.3.35}$$

will satisfy the Einstein vacuum equations on the region

$$\{(s,x,v,y) \in \mathbf{R}^4 \,|\, s \in [0,T'],\, v \in [0, T\sqrt{2}],\, \frac{1}{\sqrt{2}}(s+kv) \le kT\}.$$

This will be proved in Chapter 6.

We note a technical point. In deriving energy inequalities for $E_n[h]$ and $\overline{E}_n[\overline{\gamma}]$ it is useful to require the upper boundary of the bulk region $\Gamma$ to be a null hypersurface. Since this need not be the case in our choice of coordinates, to prove Theorem 0.3.3 we first extend $\Gamma$ to the region

$$\Gamma' = \{(\overline{s},\overline{x},\overline{v}) \in \mathbf{R}^3 \,|\, \overline{s} \in [0,2T'],\, \overline{v} \in [0,kT\sqrt{2}],\, \frac{1}{\sqrt{2}}(\overline{s}+\overline{v}) \le kT\}, \tag{0.3.36}$$

let $\chi \in C^\infty(\mathbf{R}^1)$ have support contained in $[-2,2]$ and satisfy $\chi|_{[-1,1]} = 1$, define

$$C_\chi = \max\{C_n^L \sup\{\chi^{(i)}(x) \,|\, x \in \mathbf{R}^1,\, i \in \{0,\cdots,n+1\}\}, 1\}, \tag{0.3.37}$$

where $C_n^L$ is a combinatorial constant depending only on $n$ whose exact form is not important (see equation (6.2.8) below), and replace the quantities $\overline{\delta\ell}$, $\overline{b}$ and $\overline{c}$ in the wave equation (0.2.12) with the quantities

$$\widetilde{\overline{\delta\ell}} = \chi\left(\frac{\overline{s}}{T'}\right)\overline{\delta\ell}, \qquad \widetilde{\overline{b}} = \chi\left(\frac{\overline{s}}{T'}\right)\overline{b}, \qquad \widetilde{\overline{c}} = \chi\left(\frac{\overline{s}}{T'}\right)\overline{c}, \tag{0.3.38}$$

and also define the quantities

$$\widetilde{\overline{a}} = \left[1 + k^{-1}\widetilde{\overline{\delta\ell}}\right]^2, \qquad \widetilde{\overline{\delta a}} = \widetilde{\overline{a}} - 1, \qquad \widetilde{\overline{\delta^{-1} a}} = \widetilde{\overline{a}}^{-1} - 1.$$

We then replace the wave equation (0.2.12) with

$$\begin{aligned} 0 &= \left(-2\partial_{\overline{s}}\partial_{\overline{v}}\overline{\gamma} + \partial_{\overline{x}}^2\overline{\gamma}\right) \\ &+ \frac{1}{k}\left(-\widetilde{\overline{\delta^{-1}a}}\partial_{\overline{x}}^2\overline{\gamma} - \widetilde{\overline{c}}\partial_{\overline{s}}^2\overline{\gamma} + 2\frac{\widetilde{\overline{b}}}{\widetilde{\overline{a}}}\partial_{\overline{s}}\partial_{\overline{x}}\overline{\gamma} - \frac{1}{2}\left(2\partial_{\overline{s}}\widetilde{\overline{c}} - \frac{2}{\widetilde{\overline{a}}}\partial_{\overline{x}}\widetilde{\overline{b}} + \frac{\partial_{\overline{v}}\widetilde{\overline{\delta a}}}{\widetilde{\overline{a}}}\right)\partial_{\overline{s}}\overline{\gamma} + \frac{1}{\widetilde{\overline{a}}}\partial_{\overline{s}}\widetilde{\overline{b}}\partial_{\overline{x}}\overline{\gamma} - \frac{1}{2}\frac{\partial_{\overline{x}}\widetilde{\overline{\delta a}}}{\widetilde{\overline{a}}^2}\partial_{\overline{x}}\overline{\gamma} - \frac{\partial_{\overline{s}}\widetilde{\overline{\delta a}}}{\widetilde{\overline{a}}}\partial_{\overline{v}}\overline{\gamma}\right) \\ &+ k^{-2}\left(\frac{\widetilde{\overline{b}}^2}{\widetilde{\overline{a}}}\partial_{\overline{s}}^2\overline{\gamma} - \frac{1}{2}\left(\widetilde{\overline{c}}\frac{\partial_{\overline{s}}\widetilde{\overline{\delta a}}}{\widetilde{\overline{a}}} - 4\frac{\widetilde{\overline{b}}\partial_{\overline{s}}\widetilde{\overline{b}}}{\widetilde{\overline{a}}} + \frac{\widetilde{\overline{b}}\partial_{\overline{x}}\widetilde{\overline{\delta a}}}{\widetilde{\overline{a}}^2}\right)\partial_{\overline{s}}\overline{\gamma} - \frac{1}{2}\frac{\widetilde{\overline{b}}}{\widetilde{\overline{a}}^2}\partial_{\overline{s}}\widetilde{\overline{\delta a}}\partial_{\overline{x}}\overline{\gamma}\right) - k^{-3}\left(\frac{1}{2}\frac{\widetilde{\overline{b}}^2}{\widetilde{\overline{a}}^2}\partial_{\overline{s}}\widetilde{\overline{\delta a}}\partial_{\overline{s}}\overline{\gamma}\right), \end{aligned} \tag{0.3.39}$$

similarly extend the quantities, regions, and norms in (0.3.20 – 0.3.30) to $\Gamma'$, and solve the system (0.2.9 – 0.2.11), (0.3.39) on $\Gamma'$. Existence of solutions to this system is shown by an iterative method. The results thus obtained imply those given in Theorem 0.3.3.

### 0.4. Innovations, and further comparisons to previous work

In addition to the novelty of the main results (Theorems 0.3.1 – 0.3.3), the current work contains several unique points of a technical nature. We begin with two which are unrelated to the coordinate scaling.

**Choice of gauge, including constraint equations.** Our gauge is constructed (see Section 1.2) by ruling the surface $\Sigma_0$ with null geodesics parameterised by $\overline{v}$, and then ruling the three-dimensional region $\Gamma$ with null geodesics transverse to $\Sigma_0$, parameterised by $\overline{s}$. Thus the level surfaces of $\overline{v}$ are null, while the the level surfaces of $\overline{s}$, in general, are not (with the exception of $\Sigma_0 = \{\overline{s} = 0\}$). In particular our gauge choice is definitely distinct from the double-null gauge more typically employed (see [3], [9]). (A similar gauge, but using timelike rather than null geodesics, was used in [1].) Further, for a $3+1$ metric of the form (0.2.1), the Einstein vacuum equations give ordinary differential equations (in fact Riccati equations) for $a$, $b$ and $c$ with source terms depending only on the *first* derivatives of $\gamma$. See Section 1.2 for the details.

In Chapter 2, we perform a direct evaluation of the full Ricci tensor and use it to derive the constraint equations in this gauge. We show that they can be formulated as ordinary differential equations in $\overline{v}$ on the surface $\Sigma_0$ and are preserved by the evolution under equations (0.2.2 – 0.2.5); unlike the case of wave coordinates in $3+1$ dimensions (see [11]), there are no separate constraint equations along $U_0$.*

**Utilisation of the algebraic structure of the equations.** As stated above (see Chapter 1 for the proof), the Einstein vaccum equations reduced by one polarised translational symmetry imply the system of

* Presumably, the equations on $U_0$ corresponding in some sense to the constraint equations on $\Sigma_0$ would include the Riccati equations (0.2.2 – 0.2.4), but we consider these to be evolution equations rather than constraint equations. The lack of symmetry between $\Sigma_0$ and $U_0$ appears to be due to the lack of symmetry in our gauge choice.

Riccati and wave equations (0.2.2 – 0.2.5):

$$\partial_s^2 a = \frac{(\partial_s a)^2}{2a} - 4a\,(\partial_s\gamma)^2 \tag{0.2.2}$$

$$\partial_s^2 b = \frac{1}{2a}\,(\partial_s a)\,(\partial_s b) - 4\partial_s\gamma\,(b\partial_s\gamma + \partial_x\gamma) \tag{0.2.3}$$

$$\partial_s^2 c = \frac{(\partial_s b)^2}{2a} - 2\partial_s\gamma\left(2\partial_v\gamma + \left(\frac{b^2}{a} + c\right)\partial_s\gamma + 2\frac{b}{a}\partial_x\gamma\right) - \frac{2}{a}\,(\partial_x\gamma)^2 \tag{0.2.4}$$

$$\Bigg[\left(\frac{b^2}{a} - c\right)\partial_s^2 + 2\frac{b}{a}\partial_s\partial_x - 2\partial_s\partial_v + \frac{1}{a}\partial_x^2 - \frac{1}{2}\left(\left(\frac{b^2}{a} + c\right)\frac{\partial_s a}{a} - 4\frac{b}{a}\partial_s b + \frac{b}{a^2}\partial_x a + 2\partial_s c - \frac{2}{a}\partial_x b + \frac{\partial_v a}{a}\right)\partial_s$$
$$- \frac{1}{2}\left(\frac{b}{a^2}\partial_s a - \frac{2}{a}\partial_s b + \frac{\partial_x a}{a^2}\right)\partial_x - \frac{1}{2}\frac{\partial_s a}{a}\partial_v\Bigg]\gamma = 0. \tag{0.2.5}$$

The algebraic structure of these equations is special in at least two (related) ways, which are crucial to our ability to close the energy estimates in the geodesic gauge we use. Note the top-order derivatives in $\gamma$ which occur in the wave equation (0.2.5): $\partial_s^2$, $\partial_s\partial_x$, $\partial_s\partial_v$, and $\partial_x^2$: in other words, the only second-order derivative which does not involve $\partial_s$ is $\partial_x^2$. This, together with the second-order (in $s$) nature of the Riccati equations (0.2.2 – 0.2.4), allows us to take an extra $s$ (or $\overline{s}$) derivative in the energies (0.3.30), which also define the norms with respect to which we close (see Chapter 6 for the details). Second, note the derivatives of $a$, $b$, and $c$ which appear in the first-order coefficients in the wave equation:

$$\begin{array}{lll} \partial_s a, & \partial_x a, & \partial_v a \\ \partial_s b, & \partial_x b & \\ \partial_s c. & & \end{array}$$

Now note the derivatives of $\gamma$ which appear on the right-hand sides of the Riccati equations (0.2.2 – 0.2.4):

$$\begin{array}{llll} a: & \partial_s\gamma & & \\ b: & \partial_s\gamma, & \partial_x\gamma & \\ c: & \partial_s\gamma, & \partial_x\gamma, & \partial_v\gamma. \end{array}$$

Differentiating the terms appearing in each row of the second table with respect to the derivatives appearing in the corresponding row of the first table gives exactly the second derivatives of $\gamma$ which we must bound in order to bound the coefficients appearing in the wave equation. These are precisely

$$\begin{array}{lll} \partial_s^2\gamma, & \partial_x\partial_s\gamma, & \partial_v\partial_s\gamma \\ \partial_s^2\gamma, & \partial_s\partial_x\gamma, & \partial_x^2\gamma \\ \partial_s\partial_v\gamma, & \partial_s\partial_x\gamma, & \partial_s\partial_v\gamma: \end{array}$$

which, in turn, are *exactly* those second-order derivatives of $\gamma$ appearing in the wave equation itself!

More carefully, though somewhat less dramatically, the foregoing plays out in our work below in at least the following way. Note that since we have extra $s$ derivatives, due to the nature of the energies (0.3.30), second-order derivatives involving an $s$ are not too big of a concern. On the other hand, to bound $\partial_x b$, we need to bound $\partial_x^2\gamma$, which looks like a loss of derivative. However, as noted above, we may solve the wave equation (0.2.5) for $\partial_x^2\gamma$ in terms of second-order derivatives of $\gamma$ with at least one $s$ derivative, meaning that this term is on the same footing as the others. In this vein, note that had the wave equation involved a $\partial_v b$, or any derivative of $c$ other than $\partial_s c$, we would not be able to treat it in this way, and this presumably would lead to a real loss of derivative. In other words, somehow, the derivatives of the metric components $a$, $b$, and $c$ which we need to bound are more or less exactly those we are able to bound without losing derivatives. This algebraic structure presumably reflects something deeper, but it is not readily apparent what.

**Coordinate scaling.** As noted, the foregoing innovations are independent of the coordinate scaling (0.2.7). The explicit use of scaled coordinates is a very central innovation in the present work. In other works with which we are familiar in which scaling assumptions play a role, explicitly or implicitly (such as [3], [5], [6], [9]), the scaling is at most used as an ansatz, encoded in prescribed bounds on energies, or used to define weighted norms; in none of these works are the actual equations themselves studied in a scaled coordinate system. While the use of coordinate scaling does not, by itself, give rise to new results (it is, after all, just a very simple change of coordinates), it does, in our setting at least, give rise to a much cleaner presentation, and greatly reduces the number of explicit powers of $k$ which must be carried around (compare the definitions in (0.3.5 – 0.3.14) with those in (0.3.21 – 0.3.30), for example). More significantly, by allowing us to separate out terms in both the wave (0.2.5) and Riccati (0.2.2 – 0.2.4) equations by order in $k$, it makes transparent the origin of the ansätze (0.2.14), (0.2.16). We also hope that this new approach will lead to further new results in the future.

**Comparisons on the scaling of $\hat{\chi}$.** Precise comparisons can be made between the main existence results (Theorems 0.3.2 and 0.3.3) and the results in [3], [5], [6], and [9]. Given that all four of those references work in a double null foliation, which is distinct from the geodesic gauge we use, and given also that our scaling is imposed in $2+1$ dimensions while that in [3 – 6], [9] is imposed in $3+1$ dimensions, the best way to obtain precise comparisons is through $L^\infty$ norms of initial data. In particular we may compare the second fundamental form of the spacelike surfaces $\{x = x_0, y = y_0\}$ in the outgoing initial null hypersurface $\{s = 0\}$ with analogous quantities in [3 – 6], [9]. In our case, the trace-free part of the second fundamental form is given with respect to the orthonormal frame $X = e^\gamma a^{1/2}\partial_x$, $Y = e^{-\gamma}\partial_y$ by (see Section 6.7)

$$\hat{\chi} = \begin{pmatrix} -\partial_v\gamma + \frac{\partial_v a}{4a} & 0 \\ 0 & \partial_v\gamma - \frac{\partial_v a}{4a} \end{pmatrix}.$$

We also have the following result (see Proposition 6.7.1):

0.4.1. PROPOSITION. Suppose that $\gamma|_{{}^0\Sigma_0}$ is specified as in (0.2.29), but with an overall factor of $k^{-\iota}$ instead of $k^{-1/2}$, with $\iota \geq 1/2$. (Thus, in particular, $\gamma|_{{}^0\Sigma_0}$ is supported in the double strip $\{0 < v < k^{-1}\} \cup \{T\sqrt{2} - k^{-1} < v < T\sqrt{2}\}$.) Then there are constants $C_1$, $C_2$, depending on $\ell$, $m$, and $\varpi$ but not on $k$, such that*

$$\|\partial_v^\ell \partial_x^m \gamma\|_{L^\infty({}^0\Sigma_0)} = C_1 k^{\ell + m/2 - \iota}, \qquad \|\partial_v^\ell \partial_x^m \gamma\|_{L^2({}^0\Sigma_0)} = C_2 k^{\ell + m/2 - \iota - 3/4}.$$

By Proposition 0.4.1, $\partial_v^\ell \partial_x^m \hat{\chi}$ will have size $k^{1+\ell+m/2-\iota}$ in $L^\infty$. This may be compared to (0.1.5), and also to (0.1.11) if we recall that $\hat{\chi}_0 \sim \partial_{\underline{u}}\hat{\gamma}_n$. When $\iota = 1/2$, this is exactly the scaling appearing in the

* We obtain actual *equalities*, rather than just bounds, since we are working with the initial data, which is known exactly.

right-hand column of (0.1.5) (note that a scaling for both $\omega$ and $\underline{u}$ derivatives follows in that case as well). When $\iota = 1$, it is the scaling given in (0.1.11) when $m = 0$, in accordance with what we have already noted at the end of Section 0.2 above.

**0.5. Extensions**

In this section we suggest a way in which the preceding results may admit of extension.

By domain of dependence arguments (see equation (1.2.7)), the metric $g$ arising via (0.2.1) from a solution to (0.2.2 – 0.2.5) with initial data as constructed in Lemma 0.2.2 will be Minkowskian whenever $x$ is outside of a compact set whose size is determined by $k^{-1}$, $T$, and $T'$. Let us term the closure of the set on which a solution gives rise to a metric which is not Minkowskian its *support* (in terms of the usual definition of support, this is the union

$$\operatorname{supp}(a-1) \cup \operatorname{supp} b \cup \operatorname{supp} c \cup \operatorname{supp} \gamma.)$$

Let $(a_0, b_0, c_0, \gamma_0)$ be a solution to (0.2.2 – 0.2.5) with initial data as in Lemma 0.2.2. Clearly, then, for any $\Delta x$, this solution shifted in $x$ by an amount $\Delta x$, i.e., the quadruple

$$(a_1, b_1, c_1, \gamma_1)|_{(s,x,v)} = (a_0, b_0, c_0, \gamma_0)|_{(s,x+\Delta x,v)},$$

will also be a solution to (0.2.2 – 0.2.5). For $\Delta x$ sufficiently large, depending only on $k^{-1}$, $T$, and $T'$, the support of $(a_1, b_1, c_1, \gamma_1)$ will be disjoint from that of $(a_0, b_0, c_0, \gamma_0)$. If we further define, for any integer $n$, the quadruples

$$(a_n, b_n, c_n, \gamma_n)|_{(s,x,v)} = (a_0, b_0, c_0, \gamma_0)|_{(s,x+n\Delta x,v)},$$

then clearly the supports of the $(a_n, b_n, c_n, \gamma_n)$ will be pairwise disjoint. By domain of dependence arguments, then, we may paste them together to obtain the solution

$$(a, b, c, \gamma) = \begin{cases} (a_n, b_n, c_n, \gamma_n), & x \in [0,1] + n\Delta x \\ (1, 0, 0, 0), & \text{otherwise.} \end{cases}$$

Here we can think of $n$ as varying over a finite (though arbitrarily large) subset of $\mathbf{Z}$. In particular, noting that Proposition 0.4.1 shows that $\|\gamma_0\|_{L^2({}^0\Sigma_0)}$ is of size $k^{-5/4}$, if we choose $n > k^{5/4+\alpha}$ for some $\alpha$ we can produce a solution with initial data for $\gamma$ of size at least $k^\alpha$ in $L^2({}^0\Sigma_0)$. Note that $T$ and $T'$ are completely independent of $n$, and of course still independent of $k$.

**0.6. Historical note.**

The bulk of the work through the end of Chapter 6 was completed before the connection of the key coordinate scaling (0.2.7) to the work in [6] was understood. It is possible that the current treatment could be modified to bring it closer to the method suggested there. On the other hand, the scaling here originated from a study of the structure of the equations (0.2.2 – 0.2.5), and it is possible that a further study of this structure could shed additional light on the significance of this scaling.

**0.7. Outline of current work.**

We now give a short outline of the present work. In Chapter 1 we describe how the full $3+1$ Einstein vacuum equations may be reduced to the scalar field Einstein equations in $2+1$ dimensions when a translational symmetry is present, construct a gauge choice for the reduced metric, and show how the reduced Einstein equations give rise to Riccati equations for the metric components. In Chapter 2 we determine the constraint equations, prove that they are preserved by the evolution, and show that the constraint equations on the initial hypersurface together with the Riccati and wave equations derived in Chapter 1 are equivalent to the Einstein vacuum equations. We then briefly introduce our choice of initial data. In Chapter 3 we then describe a certain coordinate scaling and rewrite the equations in the scaled coordinates. In Chapter 4 we give certain fairly straightforward algebraic and analytical background results. In Chapter 5 we construct the initial data and show how to bound their higher transverse derivatives. In Chapter 6 we prove the existence theorem Theorem 0.3.3. Finally, in Chapter 7 we apply the results of Chapter 6 to construct a family of solutions to the $3+1$ Einstein vacuum equations which are concentrated in a small region along a null 2-plane.

### 0.8. Organisation.

Each chapter is broken into sections. Propositions, lemmata, theorems, and corollaries are numbered separately in each section (so each section has, for example, a Proposition 1, Lemma 1, Theorem 1, etc.). Equations are also numbered separately in each section. References to equations and results are of the form (chapter number.section number.item number).

### 0.9. Notations and conventions.

For convenience in dealing with cases where $T' > 1$, we set $\underline{T}' = \min\{1, T'\}$; this allows us to replace quantities $\underline{T}'^{-m}$ with $\underline{T}'^{-n}$ if $n > m$.

We denote the partial derivative of a function $f$ with respect to an independent variable $x$ by either $\partial_x f$ or $f_{,x}$. We use these two notations interchangeably.

We always work with tensors in terms of their components, and employ the Einstein summation convention throughout. When writing out the covariant derivative, we use expressions like

$$\nabla_i T_{jk\ell} \tag{0.9.1}$$

to mean the $ijk\ell$ element of the 4-covariant tensor $\nabla T$ – it must be borne in mind that this is completely different from the $i$th derivative of the $jk\ell$ element of the 3-covariant tensor $T$, which would be simply $\partial_i T_{jk\ell}$ and is of course not a tensor. Where confusion might result, we employ parentheses as appropriate.

To facilitate complicated calculations, for example those of the Ricci tensor in Chapter 2, it is often convenient for us to treat rank-2 tensors (of whatever degree of co- or contravariance) as matrices; we make the convention that in doing so the first index (left to right) represents the row while the second index represents the column. When working with indices, in the unscaled picture (see Chapter 3) we always use 0 for $s$, 1 for $x$, 2 for $v$, and (in the few cases when we deal with the full $3+1$ problem directly) 3 for $y$, where $y$ is the coordinate of translational symmetry (see Section 1.1) and $sxv$ is the coordinate system described in Section 1.2. In the scaled picture (again, see Chapter 3) we use 012 for either $\overline{s}\,\overline{x}\,\overline{v}$ or $\tau\xi\zeta$ (see Chapter 3, equation (3.3.1), and Chapter 6, equation (6.2.1)); which one is intended will be indicated unless it is clear from the context. We shall not have occasion to deal with the full $3+1$ metric in the scaled picture.

Our definition of the Sobolev norm $H^m$ *includes* the $L^2$ norm; in other words, for example,

$$\|f\|^2_{H^m(\mathbf{R}^p)} = \sum_{|I|\le m} \|\partial^I f\|^2_{L^2(\mathbf{R}^p)},$$

where $I$ is a multiindex in all directions in $\mathbf{R}^m$. Similarly, if $X \subset \mathbf{R}^m$ is any open *linear* submanifold (i.e., any open subset of an affine subspace of $\mathbf{R}^m$), or any open linear submanifold together with some or all of its boundary points, we shall, unless explicitly noted otherwise, take

$$\|f\|^2_{H^m(X)} = \sum_{|J|\le m} \|\partial^J f\|^2_{L^2(X)},$$

where $J$ is a multiindex in all directions tangent to $X$. We shall write the Sobolev norm *without* the $L^2$ norm as

$$\|f\|^2_{H^m_\circ(X)} = \sum_{1\le|J|\le m} \|\partial^J f\|^2_{L^2(X)}. \tag{0.9.2}$$

When dealing with function spaces, we shall sometimes, by a slight abuse of notation, write things like $f \in L^2(\partial X)$ when $f$ is a function defined on $X$ whose restriction to $\partial X$ is in $L^2$ on $\partial X$, in place of the technically correct $f|_{\partial X} \in L^2(\partial X)$.

**1.1. Reduction of Einstein vacuum equations in the presence of one translational symmetry.**

Let $(x^i)$ denote rectangular coordinates on $\mathbf{R}^4$, and suppose that on some open set in $\mathbf{R}^4$ we are given a Lorentzian metric which has representation

$$g = e^{2\gamma(x^0,x^1,x^2)} dx^3 \otimes dx^3 + \sum_{i,j=0}^{2} {}_0h_{ij}(x^0, x^1, x^2) dx^i \otimes dx^j,$$

i.e., that $g$ has a linear translational symmetry along $x^3$. Clearly, ${}_0h_{ij}$ is a Lorentzian metric on $\mathbf{R}^3$. Suppose that $g$ satisfies the vacuum Einstein equations Ric $(g) = 0$, where Ric $(g)$ denotes the Ricci tensor of $g$, and let $h_{ij} = e^{2\gamma(x^0,x^1,x^2)} {}_0h_{ij}$. Then, letting $R_{ij}$ denote the Ricci tensor of, and $\Box_h$ the wave operator corresponding to, $h$, it can be shown (see [15], Appendix D, [1], (1.12), (1.13)) that, on an appropriate open set in $\mathbf{R}^3$,

$$\Box_h \gamma = 0, \qquad R_{ij} = 2\partial_i\gamma\partial_j\gamma. \tag{1.1.1}$$

We note that these equations are the Einstein equations coupled to a free scalar field in $2 + 1$ dimensions, but we shall not make explicit use of this fact going forwards.

**1.2. Gauge choice**

In this section we construct the gauge we shall use to analyse equation (1.1.1). We shall assume that there is a neighbourhood $O$ in the domain of $h$ and $\gamma$ which can be coordinatised in the following way. Let $p \in O$. There is a spacelike curve $\lambda : \mathbf{R}^1 \to O$ satisfying $\lambda(0) = p$ and $h(\dot\lambda(x), \dot\lambda(x)) = a_0(x)$, where $a_0 : \mathbf{R}^1 \to \mathbf{R}^+$ and the support of $a_0 - 1$ in $\mathbf{R}^1$ is compact (note that this last condition can be guaranteed by reparametrisation as long as $a$ has a uniform upper bound and a uniform positive lower bound on $\mathbf{R}^1$).* Then along $\lambda$ there are null vectors $L(x)$ and $\underline{L}(x)$, smooth in $x$, such that if $X = \dot\lambda$ then $\{\underline{L}(x), X(x), L(x)\}$ is a basis of $T_{\lambda(x)}O$ with respect to which $h$ has the matrix representation

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & a_0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

(in other words, $h(X, X) = a_0$, $h(L, \underline{L}) = h(\underline{L}, L) = -1$, and all other inner products vanish). Consider the null hypersurface ruled by the null geodesics tangent to vectors $L(x)$ (call it $\Sigma_0^0$), and extend $\underline{L}(x)$ to this hypersurface by parallel transport along these geodesics. Extend the function $x$ to $\Sigma_0^0$ by requiring it to be constant on the null geodesics tangent to $L$, let $v$ denote the affine parameter along these geodesics with $v|_\lambda = 0$, and define on $\Sigma_0^0$ the function $a = h(\partial_x, \partial_x)$. By construction, $a = a_0$ on the line $v = 0$, and therefore there is a $V' > 0$ such that $a(x, v) > 0$ for $v \in [0, V')$, $x \in \mathbf{R}^1$.

We now wish to complete the process by foliating a neighbourhood of $\Sigma_0^0$ by null geodesics. Define on $s = 0$ the function†

$$d = h(\partial_x, \underline{L}), \tag{1.2.1}$$

* Note that it would be possible, again by a reparameterisation, to require $a_0(x) = 1$ for all $x$. As we shall see in Chapter 5, however, such a condition would be too restrictive.

† Note that requiring $d = 0$ would contradict the constraint equations derived in Chapter 2.

and define a new vector field along $\Sigma_0^0$ by

$$N = \underline{L} - \frac{d}{a}\partial_x - \frac{d^2}{2a}\partial_v. \tag{1.2.2}$$

We will show in the proposition below that $N$ is null and perpendicular to $\partial_x$ on $\Sigma_0^0$.

We now complete the construction of the gauge by foliating a neighbourhood of $\Sigma_0^0$ by null hypersurfaces ruled by geodesics parallel to $N(x, v)$; call this neighbourhood $\Gamma'$. Let $s$ be an affine parameter along these satisfying $s|_{\Sigma_0^0} = 0$, and extend $x$ and $v$ to functions on $\Gamma'$ by requiring them to be constant along these geodesics. By construction, there exist $S, V > 0$ such that the region

$$\Gamma_0 = \{(s, x, v) \in \mathbf{R}^3 \mid s \in [0, S),\ v \in [0, V)\} \tag{1.2.3}$$

is contained in $\Gamma'$ and such that $a > 0$ on $\Gamma_0$. Given this coordinate system, we have the following result.

1.2.1. PROPOSITION. There exist functions $a : \Gamma_0 \to \mathbf{R}^+$, $b, c : \Gamma_0 \to \mathbf{R}^1$ such that on $\Gamma_0$ the representation of $h$ with respect to the basis $\{\partial_s, \partial_x, \partial_v\}$ is given by

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & a & b \\ -1 & b & c \end{pmatrix} \tag{1.2.4}$$

Furthermore, the functions $a$, $b$, and $c$ satisfy the following conditions, for all $x \in \mathbf{R}^1$, $v \in [0, V)$:

$$a(0, x, 0) = a_0(x), \qquad b(0, x, v) = c(0, x, v) = \partial_s c(0, x, v) = 0. \tag{1.2.5}$$

Proof. We note first that along $\lambda$ we have $X = \partial_x$ and $L = \partial_v$; thus along $\lambda$ we have $d = h(\partial_x, \underline{L}) = 0$, so by equation (1.2.2) $N = \underline{L}$, which gives, along $\lambda$, $\underline{L} = \partial_s$. Now by construction, along $\lambda$ the metric $h$ in the basis $\{\underline{L}, X, L\} = \{\partial_s, \partial_x, \partial_v\}$ has the representation (since $a = a_0$ on $\lambda$)

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & a & 0 \\ -1 & 0 & 0 \end{pmatrix}. \tag{1.2.6}$$

We claim that the metric $h$ has the representation (1.2.6) in the basis $\{\partial_s, \partial_x, \partial_v\}$ everywhere on $s = 0$. To see this, note first that since $\partial_v$ is parallel-transported along itself on $s = 0$, it is null there, so we have $h(\partial_v, \partial_v) = 0$. Next, note that $\nabla_{\partial_v}\partial_x = \nabla_{\partial_x}\partial_v$ since $\partial_x$ and $\partial_v$ are coordinate vector fields; thus

$$\partial_v h(\partial_v, \partial_x) = h(\partial_v, \nabla_{\partial_v}\partial_x) = h(\partial_v, \nabla_{\partial_x}\partial_v) = \frac{1}{2}\partial_x h(\partial_v, \partial_v) = 0,$$

so that $h(\partial_v, \partial_x)$ is constant on $s = 0$ and hence equal to zero there. Finally, we have (recalling that $\underline{L}$ is parallel-transported along $\partial_v$ on $s = 0$, so that $h(\underline{L}, \partial_v) = -1$, $h(\underline{L}, \underline{L}) = 0$ must hold everywhere on $s = 0$)

$$\begin{aligned} h(\partial_s, \partial_v) &= h(N, \partial_v) = h(\underline{L}, \partial_v) = -1 \\ h(\partial_s, \partial_x) &= h(\underline{L}, \partial_x) - \frac{d}{a}h(\partial_x, \partial_x) = d - \frac{d}{a} \cdot a = 0 \\ h(\partial_s, \partial_s) &= 2\frac{d^2}{2a} - 2\frac{d}{a}h(\underline{L}, \partial_x) + \frac{d^2}{a^2}h(\partial_x, \partial_x) = \frac{d^2}{a} - 2\frac{d^2}{a} + \frac{d^2}{a} = 0, \end{aligned}$$

showing that in the basis $\{\partial_s, \partial_x, \partial_v\}$, the metric $h$ does indeed have the representation (1.2.6) everywhere on $s = 0$.

To finish the proof, note that since $\partial_s$ is parallel-transported along itself throughout $\Gamma_0$, we have by the foregoing that $h(\partial_s, \partial_s) = 0$ everywhere in $\Gamma_0$, and thus

$$\partial_s h(\partial_x, \partial_s) = h(\partial_s, \nabla_{\partial_s}\partial_x) = h(\partial_s, \nabla_{\partial_x}\partial_s) = \frac{1}{2}\partial_x h(\partial_s, \partial_s) = 0,$$

and by the same logic $\partial_s h(\partial_v, \partial_s) = 0$. Thus throughout $\Gamma_0$ we have also $h(\partial_x, \partial_s) = 0$, $h(\partial_v, \partial_s) = -1$, and the representation of $h$ in the basis $\{\partial_s, \partial_x, \partial_v\}$ is indeed given by

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & a & b \\ -1 & b & c \end{pmatrix}$$

for functions $a$, $b$, and $c$ satisfying for all $x \in \mathbf{R}^1$, $v \in [0, V)$

$$a(0, x, 0) = h(\partial_x, \partial_x)|_\lambda = a_0(x),$$

$$b(0, x, v) = h(\partial_x, \partial_v)|_{s=0} = 0,$$

$$c(0, x, v) = h(\partial_s, \partial_v)|_{s=0} = 0.$$

The condition $a > 0$ was already obtained in the definition of $\Gamma_0$ by a suitable restriction on $S$ and $V$.

Finally, note that on $s = 0$

$$\partial_s c = \partial_s h(\partial_v, \partial_v) = 2h(\partial_v, \nabla_{\partial_s}\partial_v) = 2h(\partial_v, \nabla_{\partial_v}\partial_s) = 2\partial_v h(\partial_v, \partial_s) = 0,$$

where we have used the fact that $\partial_v$ is parallel-transported along itself. This completes the proof. QED.

Recall (see section 0.9) that when we work in the unscaled picture (as here) the indices 012 always represent the coordinates $s$, $x$, and $v$, respectively.

Conversely, let $h$ be some metric on the region $\Gamma_0$ which has the form given in equation (1.2.4) and satisfying (1.2.5), where $a_0 : \mathbf{R}^1 \to \mathbf{R}^+$ is equal to 1 outside of some compact set. From our work in Chapter 2 below, the Christoffel symbols for $h$ are given by

$$\Gamma^0_{ij} = \frac{1}{2a}\begin{pmatrix} 0 & ba_{,s} - ab_{,s} & bb_{,s} - ac_{,s} \\ ba_{,s} - ab_{,s} & -(b^2 - ac)a_{,s} + ba_{,x} - a(2b_{,x} - a_{,v}) & -(b^2 - ac)b_{,s} + ba_{,v} - ac_{,x} \\ bb_{,s} - ac_{,s} & -(b^2 - ac)b_{,s} + ba_{,v} - ac_{,x} & -(b^2 - ac)c_{,s} + b(2b_{,v} - c_{,x}) - ac_{,v} \end{pmatrix}$$

$$\Gamma^1_{ij} = \frac{1}{2a}\begin{pmatrix} 0 & a_{,s} & b_{,s} \\ a_{,s} & a_{,x} - ba_{,s} & a_{,v} - bb_{,s} \\ b_{,s} & a_{,v} - bb_{,s} & 2b_{,v} - c_{,x} - bc_{,s} \end{pmatrix}$$

$$\Gamma^2_{ij} = \frac{1}{2a}\begin{pmatrix} 0 & 0 & 0 \\ 0 & aa_{,s} & ab_{,s} \\ 0 & ab_{,s} & ac_{,s} \end{pmatrix}.$$

Since $\Gamma^k_{00} = 0$ for $k = 0, 1, 2$, it is straightforward to show that all curves $x = x_0$, $v = v_0$, $s = \sigma$ in $\Gamma_0$ are geodesics. They will be null since on $\Gamma_0$ we have $h(\partial_s, \partial_s) = 0$. Similarly, on $s = 0$ the Christoffel symbols become

$$\Gamma^0_{ij} = \frac{1}{2a}\begin{pmatrix} 0 & -ab_{,s} & 0 \\ -ab_{,s} & ab_{,v} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Gamma^1_{ij} = \frac{1}{2a}\begin{pmatrix} 0 & a_{,s} & b_{,s} \\ a_{,s} & a_{,x} & a_{,v} \\ b_{,s} & a_{,v} & 0 \end{pmatrix}$$

$$\Gamma^2_{ij} = \frac{1}{2a}\begin{pmatrix} 0 & 0 & 0 \\ 0 & aa_{,s} & ab_{,s} \\ 0 & ab_{,s} & 0 \end{pmatrix};$$

since now $\Gamma^k_{22}$ is also zero for $k = 0, 1, 2$, it is again straightforward to show that the curves $s = 0$, $x = x_0$, $v = \sigma$ in $\Gamma_0$ are geodesics, which are again null since on $s = 0$ we have $h(\partial_v, \partial_v) = 0$. Finally, the curve $s = v = 0$, $x = \sigma$ will be a spacelike curve since $a_0 > 0$. Thus if $h$ is any metric which on a set of the form $\Gamma_0 = \{(s, x, v) \in \mathbf{R}^3 \mid s \in [0, S),\ v \in [0, V)\}$ has the form given in Proposition 1.2.1, then the curve $s = v = 0$ must be spacelike, the curves $s = 0$, $x = x_0$ must be null geodesics, and the curves $x = x_0$, $v = v_0$ must also be null geodesics.

We shall work in this gauge from henceforth. For future use, we note that the determinant of the matrix in Proposition 1.2.1 is $-a = -h(\partial_x, \partial_x)$, and that its inverse (and hence the representation of $h^{-1}$) is

$$-\frac{1}{a}\begin{pmatrix} ac - b^2 & -b & a \\ -b & -1 & 0 \\ a & 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{b^2}{a} - c & \frac{b}{a} & -1 \\ \frac{b}{a} & \frac{1}{a} & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

It is worth noting that a Minkowski metric on $\mathbf{R}^3$ is of the above form with $a = 1$, $b = c = 0$, in the which case $s$ and $v$ are null coordinates and $x$ is a spatial coordinate. For future reference, we note the following related result. Suppose that on $\{s = 0\} \cup \{v = 0\}$, the functions

$$a - 1, \qquad b, \qquad c$$

have support contained in the strip $\{x_1 \le x \le x_2\}$. As just noted, the metric on $\{(s, x, v) \in \Gamma_0 \mid s = 0 \,\mathrm{or}\, v = 0 \,\mathrm{and}\, x \notin [x_1, x_2]\}$ will be Minkowskian. If we let $C_{x_1}$ and $C_{x_2}$ denote the Minkowskian null cones emanating from $(0, x_1, 0)$ and $(0, x_2, 0)$, respectively, then by domain of dependence arguments it is clear that $h$ will be Minkowskian on the set

$$\Gamma_0 \setminus [C_{x_1} \cup C_{x_2} \cup \{(s, x, v) \in \Gamma_0 \mid x \in [x_1, x_2]\}]. \tag{1.2.7}$$

We shall now derive evolution equations for $a$, $b$, and $c$, given that the metric $h$ has Ricci tensor which satisfies the second of equations (1.1.1). These equations can also be derived from the explicit representation of the Ricci tensor given in Chapter 2 below, but here we shall give a simpler geometric derivation. Let* $K_{ij} = \nabla_i N_j$. Then we see that

$$\begin{aligned} \partial_s a &= \partial_s h(\partial_x, \partial_x) = 2h(\partial_x, \nabla_{\partial_x}\partial_s) = 2K_{11}, \\ \partial_s b &= \partial_s h(\partial_x, \partial_v) = h(\nabla_{\partial_x}\partial_s, \partial_v) + h(\partial_x, \nabla_{\partial_v}\partial_s) = 2K_{12}, \\ \partial_s c &= \partial_s h(\partial_v, \partial_v) = 2h(\partial_v, \nabla_{\partial_v}\partial_s) = 2K_{22}. \end{aligned} \tag{1.2.8}$$

* If we restricted $i$ and $j$ appropriately, this would be the second fundamental form corresponding to the foliation of $\Gamma_0$ by surfaces of constant $s$.

Further, $K_{ij}$ satisfies the following equation:

1.2.2. Proposition. We have

$$\partial_s K_{ij} = h^{lk} K_{ik} K_{lj} - 2h_{ij} \left(\partial_s \gamma\right)^2 + 2h_{i0} \partial_s \gamma \partial_j \gamma + 2h_{0j} \partial_s \gamma \partial_i \gamma - h^{kl} \partial_k \gamma \partial_l \gamma h_{j0} h_{i0},$$

or alternatively

$$\partial_s K_{ij} = h^{lk} K_{ik} K_{lj} - 2h_{ij} \left(\partial_s \gamma\right)^2 - 2\delta_{i2} \partial_s \gamma \partial_j \gamma - 2\delta_{j2} \partial_s \gamma \partial_i \gamma - \delta_{i2} \delta_{j2} h^{kl} \partial_k \gamma \partial_l \gamma,$$

where $\delta_{ij}$ represents the Kronecker delta.

Proof. Let $K_i^j = h^{jk} K_{ik} = \nabla_i N^j$. Then we have

$$(\nabla_N K)_i^j = \partial_s K_i^j - \Gamma_{0i}^k K_k^j + \Gamma_{0k}^j K_i^k.$$

Now

$$\Gamma_{0i}^k = h^{kl} \left(h_{0l,i} + h_{il,0} - h_{0i,l}\right);$$

but $h_{0i}$ is constant for all $i$, so

$$\Gamma_{0i}^k = h^{kl} h_{li,0} = h^{kl} (2K_{li}) = 2K_i^k,$$

by equations (1.2.8) above. Thus

$$(\nabla_N K)_i^j = \partial_s K_i^j - 2K_i^k K_k^j + 2K_k^j K_i^k = \partial_s K_i^j.$$

But now also (remember that $N^k \nabla_k N^j = 0$ since $N = \partial_s$ is parallel-transported along itself)

$$\begin{aligned}
(\nabla_N K)_i^j &= N^k \nabla_k \nabla_i N^j \\
&= N^k \nabla_i \nabla_k N^j - R_{kil}^j N^k N^l \\
&= -\left(\nabla_i N^k\right)\left(\nabla_k N^j\right) - R_{0i0}^j \\
&= -K_i^k K_k^j - R_{0i0}^j.
\end{aligned}$$

Thus we have

$$\partial_s K_i^j = -K_i^k K_k^j - R_{0i0}^j,$$

so

$$\begin{aligned}
\partial_s K_{ij} &= \partial_s \left(h_{jk} K_i^k\right) = \left(\partial_s h_{jk}\right) K_i^k + h_{jk} \partial_s K_i^k \\
&= 2K_{jk} K_i^k + h_{jk} \left(-K_i^l K_l^k - R_{0i0}^k\right) \\
&= h^{lk} K_{ik} K_{lj} - R_{j0i0}.
\end{aligned}$$

Now we have in general (see, e.g., Wald, (3.2.28))

$$R_{ijkl} = C_{ijkl} + \frac{2}{n-2}(h_{i[k} R_{l]j} - h_{j[k} R_{l]i}) - \frac{2}{(n-1)(n-2)} R h_{i[k} h_{l]j},$$

where $C_{ijkl}$ is the Weyl tensor, [ ] denotes antisymmetrisation and $R = R_i^i$ denotes the Ricci scalar. In 3 dimensions the Weyl tensor is identically zero, and the Riemann tensor is uniquely determined by the Ricci tensor. Thus if $h$ satisfies the second of equations (1.1.1) above, we will have

$$\begin{aligned} R_{j0i0} &= h_{ji}R_{00} - h_{j0}R_{i0} - h_{0i}R_{0j} + h_{00}R_{ij} - \frac{1}{2}R\left(h_{ji}h_{00} - h_{j0}h_{i0}\right) \\ &= 2h_{ij}\left(\partial_s\gamma\right)^2 - 2h_{j0}\partial_s\gamma\partial_i\gamma - 2h_{0i}\partial_s\gamma\partial_j\gamma + h^{kl}\partial_k\gamma\partial_l\gamma h_{j0}h_{i0}. \end{aligned}$$

If we let $\delta_{ij}$ denote the Kronecker delta, then since in our basis $h_{0i} = h_{i0} = -\delta_{i2}$, this expression may also be written as

$$R_{j0i0} = 2h_{ij}\left(\partial_s\gamma\right)^2 + 2\delta_{j2}\partial_s\gamma\partial_i\gamma + 2\delta_{i2}\partial_s\gamma\partial_j\gamma + h^{kl}\partial_k\gamma\partial_l\gamma\delta_{j2}\delta_{i2}.$$

Combining everything together gives the desired results. QED.

From this we may derive the evolution equations for $a$, $b$ and $c$ as follows. Write

$$K_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & \beta & \delta \end{pmatrix}.$$

Then we see that, according to our rules for matrix representations given in Section 0.9, the tensor $h^{kl}K_{ik}K_{lj}$ has the matrix representation

$$\begin{aligned} h^{kl}K_{ik}K_{lj} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & \beta & \delta \end{pmatrix} \begin{pmatrix} \frac{b^2}{a} - c & \frac{b}{a} & -1 \\ \frac{b}{a} & \frac{1}{a} & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & \beta & \delta \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ \frac{b}{a}\alpha - \beta & \frac{1}{a}\alpha & 0 \\ \frac{b}{a}\beta - \delta & \frac{1}{a}\beta & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & \beta & \delta \end{pmatrix} \\ &= \frac{1}{a}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha^2 & \alpha\beta \\ 0 & \alpha\beta & \beta^2 \end{pmatrix}, \end{aligned}$$

whence we obtain the equations

$$\begin{aligned} \partial_s\alpha &= \frac{1}{a}\alpha^2 - 2a(\partial_s\gamma)^2, \\ \partial_s\beta &= \frac{1}{a}\alpha\beta - 2\partial_s\gamma\left(b\partial_s\gamma + \partial_x\gamma\right), \\ \partial_s\delta &= \frac{1}{a}\beta^2 - \partial_s\gamma\left(2\partial_v\gamma + \left(\frac{b^2}{a} + c\right)\partial_s\gamma + 2\frac{b}{a}\partial_x\gamma\right) - \frac{1}{a}\left(\partial_x\gamma\right)^2. \end{aligned}$$

Since we have also (see (1.2.8))

$$\partial_s a = 2\alpha, \quad \partial_s b = 2\beta, \quad \partial_s c = 2\delta,$$

we have finally the following evolution equations for $a$, $b$, and $c$:

$$\partial_s^2 a = \frac{\left(\partial_s a\right)^2}{2a} - 4a\left(\partial_s\gamma\right)^2 \tag{1.2.9}$$

$$\partial_s^2 b = \frac{1}{2a}\left(\partial_s a\right)\left(\partial_s b\right) - 4\partial_s\gamma\left(b\partial_s\gamma + \partial_x\gamma\right) \tag{1.2.10}$$

$$\partial_s^2 c = \frac{\left(\partial_s b\right)^2}{2a} - 2\partial_s\gamma\left(2\partial_v\gamma + \left(\frac{b^2}{a} + c\right)\partial_s\gamma + 2\frac{b}{a}\partial_x\gamma\right) - \frac{2}{a}\left(\partial_x\gamma\right)^2. \tag{1.2.11}$$

In the first equation above, it turns out to be very convenient to consider, instead of $a$, the quantity $\ell = \sqrt{a}$, which is easily seen to have the evolution equation

$$\partial_s^2 \ell = \partial_s \frac{\partial_s a}{2\sqrt{a}} = \frac{\partial_s^2 a}{2\sqrt{a}} - \frac{(\partial_s a)^2}{4a^{\frac{3}{2}}} = -2\ell(\partial_s \gamma)^2. \tag{1.2.12}$$

We shall refer to these four equations as the Riccati equations for the metric components.

## 1.3. Wave equation and bulk region

The coordinate form of the wave equation in the gauge in Proposition 1.2.1 can be determined as follows. The wave operator may be written

$$\Box_h \gamma = h^{ij}\partial_i \partial_j \gamma - h^{ij}\Gamma^k_{ij}\partial_k \gamma,$$

where $i, j = 0, 1, 2$ and $\Gamma^k_{ij}$ are the Christoffel symbols for $h$. We recall that $h^{ij}$ has the matrix representation

$$h^{ij} = \begin{pmatrix} \frac{b^2}{a} - c & \frac{b}{a} & -1 \\ \frac{b}{a} & \frac{1}{a} & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

As already noted, from Chapter 2 we have the following matrix representations for the Christoffel symbols in our gauge:

$$\Gamma^0_{ij} = \frac{1}{2a}\begin{pmatrix} 0 & ba_{,s} - ab_{,s} & bb_{,s} - ac_{,s} \\ ba_{,s} - ab_{,s} & -(b^2 - ac)a_{,s} + ba_{,x} - a(2b_{,x} - a_{,v}) & -(b^2 - ac)b_{,s} + ba_{,v} - ac_{,x} \\ bb_{,s} - ac_{,s} & -(b^2 - ac)b_{,s} + ba_{,v} - ac_{,x} & -(b^2 - ac)c_{,s} + b(2b_{,v} - c_{,x}) - ac_{,v} \end{pmatrix}$$

$$\Gamma^1_{ij} = \frac{1}{2a}\begin{pmatrix} 0 & a_{,s} & b_{,s} \\ a_{,s} & a_{,x} - ba_{,s} & a_{,v} - bb_{,s} \\ b_{,s} & a_{,v} - bb_{,s} & 2b_{,v} - c_{,x} - bc_{,s} \end{pmatrix}$$

$$\Gamma^2_{ij} = \frac{1}{2a}\begin{pmatrix} 0 & 0 & 0 \\ 0 & aa_{,s} & ab_{,s} \\ 0 & ab_{,s} & ac_{,s} \end{pmatrix};$$

and thus we see that

$$
\begin{aligned}
h^{ij}\Gamma^0_{ij} &= \frac{1}{2a}\mathrm{Tr}\left[\begin{pmatrix} \frac{b^2}{a}-c & \frac{b}{a} & -1 \\ \frac{b}{a} & \frac{1}{a} & 0 \\ -1 & 0 & 0 \end{pmatrix}\right. \\
&\qquad \left.\cdot\begin{pmatrix} 0 & ba_{,s}-ab_{,s} & bb_{,s}-ac_{,s} \\ ba_{,s}-ab_{,s} & -(b^2-ac)a_{,s}+ba_{,x}-a(2b_{,x}-a_{,v}) & -(b^2-ac)b_{,s}+ba_{,v}-ac_{,x} \\ bb_{,s}-ac_{,s} & -(b^2-ac)b_{,s}+ba_{,v}-ac_{,x} & -(b^2-ac)c_{,s}+b(2b_{,v}-c_{,x})-ac_{,v} \end{pmatrix}\right] \\
&= \frac{1}{2a}\left[\frac{b}{a}(ba_{,s}-ab_{,s})-(bb_{,s}-ac_{,s})\right. \\
&\qquad \left.+\frac{b}{a}(ba_{,s}-ab_{,s})+\frac{1}{a}[-(b^2-ac)a_{,s}+ba_{,x}-a(2b_{,x}-a_{,v})]-(bb_{,s}-ac_{,s})\right] \\
&= \frac{1}{2}\left[2\frac{b}{a^2}(ba_{,s}-ab_{,s})-\frac{2}{a}(bb_{,s}-ac_{,s})+\frac{1}{a^2}[(ac-b^2)a_{,s}+ba_{,x}-2ab_{,x}+aa_{,v}]\right] \\
&= \frac{1}{2}\left[-4\frac{bb_{,s}}{a}+2c_{,s}+\frac{1}{a^2}[(ac+b^2)a_{,s}+ba_{,x}-2ab_{,x}+aa_{,v}]\right] \\
&= -2\frac{bb_{,s}}{a}+c_{,s}-\frac{b_{,x}}{a}+\frac{a_{,v}}{2a}+\frac{ba_{,x}}{2a^2}+\frac{1}{2}\left(\frac{b^2}{a}+c\right)\frac{a_{,s}}{a}, \\
h^{ij}\Gamma^1_{ij} &= \frac{1}{2a}\mathrm{Tr}\left[\begin{pmatrix} \frac{b^2}{a}-c & \frac{b}{a} & -1 \\ \frac{b}{a} & \frac{1}{a} & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & a_{,s} & b_{,s} \\ a_{,s} & a_{,x}-ba_{,s} & a_{,v}-bb_{,s} \\ b_{,s} & a_{,v}-bb_{,s} & 2b_{,v}-c_{,x}-bc_{,s} \end{pmatrix}\right] \\
&= \frac{1}{2a}\left[\frac{b}{a}a_{,s}-b_{,s}+\frac{b}{a}a_{,s}+\frac{1}{a}(a_{,x}-ba_{,s})-b_{,s}\right] = \frac{b}{2a^2}a_{,s}-\frac{b_{,s}}{a}+\frac{a_{,x}}{2a^2}, \\
h^{ij}\Gamma^2_{ij} &= \frac{1}{2a}\mathrm{Tr}\left[\begin{pmatrix} \frac{b^2}{a}-c & \frac{b}{a} & -1 \\ \frac{b}{a} & \frac{1}{a} & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & aa_{,s} & ab_{,s} \\ 0 & ab_{,s} & ac_{,s} \end{pmatrix}\right] \\
&= \frac{a_{,s}}{2a},
\end{aligned}
$$

so that the wave equation takes the form

$$
\begin{aligned}
&\left[\left(\frac{b^2}{a}-c\right)\partial_s^2+2\frac{b}{a}\partial_s\partial_x-2\partial_s\partial_v+\frac{1}{a}\partial_x^2-\frac{1}{2}\left(\left(\frac{b^2}{a}+c\right)\frac{\partial_s a}{a}-4\frac{b}{a}\partial_s b+\frac{b}{a^2}\partial_x a+2\partial_s c-\frac{2}{a}\partial_x b+\frac{\partial_v a}{a}\right)\partial_s\right. \\
&\qquad\left.-\frac{1}{2}\left(\frac{b}{a^2}\partial_s a-\frac{2}{a}\partial_s b+\frac{\partial_x a}{a^2}\right)\partial_x-\frac{1}{2}\frac{\partial_s a}{a}\partial_v\right]\gamma=0. \qquad (1.3.1)
\end{aligned}
$$

We are thus led to consider the system

$$\partial_s^2 a = \frac{(\partial_s a)^2}{2a}-4a\left(\partial_s\gamma\right)^2 \qquad (1.2.9)$$

$$\partial_s^2 b = \frac{1}{2a}\left(\partial_s a\right)\left(\partial_s b\right)-4\partial_s\gamma\left(b\partial_s\gamma+\partial_x\gamma\right) \qquad (1.2.10)$$

$$\partial_s^2 c = \frac{(\partial_s b)^2}{2a}-2\partial_s\gamma\left(2\partial_v\gamma+\left(\frac{b^2}{a}+c\right)\partial_s\gamma+2\frac{b}{a}\partial_x\gamma\right)-\frac{2}{a}\left(\partial_x\gamma\right)^2 \qquad (1.2.11)$$


$$
\begin{aligned}
&\left[\left(\frac{b^2}{a}-c\right)\partial_s^2+2\frac{b}{a}\partial_s\partial_x-2\partial_s\partial_v+\frac{1}{a}\partial_x^2-\frac{1}{2}\left(\left(\frac{b^2}{a}+c\right)\frac{\partial_s a}{a}-4\frac{b}{a}\partial_s b+\frac{b}{a^2}\partial_x a+2\partial_s c-\frac{2}{a}\partial_x b+\frac{\partial_v a}{a}\right)\partial_s\right. \\
&\qquad\left.-\frac{1}{2}\left(\frac{b}{a^2}\partial_s a-\frac{2}{a}\partial_s b+\frac{\partial_x a}{a^2}\right)\partial_x-\frac{1}{2}\frac{\partial_s a}{a}\partial_v\right]\gamma=0. \qquad (1.3.1)
\end{aligned}
$$

As initial data for (1.2.9 – 1.2.11) (assuming $\gamma$ given, and considering them as standalone ordinary differential equations), it is clearly sufficient to specify $\partial_s^\ell a$, $\partial_s^\ell b$, and $\partial_s^\ell c$ on $s = 0$. Similarly, for (1.3.1) (assuming $a$, $b$, and $c$ given, and considering it as a standalone wave equation), it is sufficient to specify $\gamma$ on the two null hypersurfaces $s = 0$ and $v = 0$ (the functions specified must – see Rendall [11], Section 4 – satisfy the *consistency conditions* that

$$\begin{aligned} \lim_{v\to 0^+} \left.\frac{\partial^\ell \gamma}{\partial s^\ell}\right|_{s=0} &= \lim_{s\to 0^+} \left.\frac{\partial^\ell \gamma}{\partial s^\ell}\right|_{v=0}, \\ \lim_{v\to 0^+} \left.\frac{\partial^\ell \gamma}{\partial v^\ell}\right|_{s=0} &= \lim_{s\to 0^+} \left.\frac{\partial^\ell \gamma}{\partial v^\ell}\right|_{v=0}, \end{aligned} \tag{1.3.2}$$

where the transverse derivatives may be computed from the wave equation as explained in Chapter 5 below).

To sum up, we have shown that the equations (1.1.1) imply the Riccati equations (1.2.9 – 1.2.11) and the wave equation (1.3.1). The converse is however false: the equations $R_{ij} = 2\partial_i\gamma\partial_j\gamma$ give rise, in addition to the Riccati equations (1.2.9 – 1.2.11), to three constraint equations relating the quantities $a$, $b$, $c$ and $\gamma$. In the following chapter we shall derive these equations, and show (see Proposition 2.3.2) that the Bianchi identities imply that they are preserved by the evolution inherent in the system (1.2.9 – 1.2.11), (1.3.1), in that if the constraint equations (see Corollary 2.4.1)

$$\begin{aligned} \frac{\partial^2 a}{\partial v^2} &= \frac{(\partial_v a)^2}{2a} - 4a\,(\partial_v\gamma)^2 \\ \frac{\partial}{\partial v}\left(a^{1/2}\frac{\partial b}{\partial s}\right) &= 4a^{1/2}\partial_x\gamma\partial_v\gamma \\ 2a\frac{\partial^2 a}{\partial v\partial s} &= 2a\frac{\partial^2 b}{\partial x\partial s} - \frac{\partial b}{\partial s}\frac{\partial a}{\partial x} + \frac{\partial a}{\partial v}\partial a\partial s + a\left(\frac{\partial b}{\partial s}\right)^2 + 4a\,(\partial_x\gamma)^2 \end{aligned}$$

hold on $\Sigma_0^0$, and the system (1.2.9 – 1.2.11), (1.3.1) holds on $\Gamma_0$, then the constraints also hold on $\Gamma_0$. Finally, we shall show (again, see Corollary 2.4.1) that in this case the original Einstein equations (1.1.1) hold on $\Gamma_0$.

## 2.1. Introduction

It is well known that the Einstein equations $R_{ij} = 0$ – or, in our case, $R_{ij} = 2\partial_i\gamma\partial_j\gamma$ – represent constraints as well as evolution equations. In this chapter we shall, by an explicit computation of the Ricci tensor and a comparison with the Riccati equations derived in Chapter 1, determine the constraint equations as restricted to the initial hypersurface $s = 0$, and show, by explicit computation and an application of the Bianchi identities, that they are preserved by evolution under the Riccati and wave equations.

## 2.2. Ricci tensor

While there might be a quicker derivation of our final results by using a geometric decomposition of the Ricci tensor for $h$, certainly the most straightforward method is to simply calculate it directly. We do this now. We first find the Christoffel symbols. We recall that

$$h^{ij} = \begin{pmatrix} \frac{b^2}{a} - c & \frac{b}{a} & -1 \\ \frac{b}{a} & \frac{1}{a} & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

so we may compute, recalling our convention (see 0.9) that in the matrix representation of a rank-2 tensor the first index (left to right) will denote the row while the second will denote the column and using $\bullet$ to indicate the appropriate cross-diagonal element in a symmetric matrix,

$$\Gamma^i_{jk} = \frac{1}{2}h^{i\ell}\left(h_{\ell j,k} + h_{\ell k,j} - h_{jk,\ell}\right)$$

$$h_{0k,j} = h_{0j,k} = 0, \quad h_{jk,0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_{,s} & b_{,s} \\ 0 & b_{,s} & c_{,s} \end{pmatrix}$$

$$\Gamma^2_{jk} = -\frac{1}{2}\left(h_{0j,k} + h_{0k,j} - h_{jk,0}\right) = \frac{1}{2}h_{jk,0} = \frac{1}{2a}\begin{pmatrix} 0 & 0 & 0 \\ 0 & aa_{,s} & ab_{,s} \\ 0 & ab_{,s} & ac_{,s} \end{pmatrix}$$

$$h_{1j,k} = \begin{pmatrix} 0 & 0 & 0 \\ a_{,s} & a_{,x} & a_{,v} \\ b_{,s} & b_{,x} & b_{,v} \end{pmatrix}, \quad h_{1k,j} = \begin{pmatrix} 0 & a_{,s} & b_{,s} \\ 0 & a_{,x} & b_{,x} \\ 0 & a_{,v} & b_{,v} \end{pmatrix}, \quad h_{jk,1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_{,x} & b_{,x} \\ 0 & b_{,x} & c_{,x} \end{pmatrix}$$

$$h_{1j,k} + h_{1k,j} - h_{jk,1} = \begin{pmatrix} 0 & a_{,s} & b_{,s} \\ a_{,s} & a_{,x} & a_{,v} \\ b_{,s} & a_{,v} & 2b_{,v} - c_{,x} \end{pmatrix}$$

$$\Gamma^1_{jk} = \frac{b}{2a}\left(h_{0j,k} + h_{0k,j} - h_{jk,0}\right) + \frac{1}{2a}\left(h_{1j,k} + h_{1k,j} - h_{jk,1}\right)$$

$$= \begin{pmatrix} 0 & \frac{a_{,s}}{2a} & \frac{b_{,s}}{2a} \\ \frac{a_{,s}}{2a} & -\frac{ba_{,s}}{2a} + \frac{a_{,x}}{2a} & -\frac{bb_{,s}}{2a} + \frac{a_{,v}}{2a} \\ \frac{b_{,s}}{2a} & -\frac{bb_{,s}}{2a} + \frac{a_{,v}}{2a} & -\frac{bc_{,s}}{2a} + \frac{2b_{,v} - c_{,x}}{2a} \end{pmatrix} = \frac{1}{2a}\begin{pmatrix} 0 & a_{,s} & b_{,s} \\ a_{,s} & a_{,x} - ba_{,s} & a_{,v} - bb_{,s} \\ b_{,s} & a_{,v} - bb_{,s} & 2b_{,v} - c_{,x} - bc_{,s} \end{pmatrix}$$

$$h_{2j,k} = \begin{pmatrix} 0 & 0 & 0 \\ b_{,s} & b_{,x} & b_{,v} \\ c_{,s} & c_{,x} & c_{,v} \end{pmatrix}, \quad h_{2k,j} = \begin{pmatrix} 0 & b_{,s} & c_{,s} \\ 0 & b_{,x} & c_{,x} \\ 0 & b_{,v} & c_{,v} \end{pmatrix}, \quad h_{jk,2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_{,v} & b_{,v} \\ 0 & b_{,v} & c_{,v} \end{pmatrix}$$

$$h_{2j,k}+h_{2k,j}-h_{jk,2}=\begin{pmatrix}0 & b_{,s} & c_{,s}\\ b_{,s} & 2b_{,x}-a_{,v} & c_{,x}\\ c_{,s} & c_{,x} & c_{,v}\end{pmatrix}$$

$$\Gamma^0_{jk}=\frac{1}{2}\left(\frac{b^2}{a}-c\right)(-h_{jk,0})+\frac{b}{2a}\left(h_{1j,k}+h_{1k,j}-h_{jk,1}\right)-\frac{1}{2}\left(h_{2j,k}+h_{2k,j}-h_{jk,2}\right)$$

$$=\frac{1}{2a}\begin{pmatrix}0 & ba_{,s}-ab_{,s} & bb_{,s}-ac_{,s}\\ ba_{,s}-ab_{,s} & -(b^2-ac)a_{,s}+ba_{,x}-a(2b_{,x}-a_{,v}) & -(b^2-ac)b_{,s}+ba_{,v}-ac_{,x}\\ bb_{,s}-ac_{,s} & -(b^2-ac)b_{,s}+ba_{,v}-ac_{,x} & -(b^2-ac)c_{,s}+b(2b_{,v}-c_{,x})-ac_{,v}\end{pmatrix}$$

$$\Gamma^i_{ij}=\frac{1}{2}\partial_j\log a,\quad \Gamma^i_{ij,k}=\frac{1}{2}\left(\frac{a_{,kj}-a_{,k}a_{,j}}{a^2}\right)=\frac{1}{2a}\begin{pmatrix}a_{,ss}-{a_{,s}}^2 & a_{,sx}-a_{,s}a_{,x} & a_{,sv}-a_{,s}a_{,v}\\ a_{,sx}-a_{,s}a_{,x} & a_{,xx}-{a_{,x}}^2 & a_{,xv}-a_{,x}a_{,v}\\ a_{,sv}-a_{,s}a_{,v} & a_{,xv}-a_{,x}a_{,v} & a_{,vv}-{a_{,v}}^2\end{pmatrix}$$

$$\Gamma^k_{j0}=\frac{1}{2a}\begin{pmatrix}0 & 0 & 0\\ ba_{,s}-ab_{,s} & a_{,s} & 0\\ bb_{,s}-ac_{,s} & b_{,s} & 0\end{pmatrix},\quad \Gamma^k_{j1}=\frac{1}{2a}\begin{pmatrix}ba_{,s}-ab_{,s} & a_{,s} & 0\\ -(b^2-ac)a_{,s}+ba_{,x}-a(2b_{,x}-a_{,v}) & a_{,x}-ba_{,s} & aa_{,s}\\ -(b^2-ac)b_{,s}+ba_{,v}-ac_{,x} & a_{,v}-bb_{,s} & ab_{,s}\end{pmatrix},$$

$$\Gamma^k_{j2}=\frac{1}{2a}\begin{pmatrix}bb_{,s}-ac_{,s} & b_{,s} & 0\\ -(b^2-ac)b_{,s}+ba_{,v}-ac_{,x} & a_{,v}-bb_{,s} & ab_{,s}\\ -(b^2-ac)c_{,s}+b(2b_{,v}-c_{,x})-ac_{,v} & 2b_{,v}-c_{,x}+bc_{,s} & ac_{,s}\end{pmatrix}$$

$$\Gamma^i_{i\ell}\Gamma^\ell_{jk}=\frac{a_{,s}}{4a^2}\begin{pmatrix}0 & ba_{,s}-ab_{,s} & bb_{,s}-ac_{,s}\\ \bullet & -(b^2-ac)a_{,s}+ba_{,x}-a(2b_{,x}-a_{,v}) & -(b^2-ac)b_{,s}+ba_{,v}-ac_{,x}\\ \bullet & \bullet & -(b^2-ac)c_{,s}+b(2b_{,v}-c_{,x})-ac_{,v}\end{pmatrix}$$
$$+\frac{a_{,x}}{4a^2}\begin{pmatrix}0 & a_{,s} & b_{,s}\\ a_{,s} & a_{,x}-ba_{,s} & a_{,v}-bb_{,s}\\ b_{,s} & a_{,v}-bb_{,s} & 2b_{,v}-c_{,x}-bc_{,s}\end{pmatrix}+\frac{a_{,v}}{4a}\begin{pmatrix}0 & 0 & 0\\ 0 & a_{,s} & b_{,s}\\ 0 & b_{,s} & c_{,s}\end{pmatrix}$$

$$=\frac{1}{4a^2}\begin{pmatrix}0 & b{a_{,s}}^2+a_{,x}a_{,s}-aa_{,s}b_{,s} & bb_{,s}a_{,s}-aa_{,s}c_{,s}+a_{,x}b_{,s}\\ \bullet & \begin{matrix}ac{a_{,s}}^2-b^2{a_{,s}}^2-\\ aa_{,s}(2b_{,x}-2a_{,v})+{a_{,x}}^2\end{matrix} & \begin{matrix}caa_{,s}b_{,s}-b^2a_{,s}b_{,s}+ba_{,v}a_{,s}-\\ aa_{,s}c_{,x}+a_{,x}(a_{,v}-bb_{,s})+aa_{,v}b_{,s}\end{matrix}\\ \bullet & \bullet & \begin{matrix}aca_{,s}c_{,s}-b^2a_{,s}c_{,s}+ba_{,s}(2b_{,v}-c_{,x})-\\ aa_{,s}c_{,v}+a_{,x}(2b_{,v}-c_{,x}-bc_{,s})+aa_{,v}c_{,s}\end{matrix}\end{pmatrix}$$

$$\Gamma^m_{ij}\Gamma^i_{mk} = \mathrm{tr}\Gamma^{\cdot}_{\cdot j}\Gamma^{\cdot}_{\cdot k}$$

$$= \frac{1}{4a^2}\begin{pmatrix} {a_{,s}}^2 & \begin{array}{c} a_{,s}(ba_{,s}-ab_{,s})+ \\ (a_{,x}-ba_{,s})a_{,s}+ \\ aa_{,s}b_{,s} \end{array} & \begin{array}{c} b_{,s}(ba_{,s}-ab_{,s})+ \\ (a_{,v}-bb_{,s})a_{,s}+ \\ a{b_{,s}}^2 \end{array} \\ \bullet & \begin{array}{c} (ba_{,s}-ab_{,s})^2+ \\ 2a_{,s}\Big[-(b^2-ac)a_{,s}+ba_{,x} \\ \qquad -a(2b_{,x}-a_{,v})\Big]+ \\ (a_{,x}-ba_{,s})^2+2aa_{,s}(a_{,v}-bb_{,s})+a^2{b_{,s}}^2 \end{array} & \begin{array}{c} (ba_{,s}-ab_{,s})(bb_{,s}-ac_{,s})+ \\ a_{,s}\left(-(b^2-ac)b_{,s}+ba_{,v}-ac_{,x}\right)+ \\ \Big[-(b^2-ac)a_{,s}+ba_{,x} \\ \qquad -a(2b_{,x}-a_{,v})\Big]b_{,s}+ \\ (a_{,x}-ba_{,s})(a_{,v}-bb_{,s})+ \\ aa_{,s}(2b_{,v}-c_{,x}-bc_{,s})+ \\ a(a_{,v}-bb_{,s})b_{,s}+a^2b_{,s}c_{,s} \end{array} \\ \bullet & \bullet & \begin{array}{c} (bb_{,s}-ac_{,s})^2+ \\ 2b_{,s}\left(-(b^2-ac)b_{,s}+ba_{,v}-ac_{,x}\right)+ \\ (a_{,v}-bb_{,s})^2+ \\ 4ab_{,s}(2b_{,v}-c_{,x}-bc_{,s})+ \\ a^2{c_{,s}}^2 \end{array} \end{pmatrix}$$

$$= \frac{1}{2a}\begin{pmatrix} \frac{{a_{,s}}^2}{2a} & \frac{a_{,x}a_{,s}}{2a} & \frac{a_{,v}a_{,s}}{2a} \\ \bullet & \begin{array}{c} c{a_{,s}}^2-2a_{,s}b_{,x}+2a_{,s}a_{,v}+ \\ \frac{{a_{,x}}^2}{2a}-2ba_{,s}b_{,s}+a{b_{,s}}^2 \end{array} & \begin{array}{c} -ba_{,s}c_{,s}-b{b_{,s}}^2+ab_{,s}c_{,s}+ \\ ca_{,s}b_{,s}-a_{,s}c_{,x}-b_{,x}b_{,s}+ \\ a_{,v}b_{,s}+\frac{a_{,x}a_{,v}}{2a}+a_{,s}b_{,v} \end{array} \\ \bullet & \bullet & \begin{array}{c} a{c_{,s}}^2-2bb_{,s}c_{,s}+\frac{{a_{,v}}^2}{2a}+ \\ c{b_{,s}}^2-2b_{,s}c_{,x}+2b_{,s}b_{,v} \end{array} \end{pmatrix}$$

$$\Gamma^i_{jk,i}-\Gamma^\ell_{\ell j,k} = \Gamma^0_{jk,0}+\Gamma^1_{jk,1}+\Gamma^2_{jk,2}-\Gamma^\ell_{\ell j,k}$$

$$= \frac{1}{2a}\begin{pmatrix} 0 & \begin{array}{c} ba_{,ss}-ab_{,ss}- \\ \frac{b{a_{,s}}^2}{a}+b_{,s}a_{,s} \end{array} & \begin{array}{c} bb_{,ss}+{b_{,s}}^2-a_{,s}c_{,s} \\ -ac_{,ss}-\frac{bb_{,s}a_{,s}}{a}+a_{,s}c_{,s} \end{array} \\ \bullet & \begin{array}{c} -(2bb_{,s}-a_{,s}c-ac_{,s})a_{,s}-(b^2-ac)a_{,ss}+ \\ b_{,s}a_{,x}+ba_{,xs}-a_{,s}(2b_{,x}-a_{,v})- \\ a(2b_{,xs}-a_{,vs})+(b^2-ac)\frac{{a_{,s}}^2}{a}- \\ \frac{ba_{,x}a_{,s}}{a}+a_{,s}(2b_{,x}-a_{,v}) \end{array} & \begin{array}{c} -(2bb_{,s}-a_{,s}c-ac_{,s})b_{,s}- \\ (b^2-ac)b_{,ss}+b_{,s}a_{,v}+ba_{,vs}- \\ a_{,s}c_{,x}-ac_{,xs}+(b^2-ac)\frac{a_{,s}b_{,s}}{a} \\ -\frac{ba_{,v}a_{,s}}{a}+a_{,s}c_{,x} \end{array} \\ \bullet & \bullet & \begin{array}{c} -(2bb_{,s}-a_{,s}c-ac_{,s})c_{,s}- \\ (b^2-ac)c_{,ss}+b_{,s}(2b_{,v}-c_{,x})+ \\ b(2b_{,vs}-c_{,xs})-a_{,s}c_{,v}-ac_{,vs} \\ +(b^2-ac)c_{,s}\frac{a_{,s}}{a}-b\frac{a_{,s}}{a}(2b_{,v}-c_{,x})+ \\ a_{,s}c_{,v} \end{array} \end{pmatrix}$$

$$+\frac{1}{2a}\begin{pmatrix} 0 & a_{,xs}-\frac{a_{,s}a_{,x}}{a} & b_{,xs}-\frac{b_{,s}a_{,x}}{a} \\ \bullet & \begin{array}{c} a_{,xx}-b_{,x}a_{,s}-ba_{,xs}- \\ (a_{,x}-ba_{,s})\frac{a_{,x}}{a} \end{array} & \begin{array}{c} a_{,vx}-b_{,x}b_{,s}-bb_{,xs}- \\ (a_{,v}-bb_{,s})\frac{a_{,x}}{a} \end{array} \\ \bullet & \bullet & \begin{array}{c} 2b_{,vx}-c_{,xx}-b_{,x}c_{,s}-bc_{,xs}- \\ (2b_{,v}-c_{,x}-bc_{,s})\frac{a_{,x}}{a} \end{array} \end{pmatrix}$$

$$+\frac{1}{2a}\begin{pmatrix} 0 & 0 & 0 \\ 0 & aa_{,vs} & ab_{,vs} \\ 0 & ab_{,vs} & ac_{,vs} \end{pmatrix} - \frac{1}{2a}\begin{pmatrix} a_{,ss}-\frac{{a_{,s}}^2}{a} & a_{,xs}-\frac{a_{,x}a_{,s}}{a} & a_{,vs}-\frac{a_{,v}}{a_{,s}}a \\ \bullet & a_{,xx}-\frac{{a_{,x}}^2}{a} & a_{,vx}-\frac{a_{,v}a_{,x}}{a} \\ \bullet & \bullet & a_{,vv}-\frac{{a_{,v}}^2}{a} \end{pmatrix}$$

$$\Gamma^i_{jk,i} - \Gamma^\ell_{\ell j,k} = \frac{1}{2a}\begin{pmatrix} -a_{,ss} + \frac{{a_{,s}}^2}{a} & \begin{gathered} ba_{,ss} - ab_{,ss} - \\ \frac{b{a_{,s}}^2}{a} + b_{,s}a_{,s} \end{gathered} & \begin{gathered} bb_{,ss} + {b_{,s}}^2 - ac_{,ss} - \\ \frac{bb_{,s}a_{,s}}{a} + b_{,xs} - \\ \frac{b_{,s}a_{,x}}{a} - a_{,vs} + \frac{a_{,s}a_{,v}}{a} \end{gathered} \\ \bullet & \begin{gathered} -(2bb_{,s} - ac_{,s})a_{,s} - \\ (b^2 - ac)a_{,ss} + b_{,s}a_{,x} - \\ b_{,s}a_{,x} - a(2b_{,ss} - 2a_{,vs}) + \\ b^2\frac{{a_{,s}}^2}{a} \end{gathered} & \begin{gathered} -(2bb_{,s} - ac_{,s})b_{,s} - (b^2 - ac)b_{,ss} + \\ b_{,s}a_{,v} + ba_{,vs} + b^2\frac{a_{,s}b_{,s}}{a} - \\ \frac{ba_{,v}a_{,s}}{a} - b_{,x}b_{,s} - bb_{,xs} + \\ \frac{b}{a}b_{,s}a_{,x} + ab_{,vs} - ac_{,xs} \end{gathered} \\ \bullet & \bullet & \begin{gathered} -(2bb_{,s} - ac_{,s})c_{,s} - (b^2 - ac)c_{,ss} + \\ b_{,s}(2b_{,v} - c_{,x}) + b(2b_{,vs} - 2c_{,xs}) + \\ b^2\frac{c_{,s}a_{,s}}{a} - b\frac{a_{,s}}{a}(2b_{,v} - c_{,x}) + \\ 2b_{,vx} - c_{,xx} - b_{,x}c_{,s} - \\ (2b_{,v} - c_{,x} - bc_{,s})\frac{a_{,x}}{a} - a_{,vv} + \frac{{a_{,v}}^2}{a} \end{gathered} \end{pmatrix}$$

$$R_{jk} = \Gamma^i_{jk,i} - \Gamma^\ell_{\ell j,k} + \Gamma^\ell_{\ell m}\Gamma^m_{jk} - \Gamma^m_{ij}\Gamma^i_{mk}$$

$$= \frac{1}{2a}\begin{pmatrix} \begin{gathered} -a_{,ss} + \\ \frac{{a_{,s}}^2}{a} - \frac{{a_{,s}}^2}{2a} \end{gathered} & \begin{gathered} ba_{,ss} - ab_{,ss} - \frac{b{a_{,s}}^2}{a} + \\ b_{,s}a_{,s} - \frac{a_{,s}a_{,x}}{2a} + \frac{b{a_{,s}}^2}{2a} + \\ \frac{a_{,x}a_{,s}}{2a} - \frac{1}{2}a_{,s}b_{,s} \end{gathered} & \begin{gathered} bb_{,ss} + {b_{,s}}^2 - ac_{,ss} - \\ \frac{bb_{,s}a_{,s}}{a} + b_{,xs} - \frac{b_{,s}a_{,x}}{a} - \\ a_{,vs} + \frac{a_{,s}a_{,v}}{a} - \frac{a_{,s}a_{,v}}{2a} + \\ \frac{bb_{,s}a_{,s}}{2a} - \frac{1}{2}a_{,s}c_{,s} + \frac{a_{,x}b_{,s}}{2a} \end{gathered} \\ \bullet & \begin{gathered} -(2bb_{,s} - ac_{,s})a_{,s} - (b^2 - ac)a_{,ss} + \\ b_{,s}a_{,x} - b_{,s}a_{,x} - a(2b_{,ss} - 2a_{,vs}) + \\ b^2\frac{{a_{,s}}^2}{a} - \Big[c{a_{,s}}^2 - 2a_{,s}b_{,x} + 2a_{,s}a_{,v} + \\ \frac{{a_{,x}}^2}{2a} - 2ba_{,s}b_{,s} + a{b_{,s}}^2\Big] + \frac{1}{2}c{a_{,s}}^2 - \\ \frac{b^2{a_{,s}}^2}{2a} - \frac{1}{2}a_{,s}(2b_{,x} - 2a_{,v}) \end{gathered} & \begin{gathered} -(2bb_{,s} - ac_{,s})b_{,s} - (b^2 - ac)b_{,ss} + \\ b_{,s}a_{,v} + ba_{,vs} + b^2\frac{a_{,s}b_{,s}}{a} - \\ \frac{ba_{,v}a_{,s}}{a} - b_{,x}b_{,s} - bb_{,xs} + \\ \frac{b}{a}b_{,s}a_{,x} + ab_{,vs} - ac_{,xs} \\ -\Big[-ba_{,s}c_{,s} - b{b_{,s}}^2 + ab_{,s}c_{,s} + \\ ca_{,s}b_{,s} - a_{,s}c_{,x} - b_{,x}b_{,s} + \\ a_{,v}b_{,s} + \frac{a_{,x}a_{,v}}{2a} + a_{,s}b_{,v}\Big] \\ +\frac{1}{2a}\Big[caa_{,s}b_{,s} - b^2a_{,s}b_{,s} + ba_{,v}a_{,s} - \\ aa_{,s}c_{,x} + a_{,x}(a_{,v} - bb_{,s}) + aa_{,v}b_{,s}\Big] \end{gathered} \\ \bullet & \bullet & \begin{gathered} -(2bb_{,s} - ac_{,s})c_{,s} - (b^2 - ac)c_{,ss} + \\ b_{,s}(2b_{,v} - c_{,x}) + b(2b_{,vs} - 2c_{,xs}) + \\ b^2\frac{c_{,s}a_{,s}}{a} - b\frac{a_{,s}}{a}(2b_{,v} - c_{,x}) + \\ 2b_{,vx} - c_{,xx} - b_{,x}c_{,s} - \\ (2b_{,v} - c_{,x} - bc_{,s})\frac{a_{,x}}{a} - a_{,vv} + \\ \frac{{a_{,v}}^2}{a} - \Big[a{c_{,s}}^2 - 2bb_{,s}c_{,s} + \frac{{a_{,v}}^2}{2a} + \\ c{b_{,s}}^2 - 2b_{,s}c_{,x} + 2b_{,s}b_{,v}\Big] + \frac{1}{2a}\Big[aca_{,s}c_{,s} - \\ b^2a_{,s}c_{,s} + ba_{,s}(2b_{,v} - c_{,x}) - \\ aa_{,s}c_{,v} + a_{,x}(2b_{,v} - c_{,x} - bc_{,s}) + aa_{,v}c_{,s}\Big] \end{gathered} \end{pmatrix}$$

$$R_{jk} = \frac{1}{2a}\begin{pmatrix} -a_{,ss} + \frac{a_{,s}{}^2}{2a} & \begin{matrix} ba_{,ss} - ab_{,ss} - \\ \frac{1}{2}\left(b\frac{a_{,s}{}^2}{a} - b_{,s}a_{,s}\right)\end{matrix} & \begin{matrix} bb_{,ss} - ac_{,ss} + b_{,xs} - a_{,vs} + \\ b_{,s}{}^2 - \frac{bb_{,s}a_{,s}}{2a} - \frac{b_{,s}a_{,x}}{2a} + \\ \frac{a_{,s}a_{,v}}{2a} - \frac{1}{2}a_{,s}c_{,s}\end{matrix} \\ \bullet & \begin{matrix} ac_{,s}a_{,s} - (b^2 - ac)a_{,ss} + b_{,s}a_{,x} - \\ a(2b_{,xs} - 2a_{,vs}) + \frac{b^2 a_{,s}{}^2}{2a} - \\ \frac{1}{2}ca_{,s}{}^2 - a_{,s}a_{,v} - ab_{,s}{}^2\end{matrix} & \begin{matrix} -bb_{,s}{}^2 - (b^2 - ac)b_{,ss} + b(a_{,vs} - b_{,xs}) + \\ \frac{b^2 a_{,s}b_{,s}}{2a} - \frac{ba_{,v}a_{,s}}{2a} + \frac{bb_{,s}a_{,x}}{2a} + \\ a(b_{,vs} - c_{,xs}) + ba_{,v}c_{,s} - \frac{1}{2}ca_{,s}b_{,s} + \\ \frac{1}{2}a_{,s}c_{,x} + \frac{1}{2}a_{,v}b_{,s}\end{matrix} \\ \bullet & \bullet & \begin{matrix} -(b^2 - ac)c_{,ss} + b_{,s}c_{,x} + 2b(b_{,vs} - c_{,xs}) + \\ \frac{b^2 a_{,s}c_{,s}}{2a} - \frac{ba_{,s}}{2a}(2b_{,v} - c_{,x}) + \\ 2b_{,vx} - c_{,xx} - a_{,vv} - b_{,x}c_{,s} - \\ \frac{a_{,x}}{2a}(2b_{,v} - c_{,x} + bc_{,x}) + \frac{a_{,v}{}^2}{2a} - \\ cb_{,s}{}^2 + \frac{1}{2}a_{,s}cc_{,s} - \frac{1}{2}a_{,s}c_{,v} + \frac{1}{2}a_{,v}c_{,s}\end{matrix} \end{pmatrix}. \tag{2.2.1}$$

**2.3. Equations of motion and constraints**

From the expression for the Ricci tensor in (2.2.1) we see that the equations $R_{00} = 2\gamma_{,s}{}^2$, $R_{01} = 2\gamma_{,s}\gamma_{,x}$, $R_{02} = 2\gamma_{,s}\gamma_{,v}$, and $R_{11} = 2\gamma_{,x}{}^2$ give, respectively,

$$\begin{aligned} a_{,ss} &= \frac{a_{,s}{}^2}{2a} - 4a\gamma_{,s}{}^2, \\ ba_{,ss} - ab_{,ss} - \frac{1}{2}\left(b\frac{a_{,s}{}^2}{a} - b_{,s}a_{,s}\right) &= 4a\gamma_{,s}\gamma_{,x}, \\ c_{,ss} &= \frac{1}{a}\left(bb_{,ss} + b_{,xs} - a_{,vs} + b_{,s}{}^2 - \frac{bb_{,s}a_{,s}}{2a} - \frac{b_{,s}a_{,x}}{2a} + \frac{a_{,s}a_{,v}}{2a} - \frac{1}{2}a_{,s}c_{,s}\right) - 4\gamma_{,s}\gamma_{,v}, \\ b_{,xs} - a_{,vs} &= \frac{1}{2a}\left(ac_{,s}a_{,s} - (b^2 - ac)a_{,ss} + b_{,s}a_{,x} + \frac{b^2 a_{,s}{}^2}{2a} - \frac{1}{2}ca_{,s}{}^2 - a_{,s}a_{,v} - ab_{,s}{}^2\right) - 2\gamma_{,x}{}^2. \end{aligned}$$

From these equations we obtain the following proposition. We recall for reference the Riccati equations (1.2.9 – 1.2.11):

$$a_{,ss} = \frac{a_{,s}{}^2}{2a} - 4a\gamma_{,s}{}^2 \qquad b_{,ss} = \frac{1}{2a}a_{,s}b_{,s} - 4\gamma_{,s}\left(b\gamma_{,s} + \gamma_{,x}\right) \tag{2.3.1) – (2.3.2}$$

$$c_{,ss} = \frac{b_{,s}{}^2}{2a} - 2\gamma_{,s}\left(2\gamma_{,v} + \left(\frac{b^2}{a} + c\right)\gamma_{,s} + 2\frac{b}{a}\gamma_{,x}\right) - \frac{2}{a}\gamma_{,x}{}^2. \tag{2.3.3}$$

2.3.1. Proposition. We have the following equivalences:

(i) Equation (2.3.1) holds if and only if $R_{00} = 2\gamma_{,s}{}^2$.

(ii) Equations (2.3.1) and (2.3.2) hold if and only if $R_{00} = 2\gamma_{,s}{}^2$ and $R_{01} = 2\gamma_{,s}\gamma_{,x}$

(iii) Equations (2.3.1) – (2.3.3) hold if and only if $R_{00} = 2\gamma_{,s}{}^2$, $R_{01} = 2\gamma_{,s}\gamma_{,x}$, and $-2R_{02} + \frac{1}{a}R_{11} = -2(2\gamma_{,s}\gamma_{,v}) + \frac{2}{a}\gamma_{,x}{}^2$

Proof. (i) is clear. $R_{00} = 2\gamma_{,s}{}^2$ and $R_{01} = 2\gamma_{,s}\gamma_{,x}$ together imply

$$\begin{aligned} 4a\gamma_{,s}\gamma_{,x} &= ba_{,ss} - ab_{,ss} - \frac{1}{2}\left(\frac{ba_{,s}{}^2}{a} - b_{,s}a_{,s}\right) \\ &= \frac{ba_{,s}{}^2}{2a} - 4ab\gamma_{,s}{}^2 - ab_{,ss} - \frac{ba_{,s}{}^2}{2a} + \frac{1}{2}b_{,s}a_{,s} \end{aligned}$$

so

$$b_{,ss} = \frac{b_{,s}a_{,s}}{2a} - 4\gamma_{,s}(b\gamma_{,s} + \gamma_{,x}),$$

and the converse is also clearly true, thus establishing (ii). Furthermore, if $R_{00} = 2{\gamma_{,s}}^2$ and $R_{01} = 2\gamma_{,s}\gamma_{,x}$, then multiplying $R_{02} = 2\gamma_{,s}\gamma_{,v}$ by $-2$ and simplifying gives

$$\begin{aligned} c_{,ss} &= -4\gamma_{,s}\gamma_{,v} - \frac{a_{,x}b_{,s}}{2a^2} - \frac{b}{2a^2}a_{,s}b_{,s} + \frac{1}{a}{b_{,s}}^2 + \frac{a_{,s}a_{,v}}{2a^2} - \frac{1}{2a}a_{,s}c_{,s} + \frac{b}{a}b_{,ss} + \frac{1}{a}b_{,xs} - \frac{1}{a}a_{,vs} \\ &= -4\gamma_{,s}\gamma_{,v} - \frac{a_{,x}b_{,s}}{2a^2} - \frac{b}{2a^2}a_{,s}b_{,s} + \frac{1}{a}{b_{,s}}^2 + \frac{a_{,s}a_{,v}}{2a^2} - \frac{1}{2a}a_{,s}c_{,s} \\ &\qquad + \frac{b}{a}\left(\frac{a_{,s}b_{,s}}{2a} - 4b{\gamma_{,s}}^2 - 4\gamma_{,s}\gamma_{,x}\right) + \frac{1}{a}b_{,xs} - \frac{1}{a}a_{,vs} \\ &= -4\gamma_{,s}\left[\gamma_{,v} + \frac{b^2}{a}\gamma_{,s} + \frac{b}{a}\gamma_{,x}\right] + \frac{1}{a}b_{,xs} - \frac{a_{,x}b_{,s}}{2a^2} - \frac{1}{a}a_{,vs} + \frac{a_{,x}a_{,v}}{2a^2} + \frac{1}{a}{b_{,s}}^2 - \frac{1}{2a}a_{,s}c_{,s}, \end{aligned} \tag{2.3.4}$$

while multiplying

$$R_{11} = 2{\gamma_{,x}}^2$$

by $2a$ gives

$$\begin{aligned} 4a{\gamma_{,x}}^2 &= 2aa_{,vs} - a_{,s}a_{,v} - 2ab_{,xs} + b_{,s}a_{,x} + \frac{{a_{,s}}^2b^2}{2a} + ac_{,s}a_{,s} - \frac{1}{2}c{a_{,s}}^2 - a{b_{,s}}^2 - (b^2 - ac)\left(\frac{{a_{,s}}^2}{2a} - 4a{\gamma_{,s}}^2\right) \\ &= 2a^2\left(\frac{1}{a}a_{,vs} - \frac{a_{,s}a_{,v}}{2a^2} - \frac{1}{a}b_{,xs} + \frac{a_{,x}b_{,s}}{2a^2}\right) + ac_{,s}a_{,s} - a{b_{,s}}^2 + 4(b^2 - ac)a{\gamma_{,s}}^2 \\ &= 2a^2\left(\frac{1}{a}a_{,vs} - \frac{a_{,s}a_{,v}}{2a^2} - \frac{1}{a}b_{,xs} + \frac{a_{,x}b_{,s}}{2a^2} + \frac{c_{,s}a_{,s}}{2a} - \frac{{b_{,s}}^2}{2a} - 2\left(c - \frac{b^2}{a}\right){\gamma_{,s}}^2\right) \end{aligned}$$

so that adding $1/(2a)^2$ times this to equation (2.3.4) gives

$$\begin{aligned} c_{,ss} &= -\frac{2}{a}{\gamma_{,x}}^2 - 4\gamma_{,s}\left(\gamma_{,v} + \frac{b^2}{a}\gamma_{,s} + \frac{b}{a}\gamma_{,x} + \frac{1}{2}\left(c - \frac{b^2}{a}\right)\gamma_{,s}\right) + \frac{{b_{,s}}^2}{2a} \\ &= -\frac{2}{a}{\gamma_{,x}}^2 - 4\gamma_{,s}\left(\gamma_{,v} + \frac{b}{a}\gamma_{,x} + \frac{1}{2}\left(c + \frac{b^2}{a}\right)\gamma_{,s}\right) + \frac{{b_{,s}}^2}{2a}, \end{aligned}$$

which is the third of the Riccati equations.

Since the steps here can evidently be carried out in reverse, this establishes the proposition. QED.

This proposition can be rephrased by saying that the three Riccati equations are equivalent to the system

$$R_{00} = 2{\gamma_{,s}}^2 \tag{2.3.5}$$

$$R_{01} = 2\gamma_{,s}\gamma_{,x} \tag{2.3.6}$$

$$-2R_{02} + \frac{1}{a}R_{11} = -2(2\gamma_{,s}\gamma_{,v}) + \frac{2}{a}{\gamma_{,x}}^2. \tag{2.3.7}$$

Thus for the full equation $R_{ij} = 2\gamma_{,i}\gamma_{,j}$ to hold, it is sufficient for the equations $R_{12} = 2\gamma_{,x}\gamma_{,v}$, $R_{22} = 2{\gamma_{,v}}^2$, and any linear combination of the equations involving either $R_{02}$ or $R_{11}$ which is linearly independent of (2.3.7), to hold. We shall term any such set a system of constraint equations. We shall next show that, if the Riccati equations (2.3.5 – 2.3.7) and the wave equation (1.3.1) hold on $\Gamma_0$, then for a particular system of constraint equations, the Bianchi identities allow us to conclude that the system holds on all of $\Gamma_0$ if it holds on $\Sigma_0^0 = \{(s,x,v) \in \Gamma_0 \,|\, s=0\}$.

2.3.2. Proposition. If the system (2.3.5 – 2.3.7) and the wave equation (1.3.1) hold on $\Gamma_0$, then the equations

$$R_{11} = 2{\gamma_{,x}}^2, \qquad R_{12} = 2\gamma_{,x}\gamma_{,v}, \qquad R_{22} = 2{\gamma_{,v}}^2$$

hold on $\Gamma_0$ if they hold on $\Sigma_0^0$.

Proof. Given (2.3.7), $R_{11} = 2{\gamma_{,x}}^2$ holds at any given point if and only if $R_{02} = 2\gamma_{,s}\gamma_{,v}$ does; thus it suffices to work with this equation instead. To begin, we note that, using the Riccati equations, we may write $2aR_{20}$ as

$$-4b^2{\gamma_{,s}}^2 - 4b\gamma_{,s}\gamma_{,x} + \frac{1}{2}{b_{,s}}^2 + 2a\gamma_{,s}\left[\left(\frac{b^2}{a}+c\right)\gamma_{,s} + 2\gamma_{,v} + 2\frac{b}{a}\gamma_{,x}\right] + 2{\gamma_{,x}}^2 + b_{,xs} - a_{,vs} - \frac{b_{,s}a_{,x}}{2a} + \frac{a_{,s}a_{,v}}{2a} - \frac{1}{2}a_{,s}c_{,s},$$

so

$$
\begin{aligned}
\partial_s\left(2aR_{20}-4a\gamma_{,s}\gamma_{,v}\right) &= \partial_s\left[\frac{1}{2}{b_{,s}}^2+2{\gamma_{,s}}^2(ac-b^2)+2{\gamma_{,x}}^2+b_{,xs}-a_{,vs}-\frac{b_{,s}a_{,x}}{2a}+\frac{a_{,s}a_{,v}}{2a}-\frac{1}{2}a_{,s}c_{,s}\right]\\
&= b_{,s}b_{,ss}+4\gamma_{,s}\gamma_{,ss}(ac-b^2)+2{\gamma_{,s}}^2\left(a_{,s}c+ac_{,s}-2bb_{,s}\right)+4\gamma_{,x}\gamma_{,sx}+b_{,ssx}-a_{,ssv}\\
&\quad+\frac{b_{,s}a_{,s}a_{,x}}{2a^2}-\frac{{a_{,s}}^2a_{,v}}{2a^2}+\frac{1}{2a}\left(-b_{,ss}a_{,x}-b_{,s}a_{,xs}+a_{,ss}a_{,v}+a_{,s}a_{,vs}\right)-\frac{1}{2}a_{,ss}c_{,s}-\frac{1}{2}a_{,s}c_{,ss}\\
&= \frac{1}{2a}a_{,s}{b_{,s}}^2-4bb_{,s}{\gamma_{,s}}^2-4b_{,s}\gamma_{,s}\gamma_{,x}\\
&\quad+4\gamma_{,s}\Bigg[2b\gamma_{,sx}-2a\gamma_{,sv}+\gamma_{,xx}-\frac{1}{2}\left(a_{,s}\left(c+\frac{b^2}{a}\right)-4bb_{,s}+\frac{b}{a}a_{,x}+2ac_{,s}-2b_{,x}+a_{,v}\right)\gamma_{,s}\\
&\qquad\qquad-\frac{1}{2}\left(\frac{b}{a}a_{,s}-2b_{,s}+\frac{a_{,x}}{a}\right)\gamma_{,x}-\frac{1}{2}a_{,s}\gamma_{,v}\Bigg]\\
&\quad+2{\gamma_{,s}}^2(a_{,s}c+ac_{,s}-2bb_{,s})+4\gamma_{,x}\gamma_{,sx}\\
&\quad+\left[-\frac{a_{,x}}{2a^2}a_{,s}b_{,s}+\frac{1}{2a}(a_{,xs}b_{,s}+a_{,s}b_{,xs})-4\gamma_{,sx}(b\gamma_{,s}+\gamma_{,x})-4\gamma_{,s}(b_{,x}\gamma_{,s}+b\gamma_{,sx}+\gamma_{,xx})\right]\\
&\quad-\left[-\frac{a_{,v}}{2a^2}{a_{,s}}^2+\frac{a_{,s}a_{,vs}}{a}-4a_{,v}{\gamma_{,s}}^2-8a\gamma_{,s}\gamma_{,sv}\right]+\frac{b_{,s}a_{,s}a_{,x}}{2a^2}-\frac{a_{,v}{a_{,s}}^2}{2a^2}\\
&\quad+\frac{1}{2a}\left[-b_{,s}a_{,xs}-\left(\frac{a_{,s}b_{,s}}{2a}-4b{\gamma_{,s}}^2-4\gamma_{,s}\gamma_{,x}\right)a_{,x}+\left(\frac{{a_{,s}}^2}{2a}-4a{\gamma_{,s}}^2\right)a_{,v}+a_{,s}a_{,vs}\right]\\
&\quad-\frac{1}{2}c_{,s}\left(\frac{{a_{,s}}^2}{2a}-4a{\gamma_{,s}}^2\right)-\frac{1}{2}a_{,s}\left(\frac{{b_{,s}}^2}{2a}-2\gamma_{,s}\left(2\gamma_{,v}+\left(\frac{b^2}{a}+c\right)\gamma_{,s}+2\frac{b}{a}\gamma_{,x}\right)-\frac{2}{a}{\gamma_{,x}}^2\right)\\
&= \frac{1}{2a}a_{,s}{b_{,s}}^2+4\gamma_{,s}\left[-\frac{1}{2}\left(a_{,s}\frac{b^2}{a}+2ac_{,s}-2b_{,x}\right)\gamma_{,s}-\frac{ba_{,s}}{2a}\gamma_{,x}-\frac{1}{2}a_{,s}\gamma_{,v}\right]+2ac_{,s}{\gamma_{,s}}^2+\frac{a_{,s}b_{,xs}}{2a}\\
&\quad-4b_{,x}{\gamma_{,s}}^2-\frac{a_{,s}a_{,vs}}{a}-\frac{a_{,s}b_{,s}a_{,x}}{4a^2}+\frac{{a_{,s}}^2a_{,v}}{4a^2}+\frac{a_{,s}a_{,vs}}{2a}-\frac{c_{,s}{a_{,s}}^2}{4a}+2ac_{,s}{\gamma_{,s}}^2-\frac{a_{,s}{b_{,s}}^2}{4a}+2a_{,s}\gamma_{,s}\gamma_{,v}\\
&\quad+a_{,s}\left(\frac{b^2}{a}+c\right){\gamma_{,s}}^2+2\frac{b}{a}a_{,s}\gamma_{,s}\gamma_{,x}+\frac{a_{,s}}{a}{\gamma_{,x}}^2\\
&= \frac{1}{4a}a_{,s}{b_{,s}}^2+\frac{a_{,s}b_{,xs}}{2a}-\frac{a_{,s}a_{,vs}}{2a}-\frac{a_{,s}b_{,s}a_{,x}}{4a^2}+\frac{{a_{,s}}^2a_{,v}}{4a^2}-\frac{c_{,s}{a_{,s}}^2}{4a}+{\gamma_{,s}}^2\left[-a_{,s}\frac{b^2}{a}+a_{,s}c-2b_{,x}\right]+\frac{a_{,s}}{a}{\gamma_{,x}}^2\\
&= a_{,s}R_{20}-\frac{a_{,s}}{2a}\left(bb_{,ss}-ac_{,ss}+{b_{,s}}^2-\frac{bb_{,s}a_{,s}}{2a}-\frac{b_{,s}a_{,x}}{2a}+\frac{a_{,s}a_{,v}}{2a}-\frac{1}{2}a_{,s}c_{,s}\right)+\frac{1}{4a}a_{,s}{b_{,s}}^2-\frac{a_{,s}b_{,s}a_{,x}}{4a^2}\\
&\quad+\frac{{a_{,s}}^2a_{,v}}{4a^2}-\frac{c_{,s}{a_{,s}}^2}{4a}+{\gamma_{,s}}^2\left[-a_{,s}\frac{b^2}{a}+a_{,s}c\right]+\frac{a_{,s}}{a}{\gamma_{,x}}^2\\
&= a_{,s}R_{20}-\frac{a_{,s}}{2a}\left(-4b^2{\gamma_{,s}}^2-4b\gamma_{,s}\gamma_{,x}+\frac{1}{2}{b_{,s}}^2+2(-b^2+ac){\gamma_{,s}}^2+4a\gamma_{,s}\gamma_{,v}+4b\gamma_{,s}\gamma_{,x}+2{\gamma_{,x}}^2\right)\\
&\quad+\frac{1}{4a}a_{,s}{b_{,s}}^2+{\gamma_{,s}}^2\left[-\frac{a_{,s}}{a}(b^2-ac)\right]+\frac{a_{,s}}{a}{\gamma_{,x}}^2=a_{,s}\left(R_{20}-2\gamma_{,s}\gamma_{,v}\right).
\end{aligned}
$$

Thus if $a$, $b$, and $c$ satisfy the Riccati equations (2.3.5 – 2.3.7) and $\gamma$ satisfies the wave equation (1.3.1), and if the values of these functions on $\Sigma_0^0$ satisfy the equation $R_{20}=2\gamma_{,s}\gamma_{,v}$, then this equation will continue to hold for all $s>0$. Again, by (2.3.7), the equation $R_{11}=2{\gamma_{,x}}^2$ is then also preserved by the evolution.

That the other two equations, $R_{12}=2\gamma_{,x}\gamma_{,v}$ and $R_{22}=2{\gamma_{,v}}^2$, are likewise preserved by the evolution may be proved using the Bianchi identity and the foregoing, as follows. Now note that

$$R=g^{ij}R_{ij}=\left(\frac{b^2}{a}-c\right)R_{00}+2\frac{b}{a}R_{01}-2R_{02}+\frac{1}{a}R_{11},$$

while

$$g^{ij}\left(2\gamma_{,i}\gamma_{,j}\right) = 2\left[\left(\frac{b^2}{a} - c\right)\gamma_{,s}{}^2 + 2\frac{b}{a}\gamma_{,s}\gamma_{,x} - 2\gamma_{,s}\gamma_{,v} + \frac{1}{a}\gamma_{,x}{}^2\right],$$

so by our work in the previous paragraph, if $R = 2g^{ij}\gamma_{,i}\gamma_{,j}$ holds on $\Sigma_0^0$, then it will hold on all of $\Gamma_0$. (Recall that we are assuming that the equations (2.3.5 – 2.3.7) hold throughout $\Gamma_0$.) Further, note that the equation

$$\nabla^j\left(\gamma_{,i}\gamma_{,j}\right) = \frac{1}{2}\nabla_i g^{jk}\gamma_{,j}\gamma_{,k}$$

holds since $\gamma$ satisfies the wave equation (this is just conservation of the stress-energy tensor).

Now the Bianchi identity gives

$$\nabla^j R_{ji} = \frac{1}{2}\nabla_i R.$$

Suppose that $(a, b, c, \gamma)$ is a solution to the system (2.3.5 – 2.3.7), (1.3.1) which moreover satisfies all six equations $R_{ij} = 2\gamma_{,i}\gamma_{,j}$ on the surface $\Sigma_0^0$. Then we have on $\Gamma_0$

$$\nabla^j R_{ji} = \frac{1}{2}\nabla_i R = \frac{1}{2}\nabla_i\left(2g^{jk}\gamma_{,j}\gamma_{,k}\right) = \nabla^j\left(2\gamma_{,j}\gamma_{,i}\right),$$

so, writing $T_{ij} = R_{ij} - 2\gamma_{,i}\gamma_{,j}$ for convenience, and letting, as per our note in 0.9,

$$\nabla_k T_{ij} = \partial_k T_{ij} - \Gamma^{\ell}_{ik}T_{\ell j} - \Gamma^{\ell}_{jk}T_{i\ell}$$

denote the $kij$ component of the covariant derivative of the tensor $T$,

$$\begin{aligned}
\nabla^j T_{j2} = \left(\frac{b^2}{a} - c\right)\nabla_0 T_{02} + \frac{b}{a}\nabla_0 T_{12} - \nabla_0 T_{22}& \\
+ \frac{b}{a}\nabla_1 T_{02} + \frac{1}{a}\nabla_1 T_{12} - \nabla_2 T_{02} &= 0, \\
\nabla^j T_{j1} = \left(\frac{b^2}{a} - c\right)\nabla_0 T_{01} + \frac{b}{a}\nabla_0 T_{11} - \nabla_0 T_{21}& \\
+ \frac{b}{a}\nabla_1 T_{01} + \frac{1}{a}\nabla_1 T_{11} - \nabla_2 T_{01} &= 0.
\end{aligned}$$

Now we already know that $T_{01} = T_{02} = T_{11} = 0$ on $\Gamma_0$, so these two equations become simply

$$\frac{b}{a}\nabla_0 T_{12} - \nabla_0 T_{22} + \frac{1}{a}\nabla_1 T_{12} = 0 \tag{2.3.8}$$

$$-\nabla_0 T_{12} = 0. \tag{2.3.9}$$

The second of these equations is, in full,

$$\begin{aligned}
0 &= \partial_0 T_{12} - \Gamma^{\ell}_{01}T_{\ell 2} - \Gamma^{\ell}_{02}T_{\ell 1} \\
&= \partial_s T_{12} - \frac{1}{2a}\left[(ba_{,s} - ab_{,s})T_{02} - a_{,s}T_{12} - (bb_{,s} - ac_{,s})T_{10} - b_{,s}T_{20}\right] \\
&= \partial_s T_{12} - \frac{1}{2a}[-a_{,s}T_{12}],
\end{aligned}$$

from which we see that

$$\partial_s T_{12} = -\frac{a_{,s}}{2a}T_{12},$$

so that if $T_{12}=0$ on $\Sigma_0^0$ then $T_{12}$ will vanish on $\Gamma_0$. Substituting this into equation (2.3.8) above, we obtain $-\nabla_0 T_{22}=0$, or, in full,

$$\begin{aligned} 0 &= \partial_0 T_{22} - \Gamma^\ell_{02} T_{\ell 2} - \Gamma^\ell_{02} T_{2\ell} \\ &= \partial_s T_{22} - \frac{1}{a}\left[(bb_{,s}-ac_{,s})T_{02} + b_{,s}T_{12}\right] = \partial_s T_{22}, \end{aligned}$$

so that if $T_{22}=0$ on $s=0$, then $T_{22}$ will vanish for all $s$. This completes the demonstration of the preservation of the constraint equations by the evolution. QED.

## 2.4. Initial data

From the foregoing we obtain the following corollary.

2.4.1. Corollary. Suppose that on $\Gamma_0$ the system

$$a_{,ss} = \frac{{a_{,s}}^2}{2a} - 4a{\gamma_{,s}}^2 \qquad b_{,ss} = \frac{1}{2a}a_{,s}b_{,s} - 4\gamma_{,s}\left(b\gamma_{,s}+\gamma_{,x}\right) \tag{2.3.1) – (2.3.2}$$

$$c_{,ss} = \frac{{b_{,s}}^2}{2a} - 2\gamma_{,s}\left(2\gamma_{,v} + \left(\frac{b^2}{a}+c\right)\gamma_{,s} + 2\frac{b}{a}\gamma_{,x}\right) - \frac{2}{a}{\gamma_{,x}}^2 \tag{2.3.3}$$

$$\left[\left(\frac{b^2}{a}-c\right)\partial_s^2 + 2\frac{b}{a}\partial_s\partial_x - 2\partial_s\partial_v + \frac{1}{a}\partial_x^2 - \frac{1}{2}\left(\left(\frac{b^2}{a}+c\right)\frac{a_{,s}}{a} - 4\frac{b}{a}b_{,s} + \frac{b}{a^2}a_{,x} + 2c_{,s} - \frac{2}{a}b_{,x} + \frac{a_{,v}}{a}\right)\partial_s \right.$$
$$\left. - \frac{1}{2}\left(\frac{b}{a^2}a_{,s} - \frac{2}{a}b_{,s} + \frac{a_{,x}}{a^2}\right)\partial_x - \frac{1}{2}\frac{a_{,s}}{a}\partial_v\right]\gamma = 0 \tag{1.3.1}$$

holds, and that on $\Sigma_0^0$ the system

$$-a_{,vv} + \frac{{a_{,v}}^2}{2a} = 4a{\gamma_{,v}}^2 \tag{2.4.1}$$

$$b_{,s}a_{,x} - 2a(b_{,xs}-a_{,vs}) - a_{,s}a_{,v} - a{b_{,s}}^2 = 4a{\gamma_{,x}}^2 \tag{2.4.2}$$

$$ab_{,vs} + \frac{1}{2}a_{,v}b_{,s} = 4a\gamma_{,x}\gamma_{,v} \tag{2.4.3}$$

as well as the conditions

$$b = c = c_{,s} = 0 \tag{2.4.4}$$

hold. Then the equations (1.1.1)

$$\Box_h\gamma = 0, \qquad R_{ij} = 2\partial_i\gamma\partial_j\gamma$$

hold on $\Gamma_0$.

Proof. By (2.4.4), the equations $R_{11} = 2{\gamma_{,x}}^2$, $R_{12} = 2\gamma_{,x}\gamma_{,v}$, and $R_{22} = 2{\gamma_{,v}}^2$ are equivalent to the system (2.4.1 – 2.4.3) on $\Sigma_0^0$. The result then follows directly from Proposition 2.3.2. QED.

We note that in terms of the quantity $\ell = \sqrt{a}$ the equation (2.4.1) becomes

$$\partial_v^2\ell = -2\ell(\partial_v\gamma)^2. \tag{2.4.5}$$

We have the following proposition, which will be sharpened considerably after we have introduced the coordinate scaling in the next chapter.

2.4.1. PROPOSITION. Suppose that on $\Sigma_0^0$ the quantity $\gamma$ is specified, that on $\Sigma_0^0$ the condition (2.4.4) holds and the quantities

$$a|_{\Sigma_0^0}, \quad a_{,v}|_{\Sigma_0^0}, \quad a_{,s}|_{\Sigma_0^0}, \quad b_{,s}|_{\Sigma_0^0}$$

are specified on a line $v = v_0$, and that $a|_{\Sigma_0^0 \cap \{v=v_0\}}$ has a positive lower bound. Then equations (2.4.1 – 2.4.3) give a unique set of initial data for the Riccati equations (2.3.1 – 2.3.3) on some neighbourhood of $v = v_0$ in $\Sigma_0^0$.

Proof. Recall that a full set of initial data for (2.3.1 – 2.3.3) is a specification on $s = 0$ of

$$a, \quad b, \quad c, \quad a_{,s}, \quad b_{,s}, \quad c_{,s}.$$

By the gauge choice, $b = c = c_{,s} = 0$ on $s = 0$, so that only the three remaining quantities

$$a, \quad a_{,s}, \quad b_{,s}$$

are free. Suppose that these three quantities, together with $a_{,v}$ (this is needed since (2.4.1) is second order) are specified on $v = v_0$ for some $v_0 \geq 0$, and that $a$ on that line has a uniform lower bound. Then on some open set $U$ in $\Sigma_0^0$ containing $v = v_0$ equation (2.4.1) can be solved for $a$, and we may moreover assume, shrinking $U$ if necessary, that $a|_U$ has an upper bound and a positive lower bound on $U$. (Note that our only requirement on $U$ is that it be a neighbourhood of the line $v = v_0$.) Given $a|_U$, equation (2.4.3) then gives a linear, first-order equation for $b_{,s}|_U$, which has a unique solution on $U$ once $b_{,s}|_{U\cap\{v=v_0\}}$ is specified. Finally, given $a|_U$ and $b_{,s}|_U$, equation (2.4.2) becomes a linear equation for $a_{,s}|_U$, which has a unique solution on $U$ once $a_{,s}|_{U\cap\{v=v_0\}}$ is specified. (It should also be noted that equations (2.4.2 – 2.4.3) do not give rise to singularities on $U$ since $a|_U$ has a uniform positive lower bound.) This completes the proof. QED.

As stated already, we shall later (see Chapter 5, especially  and Proposition 5.4.1) give a much more precise treatment of the solutions of (2.4.1 – 2.4.3).

A slightly similar setting has been considered in Rendall's paper [11]. In section 5 of [11] the 3 + 1 vacuum Einstein equations are studied in harmonic coordinates which, somewhat like our situation here, originate from data prescribed on two transverse null hypersurfaces. In this setting one starts out with a 2-dimensional spacelike hypersurface $S$, corresponding to our spacelike curve $\lambda(x)$ (see Section 1.2), from which null hypersurfaces $N_1$ and $N_2$ are developed, exactly as in our case. For purposes of comparison we identify $x^1$ (a null geodesic coordinate along $N_1$) with $s$ and $x^2$ (an analogous coordinate along $N_2$) with $v$. Subsequently an equivalence class of positive-definite metrics on $S$ is considered, and a conformal factor $\Omega$ introduced to define a particular element of this class; since up to conformal equivalence there is only one metric on a one-dimensional spacelike curve, $\Omega$ can be identified with $a$ in our treatment. Theorem 3 in [11] then guarantees the existence of solutions to the Einstein equations given the conformal equivalence class of the metric on $S$ and the specification on $S$ of the quantities $\Omega$, $\Omega_{,1}$, $\Omega_{,2}$, $g_{23,1}$ and $g_{24,1}$, where $x^3$ and $x^4$ are coordinates along $S$. We may indicate the correspondence between these quantities and quantities in our present work as follows:

$$\Omega \sim a, \qquad \Omega_{,1} \sim a_{,s}, \qquad \Omega_{,2} \sim a_{,v}$$

$$g_{23,1}, g_{24,1} \sim g_{vx,s} = b_{,s},$$

and their specification on $S$ in [11] corresponds to their specification along $\lambda(x)$, i.e., along $s = v = 0$, in the present case.

While our setting is sufficiently different from that in [11] to not allow for any direct application of the results therein, the above comparison suggests that we have not gone too far off. (For purposes of comparison, we remind the reader that – as noted in the footnote after Lemma 0.2.1 above – in our case there are distinct constraint equations only along the null hypersurface $s = 0$, and not along the hypersurface $v = 0$, the role of the latter being played by the Riccati equations.)

This completes those portions of our work which are independent of the coordinate scaling.

## 3.1. Introduction and summary

In the previous two chapters we have dealt with the general problem of solving the reduced Einstein vacuum equations (1.1.1) in our particular gauge, without considering the special problem of finding highly localised solutions. Our ultimate goal is to find solutions to the system (1.1.1) which are tightly localised (in $xv$) on a scale of size $k^{-1/2} \times k^{-1}$. We shall proceed by first obtaining existence theorems for solutions with initial data which are so localised, and which have therefore large derivatives in the $v$ direction. It turns out that the structure of the equations (1.2.9 – 1.2.11), (1.3.1), and (2.4.1 – 2.4.3) allows us to introduce a scaling of the coordinates $x$ and $v$, as well as the quantities $a-1$, $b$, $c$, and $\gamma$, which greatly simplifes this task: with respect to these scaled coordinates, all derivatives will remain *bounded* (in spaces to be specified) with respect to $k$. In particular, we shall ultimately (see Chapter 6) be able to define energies in this scaled picture which remain bounded, uniformly in $k$, up to a time proportional to $k$, and from this the desired solutions can be readily derived.

In the present chapter we provide some motivation for the scaling we shall use, and then determine the equations of motion (1.2.9 – 1.2.11), (1.3.1) as well as the constraint equations (2.4.1 – 2.4.3) in the scaled picture. In Chapter 5 we shall construct a particular class of initial data in the scaled picture and show that it satisfies the bounds which shall be necessary in Chapter 6. Finally, in Chapter 6 we define energies in the scaled picture and show that they remain bounded up to a time proportional to $k$.

## 3.2. Motivation

Before giving this scaling, we provide some motivation. For convenience in reference, recall that the Riccati equations for the metric components are (see (1.2.9 – 1.2.11))

$$\partial_s^2 a = \frac{(\partial_s a)^2}{2a} - 4a\,(\partial_s\gamma)^2 \qquad \partial_s^2 b = \frac{1}{2a}(\partial_s a)(\partial_s b) - 4\partial_s\gamma\,(b\partial_s\gamma + \partial_x\gamma) \qquad (3.2.1) – (3.2.2)$$

$$\partial_s^2 c = \frac{(\partial_s b)^2}{2a} - 2\partial_s\gamma\left(2c\partial_s\gamma + 2\partial_v\gamma + \left(\frac{b^2}{a} - c\right)\partial_s\gamma + 2\frac{b}{a}\partial_x\gamma\right) - \frac{2}{a}(\partial_x\gamma)^2, \qquad (3.2.3)$$

while the wave equation for $\gamma$ is (see (1.3.1))

$$\left[\left(\frac{b^2}{a} - c\right)\partial_s^2 + 2\frac{b}{a}\partial_s\partial_x - 2\partial_s\partial_v + \frac{1}{a}\partial_x^2 - \frac{1}{2}\left(\frac{\partial_s a}{a}\left(c + \frac{b^2}{a}\right) - 4\frac{b}{a}\partial_s b + \frac{b}{a^2}\partial_x a + 2\partial_s c - \frac{2}{a}\partial_x b + \frac{\partial_v a}{a}\right)\partial_s \right.$$
$$\left. - \frac{1}{2}\left(\frac{b}{a^2}\partial_s a - \frac{2}{a}\partial_s b + \frac{\partial_x a}{a^2}\right)\partial_x - \frac{1}{2}\frac{\partial_s a}{a}\partial_v\right]\gamma = \Box_h\gamma = 0. \qquad (3.2.4)$$

The scaling we shall use is suggested by the theory of Gaussian beams.* In particular, it is quite straightforward to show that an approximate Gaussian beam solution to the wave equation $\Box_h\gamma = 0$ along

* We shall show in Chapter 7 below that it is possible to find a Gaussian beam-like solution in our current setting, but depending on two parameters instead of one: first, $k$, which controls the spatial extent of the beam; second, a parameter we shall call $r$, which controls to what extent the energy is concentrated. While

the null geodesic $x = v = 0$ is given by (see Lemma 7.2.3)

$$\gamma_{GB} = k^{-\iota}\phi(x,v)\Re\left\{ A\left[\frac{iC - \frac{1}{E}\int_0^s \frac{1}{a}}{C^2 + \left(\frac{1}{E}\int_0^s \frac{1}{a}\right)^2}\right]^{\frac{1}{2}} e^{ikEv - \frac{1}{2}ik\frac{\frac{1}{E}\int_0^s \frac{1}{a}}{C^2+\left(\frac{1}{E}\int_0^s \frac{1}{a}\right)^2}x^2 - \frac{1}{2}k\left[\frac{C}{C^2+\left(\frac{1}{E}\int_0^s \frac{1}{a}\right)^2}x^2 + Dv^2\right]}\right\},$$

where $A$, $C$, $D$, and $E$ are arbitrary positive constants, $a = a(s, 0, 0)$ is the metric component $a$ along the geodesic, and $\phi$ is some $C^\infty$ function with compact support satisfying $\phi = 1$ on a neighbourhood of $(0, 0)$. The important point for us is the scaling in the exponential; note that the terms involving $x$ and $v$ are of the form $kv$, $kv^2$, and $kx^2$. In terms of Sobolev norms, then, a derivative in $v$ counts essentially for one power of $k$, while a derivative in $x$ counts for half a power of $k$.*

Another motivation for the scaling we shall make comes from a scaling symmetry of the system (3.2.1 – 3.2.4). Before presenting this, we wish to clarify our perspective. The components of the metric $h$, which in the coordinate system $sxv$ include $a$, $b$, and $c$, have geometric significance, and, hence, will change in a well-defined way if we scale the coordinates. To put it more simply, $a$, $b$, and $c$ might properly be considered as components of a tensor (namely the metric tensor $h$) rather than as scalars. For the purposes of this current section, however, we ignore this and consider system (3.2.1 – 3.2.3) from a purely analytical standpoint; thus we consider the functions $a$, $b$, and $c$ as scalars which do not change under a coordinate transformation. We shall discuss how the coordinate scaling impacts the actual components of the metric $h$ later (see Section 6.2).

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the solutions we construct in Chapter 6, and hence also in Chapter 7, have an existence time which is independent of $k$, the existence time of the Gaussian beam-like solutions we construct in Chapter 7 does in principle depend on $r$.

It is felt that taking full Gaussian beam initial data may produce more sharply peaked solutions than those we are able to give here – for example, solutions with $\iota = 1/2$ but $\|\gamma\|_{H^2}$ scaling like $k$ instead of $k^{3/4}$ – but attempting to integrate the usual development of Gaussian beams directly into the system (3.2.1 – 3.2.4) we have here causes difficulties due to the coupling between the coefficients in the wave equation and $\gamma$. It is felt that it might be possible to continue by expanding all quantities in a series in $k$, but that is beyond the scope of the present thesis.

* In terms of our previous footnote, it is worth noting that the inconsistent scaling in $v$ – that in one term $v$ scales with $k$ while in the other it scales with $k^{1/2}$ – is a major cause of the difficulties mentioned when attempting to extend the standard Gaussian beam treatment to the coupled system (3.2.1 – 3.2.3), (3.2.4). In particular, note that if we replace $x$ and $v$ by the scaled variables – see equation (3.3.1) below – $\overline{x} = k^{1/2}x$, $\overline{v} = kv$, then the term $kv^2$ will go to $\overline{v}^2/k$, which goes to zero as $k$ increases, meaning that for large $k$ the function $\gamma_{GB}$ is only weakly peaked in $\overline{v}$. This would render the $L^2$ norm of $\gamma_{GB}$ along the initial hypersurface $\Sigma_0^0$ of size $k^{1/4}$, which – as we shall see in Chapter 6 below – would cause great difficulties for our present method.

With this understood, suppose now that $(a, b, c, \gamma)$ is any solution to the system (3.2.1 – 3.2.4), and consider the general scaling of coordinates and dependent variables

$$\begin{gathered}
\overline{s} = k^{\delta} s, \qquad \overline{x} = k^{\alpha} x, \qquad \overline{v} = k^{\beta} v, \\
\partial_s = k^{\delta} \partial_{\overline{s}}, \qquad \partial_x = k^{\alpha} \partial_{\overline{x}}, \qquad \partial_v = k^{\beta} \partial_{\overline{v}}, \\
\overline{a} = k^{\zeta} a(\overline{s}, \overline{x}, \overline{v}), \qquad \overline{b} = k^{\eta} b(\overline{s}, \overline{x}, \overline{v}), \qquad \overline{c} = k^{\theta} c(\overline{s}, \overline{x}, \overline{v}), \\
\overline{\gamma} = k^{\iota} \gamma(\overline{s}, \overline{x}, \overline{v}),
\end{gathered} \tag{3.2.5}$$

and define also $\overline{\ell} = \sqrt{\overline{a}}$. The system (3.2.1 – 3.2.3) then gives for $\overline{a}$, $\overline{b}$, and $\overline{c}$

$$\begin{aligned}
\partial_{\overline{s}}^2 \overline{a} &= \frac{(\partial_{\overline{s}} \overline{a})^2}{2\overline{a}} - 4\overline{a} k^{-2\iota} (\partial_{\overline{s}} \overline{\gamma})^2, \qquad \partial_{\overline{s}}^2 \overline{\ell} = -2\overline{\ell} k^{-2\iota} (\partial_{\overline{s}} \overline{\gamma})^2, \\
\partial_{\overline{s}}^2 \overline{b} &= k^{\eta - 2\delta} \partial_s^2 b = k^{\eta - 2\delta} \left[ \frac{1}{2\overline{a}} k^{-\delta} (\partial_{\overline{s}} \overline{a}) k^{-\eta + \delta} (\partial_{\overline{s}} \overline{b}) - 4 k^{-2\iota + \delta} (\partial_{\overline{s}} \overline{\gamma}) (k^{-\eta + \delta} \overline{b} \partial_{\overline{s}} \overline{\gamma} + k^{\alpha} \partial_{\overline{x}} \overline{\gamma}) \right] \\
&= \frac{1}{2\overline{a}} (\partial_{\overline{s}} \overline{a})(\partial_{\overline{s}} \overline{b}) - 4 k^{-2\iota} \partial_{\overline{s}} \overline{\gamma} (\overline{b} \partial_{\overline{s}} \overline{\gamma} + k^{\alpha + \eta - \delta} \partial_{\overline{x}} \overline{\gamma}), \\
\partial_{\overline{s}}^2 \overline{c} &= k^{\theta} k^{-2\delta} \partial_s c = k^{\theta - 2\delta} \Bigg[ k^{-2\eta + 2\delta + \zeta} \frac{\left(\partial_{\overline{s}} \overline{b}\right)^2}{2\overline{a}} \\
&\qquad - 2k^{-2\iota + \delta} \partial_{\overline{s}} \overline{\gamma} \left( 2k^{-\theta + \delta} \overline{c} \partial_{\overline{s}} \overline{\gamma} + 2k^{\beta} \partial_{\overline{v}} \overline{\gamma} + \left( k^{-2\eta + \zeta} \frac{\overline{b}^2}{\overline{a}} - k^{-\theta} \overline{c} \right) k^{\delta} \partial_{\overline{s}} \overline{\gamma} + 2k^{-\eta + \zeta} \frac{\overline{b}}{\overline{a}} k^{\alpha} \partial_{\overline{x}} \overline{\gamma} \right) \\
&\qquad - k^{2(\alpha - \iota + \zeta)} \frac{2}{\overline{a}} \left(\partial_{\overline{x}} \overline{\gamma}\right)^2 \Bigg] \\
&= k^{\theta - 2\eta + \zeta} \frac{(\partial_{\overline{s}} \overline{b})^2}{2\overline{a}} - 2k^{-2\iota} \partial_{\overline{s}} \overline{\gamma} \left( 2k^{\beta + \theta - \delta} \partial_{\overline{v}} \overline{\gamma} + \left( k^{\zeta + \theta - 2\eta} \frac{\overline{b}^2}{\overline{a}} + \overline{c} \right) \partial_{\overline{s}} \overline{\gamma} + 2k^{\theta + \alpha + \zeta - \eta - \delta} \frac{\overline{b}}{\overline{a}} \partial_{\overline{x}} \overline{\gamma} \right) \\
&\qquad - k^{2(\alpha - \iota + \zeta - \delta) + \theta} \frac{2}{\overline{a}} (\partial_{\overline{x}} \overline{\gamma})^2,
\end{aligned}$$

while the wave equation (3.2.4) becomes

$$\begin{aligned}
&\Bigg[ \left( k^{2\delta - 2\eta + \zeta} \frac{\overline{b}^2}{\overline{a}} - k^{2\delta - \theta} \overline{c} \right) \partial_{\overline{s}}^2 + 2k^{-\eta + \zeta + \delta + \alpha} \frac{\overline{b}}{\overline{a}} \partial_{\overline{s}} \partial_{\overline{x}} - 2k^{\delta + \beta} \partial_{\overline{s}} \partial_{\overline{v}} + k^{\zeta + 2\alpha} \frac{1}{\overline{a}} \partial_{\overline{x}}^2 \\
&\quad - \frac{1}{2} \Bigg[ \left( k^{2\delta - \theta} \overline{c} + k^{2\delta + \zeta - 2\eta} \frac{\overline{b}^2}{\overline{a}} \right) \frac{\partial_{\overline{s}} \overline{a}}{\overline{a}} - 4k^{2\delta + \zeta - 2\eta} \frac{\overline{b}}{\overline{a}} \partial_{\overline{s}} \overline{b} + k^{\delta + \alpha + \zeta - \eta} \frac{\overline{b}}{\overline{a}^2} \partial_{\overline{x}} \overline{a} \\
&\qquad + 2k^{2\delta - \theta} \partial_{\overline{s}} \overline{c} - 2k^{\delta + \alpha + \zeta - \eta} \frac{1}{\overline{a}} \partial_{\overline{x}} \overline{b} + k^{\beta + \delta} \frac{\partial_{\overline{v}} \overline{a}}{\overline{a}} \Bigg] \partial_{\overline{s}} \\
&\qquad - \frac{1}{2} \left( k^{\alpha + \delta + \zeta - \eta} \frac{\overline{b}}{\overline{a}^2} \partial_{\overline{s}} \overline{a} - 2k^{\delta + \alpha + \zeta - \eta} \frac{1}{\overline{a}} \partial_{\overline{s}} \overline{b} + k^{2\alpha + \zeta} \frac{\partial_{\overline{x}} \overline{a}}{\overline{a}^2} \right) \partial_{\overline{x}} - \frac{1}{2} k^{\delta + \beta} \frac{\partial_{\overline{s}} \overline{a}}{\overline{a}} \partial_{\overline{v}} \Bigg] \overline{\gamma} = 0.
\end{aligned}$$

For these transformed equations to be formally identical to the original system (3.2.1 – 3.2.4), it is necessary and sufficient that we be able to cancel all powers of $k$. From this we see that it is necessary and sufficient that $\iota = 0$, that

$$\begin{aligned} \alpha + \eta - \delta &= 0 \\ \theta - 2\eta + \zeta &= 0 \\ \beta + \theta - \delta &= 0 \\ \theta + \alpha + \zeta - \eta - \delta &= 0 \\ 2(\alpha + \zeta - \delta) + \theta &= 0, \end{aligned} \tag{3.2.6}$$

and that there be a constant $\lambda$ such that

$$\begin{aligned} 2\delta - 2\eta + \zeta &= \lambda \\ 2\delta - \theta &= \lambda \\ -\eta + \zeta + \delta + \alpha &= \lambda \\ \delta + \beta &= \lambda \\ \zeta + 2\alpha &= \lambda. \end{aligned} \tag{3.2.7}$$

From these, it is straightforward, if slightly tedious, to show that we must have $\zeta = 0$, while

$$\begin{aligned} \alpha &= \frac{1}{2}\lambda, \qquad \beta = -\delta + \lambda \\ \eta &= \delta - \frac{1}{2}\lambda, \qquad \theta = 2\delta - \lambda. \end{aligned} \tag{3.2.8}$$

Now if we do not scale the $s$ coordinate – motivated by the Gaussian beam situation described above – then we must take $\delta = 0$, whence the first of these give $\beta = 2\alpha$. In other words, there is some sort of intrinsic scaling symmetry to the system which requires that the $v$ coordinate scale with an exponent equal to twice that of the $x$ coordinate, exactly as occurs with the approximate Gaussian beam.

We note finally that the wave equation in free space in $2 + 1$ dimensions, written in terms of null coordinates $sxv$ as here, is simply

$$-2\partial_s\partial_v\gamma + \partial_x^2\gamma = 0,$$

which is clearly also invariant under the scaling transformation $\overline{s} = s$, $\overline{x} = k^\alpha x$, $\overline{v} = k^{2\alpha} v$.

### 3.3. Scaling and scaled equations

With the foregoing as motivation, we now give the coordinate scaling we shall use in the rest of this work. In terms of the parameters defined in (3.2.5), we take $\alpha = 1/2$, $\beta = 1$, $\delta = 0$, $\zeta = 0$, $\eta = 1/2$, $\theta = 0$; finally, we require $\iota \geq 1/2$ but leave it otherwise unspecified for the moment. It is worth noting that these choices satisfy the first two equations in (3.2.8) if we take $\lambda = 1$, but the equations for $\eta$ and $\theta$ – which we recall controls the scaling of $b$ – will then fail, and of course the choice $\iota \geq 1/2$ also breaks the scaling. As we shall in great detail in Chapter 5 and Chapter 6 below, though, the extra terms resulting will all be of lower order in $k$, and thus in some sense unimportant.

The above choice of scaling parameters gives the scaled coordinates

$$\begin{aligned} \overline{s} &= s, \qquad \overline{x} = k^{1/2}x, \qquad \overline{v} = kv, \\ \partial_s &= \partial_{\overline{s}}, \qquad \partial_x = k^{1/2}\partial_{\overline{x}}, \qquad \partial_v = k\partial_{\overline{v}}, \end{aligned} \tag{3.3.1}$$

and scaled dependent variables

$$\overline{a} = a, \qquad \overline{\ell} = \ell, \qquad \overline{b} = k^{1/2} b, \qquad \overline{c} = c, \tag{3.3.2}$$

$$\overline{\gamma} = k^{\iota}\gamma. \tag{3.3.3}$$

We further define quantities $\overline{\delta a}$, $\overline{\delta \ell}$, and $\overline{\delta^{-1} a}$ by

$$\overline{a} = 1 + k^{-1}\overline{\delta a}, \qquad \overline{\ell} = 1 + k^{-1}\overline{\delta \ell}, \qquad \overline{a}^{-1} = 1 + k^{-1}\overline{\delta^{-1} a}. \tag{3.3.4}$$

Note that all five of the quantities $\overline{a}$, $\overline{\delta a}$, $\overline{\ell}$, $\overline{\delta \ell}$, and $\overline{\delta^{-1} a}$ uniquely determine $a$; we shall use whichever of them is most convenient for the purpose at hand.

As our work in the sequel will show, given suitable* initial data, the above scaling allows us to obtain $k$-independent bounds for $\overline{\delta \ell}$, $\overline{b}$, and $\overline{c}$ in appropriate Sobolev spaces. (See Chapter 6 below for the details.) For comparison, we note that this implies the following ansätze on $a$, $b$, and $c$:

$$a = 1 + k^{-1}\overline{\delta a}, \qquad b = k^{-1/2}\overline{b}, \qquad c = \overline{c}.$$

We shall not have occasion to explicitly use these ansätze in the following, however.

We shall always assume that $k \geq 1$. This is permissible since we are interested in what happens when $k$ is arbitrarily large.

The wave and Riccati equations may now be written out in the scaled coordinates. The above choice of scaling parameters gives for the scaling exponents in the wave equation (cf. (3.2.6), (3.2.7))

$$\begin{aligned} 2\delta - 2\eta + \zeta &= -1 \\ 2\delta - \theta &= 0 \\ -\eta + \zeta + \delta + \alpha &= 0 \\ \delta + \beta &= 1 \\ \zeta + 2\alpha &= 1. \end{aligned}$$

The wave equation thus becomes

$$\begin{aligned} \Bigg[ \left( k^{-1}\frac{\overline{b}^2}{\overline{a}} - \overline{c} \right) \partial_{\overline{s}}^2 + 2\frac{\overline{b}}{\overline{a}}\partial_{\overline{s}}\partial_{\overline{x}} - 2k\partial_{\overline{s}}\partial_{\overline{v}} + k\frac{1}{\overline{a}}\partial_{\overline{x}}^2 \\ - \frac{1}{2}\left[ \left( \overline{c} + k^{-1}\frac{\overline{b}^2}{\overline{a}} \right) \frac{\partial_{\overline{s}}\overline{a}}{\overline{a}} - 4k^{-1}\frac{\overline{b}}{\overline{a}}\partial_{\overline{s}}\overline{b} + \frac{\overline{b}}{\overline{a}^2}\partial_{\overline{x}}\overline{a} + 2\partial_{\overline{s}}\overline{c} - 2\frac{1}{\overline{a}}\partial_{\overline{x}}\overline{b} + k\frac{\partial_{\overline{v}}\overline{a}}{\overline{a}} \right] \partial_{\overline{s}} \\ - \frac{1}{2}\left( \frac{\overline{b}}{\overline{a}^2}\partial_{\overline{s}}\overline{a} - 2\frac{1}{\overline{a}}\partial_{\overline{s}}\overline{b} + k\frac{\partial_{\overline{x}}\overline{a}}{\overline{a}^2} \right)\partial_{\overline{x}} - \frac{1}{2}k\frac{\partial_{\overline{s}}\overline{a}}{\overline{a}}\partial_{\overline{v}} \Bigg] \overline{\gamma} = 0. \end{aligned}$$

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* The particular choice of initial data will affect the exponent $\iota$. For the initial data we construct in Chapter 5 below, it is sufficient to take $\iota = 1/2$.

Now since $\overline{a} = \overline{\ell}^2 = (1 + k^{-1}\overline{\delta\ell})^2$, we have

$$\partial_i \overline{a} = 2\overline{\ell}\partial_i\overline{\ell} = 2k^{-1}\overline{\ell}\partial_i\overline{\delta\ell};$$

substituting this into the foregoing, collecting powers of $k$, and multiplying by $k^{-1}$, we obtain

$$\Bigg[ (-2\partial_{\overline{s}}\partial_{\overline{v}} + \partial_{\overline{x}}^2) + \frac{1}{k}\left( 2\frac{\overline{b}}{\overline{a}}\partial_{\overline{s}}\partial_{\overline{x}} - \overline{c}\partial_{\overline{s}}^2 - \overline{\delta^{-1}a}\partial_{\overline{x}}^2 - \left( \partial_{\overline{s}}\overline{c} - \frac{1}{\overline{a}}\partial_{\overline{x}}\overline{b} + \frac{\partial_{\overline{v}}\overline{\delta\ell}}{\overline{\ell}} \right)\partial_{\overline{s}} + \left( \frac{1}{\overline{a}}\partial_{\overline{s}}\overline{b} - \frac{\overline{\ell}\partial_{\overline{x}}\overline{\delta\ell}}{\overline{a}^2} \right)\partial_{\overline{x}} - \frac{\partial_{\overline{s}}\overline{\delta\ell}}{\overline{\ell}}\partial_{\overline{v}} \right)$$

$$+ \frac{1}{k^2}\left( \frac{\overline{b}^2}{\overline{a}}\partial_{\overline{s}}^2 - \left( \overline{c}\frac{\partial_{\overline{s}}\overline{\delta\ell}}{\overline{\ell}} - 2\frac{\overline{b}}{\overline{a}}\partial_{\overline{s}}\overline{b} \right)\partial_{\overline{s}} - \frac{\overline{b\ell}\partial_{\overline{s}}\overline{\delta\ell}}{\overline{a}^2}\partial_{\overline{x}} \right) - \frac{1}{k^3}\frac{\overline{b}^2\overline{\ell}\partial_{\overline{s}}\overline{\delta\ell}}{\overline{a}^2}\partial_{\overline{s}} \Bigg]\overline{\gamma} = 0. \tag{3.3.5}$$

Note that the leading-order term is simply the free-space (Minkowski) wave operator.

We may similarly write out the scaled Riccati equations. We note that the scaling exponents for these equations are (again, cf. (3.2.6), (3.2.7))

$$\alpha + \eta - \delta = 1$$
$$\theta - 2\eta + \zeta = -1$$
$$\beta + \theta - \delta = 1$$
$$\theta + \alpha + \zeta - \eta - \delta = 0$$
$$2(\alpha + \zeta - \delta) + \theta = 1,$$

so that the Riccati equations become

$$\partial_{\overline{s}}^2\overline{a} = \frac{(\partial_{\overline{s}}\overline{a})^2}{2\overline{a}} - 4\overline{a}k^{-2\iota}(\partial_{\overline{s}}\overline{\gamma})^2, \qquad \partial_{\overline{s}}^2\overline{\ell} = -2\overline{\ell}k^{-2\iota}(\partial_{\overline{s}}\overline{\gamma})^2, \qquad \partial_{\overline{s}}^2\overline{\delta\ell} = -2\overline{\ell}k^{1-2\iota}(\partial_{\overline{s}}\overline{\gamma})^2, \tag{3.3.6}$$

$$\partial_{\overline{s}}^2\overline{b} = \frac{1}{2\overline{a}}(\partial_{\overline{s}}\overline{a})(\partial_{\overline{s}}\overline{b}) - 4k^{-2\iota}\partial_{\overline{s}}\overline{\gamma}(\overline{b}\partial_{\overline{s}}\overline{\gamma} + k\partial_{\overline{x}}\overline{\gamma}) = \frac{1}{\overline{\ell}}k^{-1}(\partial_{\overline{s}}\overline{\delta\ell})(\partial_{\overline{s}}\overline{b}) - 4k^{1-2\iota}\partial_{\overline{s}}\overline{\gamma}(\partial_{\overline{x}}\overline{\gamma} + k^{-1}\overline{b}\partial_{\overline{s}}\overline{\gamma}), \tag{3.3.7}$$

$$\partial_{\overline{s}}^2\overline{c} = k^{-1}\frac{(\partial_{\overline{s}}\overline{b})^2}{2\overline{a}} - 2k^{1-2\iota}\partial_{\overline{s}}\overline{\gamma}\left( 2\partial_{\overline{v}}\overline{\gamma} + 2k^{-1}\frac{\overline{b}}{\overline{a}}\partial_{\overline{x}}\overline{\gamma} + k^{-1}\left( \overline{c} + k^{-1}\frac{\overline{b}^2}{\overline{a}} \right)\partial_{\overline{s}}\overline{\gamma} \right) - k^{1-2\iota}\frac{2}{\overline{a}}(\partial_{\overline{x}}\overline{\gamma})^2. \tag{3.3.8}$$

Note that the restriction $\iota \geq 1/2$ means that the right-hand sides of these equations will not have any positive powers of $k$.

### 3.4. Constraints in the scaled coordinates

Recall the constraint equations (2.4.2 – 2.4.1):

$$b_{,s}a_{,x} - 2a(b_{,xs} - a_{,vs}) - a_{,s}a_{,v} - a{b_{,s}}^2 = 4a{\gamma_{,x}}^2,$$
$$ab_{,vs} + \frac{1}{2}a_{,v}b_{,s} = 4a\gamma_{,x}\gamma_{,v},$$
$$-a_{,vv} + \frac{{a_{,v}}^2}{2a} = 4a{\gamma_{,v}}^2.$$

We shall now derive the corresponding equations in the scaled coordinates. It is often convenient to work with the quantity $\overline{\ell} = \sqrt{\overline{a}}$ instead of $\overline{a}$ itself. We have the following proposition.

3.4.1. PROPOSITION. In the coordinates $\overline{s}\,\overline{x}\,\overline{v}$, the equations (2.4.2 – 2.4.1) become

$$2\overline{\ell}\cdot\overline{\delta\ell}_{,\overline{vs}} - \overline{\ell}\partial_{\overline{x}}\left(\overline{\ell}^{-1}\overline{b}_{,\overline{s}}\right) - \frac{1}{2k}\overline{b}_{,\overline{s}}{}^2 = 2k^{1-2\iota}\overline{\gamma}_{,\overline{x}}{}^2 \tag{3.4.1}$$

$$\partial_{\overline{v}}\left(\overline{\ell}\,\overline{b}_{,\overline{s}}\right) = 4\overline{\ell}k^{1-2\iota}\overline{\gamma}_{,\overline{v}}\overline{\gamma}_{,\overline{x}} \tag{3.4.2}$$

$$-\overline{\delta\ell}_{,\overline{vv}} = 2\overline{\ell}k^{1-2\iota}\overline{\gamma}_{,\overline{v}}{}^2. \tag{3.4.3}$$

Proof. We note the following derivatives of $\overline{\ell}$ and $\overline{\delta\ell}$:

$$\partial_{\overline{v}}\overline{\ell} = \frac{\overline{a}_{,\overline{v}}}{2\sqrt{\overline{a}}}, \qquad \partial_{\overline{v}}^2\overline{\ell} = \frac{\overline{a}_{,\overline{vv}}}{2\sqrt{\overline{a}}} - \frac{\overline{a}_{,\overline{v}}{}^2}{4\overline{a}^{3/2}} = \frac{1}{2\sqrt{\overline{a}}}\left(\overline{a}_{,\overline{vv}} - \frac{\overline{a}_{,\overline{v}}{}^2}{2\overline{a}}\right),$$

$$\partial_{\overline{s}}\partial_{\overline{v}}\overline{\ell} = \frac{\overline{a}_{,\overline{vs}}}{2\sqrt{\overline{a}}} - \frac{\overline{a}_{,\overline{v}}\overline{a}_{,\overline{s}}}{4\overline{a}^{3/2}} = \frac{1}{2\sqrt{\overline{a}}}\left(\overline{a}_{,\overline{vs}} - \frac{\overline{a}_{,\overline{v}}\overline{a}_{,\overline{s}}}{2\overline{a}}\right),$$

$$\partial_{\overline{s}}\partial_{\overline{v}}\overline{\delta\ell} = k\partial_{\overline{s}}\partial_{\overline{v}}\overline{\ell} = \frac{k}{2\sqrt{\overline{a}}}\left(\overline{a}_{,\overline{vs}} - \frac{\overline{a}_{,\overline{v}}\overline{a}_{,\overline{s}}}{2\overline{a}}\right).$$

Recall also the following derivatives (see (3.3.1 – 3.3.4)):

$$\partial_s = \partial_{\overline{s}}, \qquad \partial_x = k^{1/2}\partial_{\overline{x}}, \qquad \partial_v = k\partial_{\overline{v}},$$

$$\partial_x b = \partial_{\overline{x}}\overline{b}, \qquad \partial_v b = k^{1/2}\partial_{\overline{v}}\overline{b},$$

$$\partial_x\gamma = k^{1/2-\iota}\partial_{\overline{x}}\overline{\gamma}, \qquad \partial_v\gamma = k^{1-\iota}\partial_{\overline{v}}\overline{\gamma}.$$

Given these, the equations (2.4.2 – 2.4.1) become

$$-\left(k^{1/2}\partial_{\overline{x}}\right)\left(k^{-1/2}\overline{b}_{,\overline{s}}\right) + \frac{1}{2\overline{a}}\left(k^{-1/2}\overline{b}_{,\overline{s}}\left(k^{1/2}\partial_{\overline{x}}\right)\overline{a}\right)$$
$$+k\overline{a}_{,\overline{vs}} - k\frac{\overline{a}_{,\overline{s}}\overline{a}_{,\overline{v}}}{2\overline{a}} - \frac{1}{2}\left(k^{-1/2}\overline{b}_{,\overline{s}}\right)^2 = 2\left(\left(k^{1/2}\partial_{\overline{x}}\right)k^{-\iota}\overline{\gamma}\right)^2$$
$$-\overline{b}_{,\overline{xs}} + \frac{\overline{b}_{,\overline{s}}\overline{a}_{,\overline{x}}}{2\overline{a}} + k\left(\overline{a}_{,\overline{vs}} - \frac{\overline{a}_{,\overline{s}}\overline{a}_{,\overline{v}}}{2\overline{a}}\right) - \frac{1}{2k}\overline{b}_{,\overline{s}}{}^2 = 2\overline{\gamma}_{,\overline{x}}{}^2k^{1-2\iota}$$
$$2\overline{\ell}\cdot\overline{\delta\ell}_{,\overline{vs}} - \overline{\ell}\partial_{\overline{x}}\left(\overline{\ell}^{-1}\overline{b}_{,\overline{s}}\right) - \frac{1}{2k}\overline{b}_{,\overline{s}}{}^2 = 2k^{1-2\iota}\overline{\gamma}_{,\overline{x}}{}^2,$$

which is equation (3.4.1);

$$k^{1/2}\overline{b}_{,\overline{vs}} + k^{1/2}\frac{\overline{a}_{,\overline{v}}\overline{b}_{,\overline{s}}}{2\overline{a}} = 4k^{3/2-2\iota}\overline{\gamma}_{,\overline{v}}\overline{\gamma}_{,\overline{x}}$$
$$\overline{b}_{,\overline{vs}} + \frac{\overline{a}_{,\overline{v}}\overline{b}_{,\overline{s}}}{2\overline{a}} = 4k^{1-2\iota}\overline{\gamma}_{,\overline{v}}\overline{\gamma}_{,\overline{x}}$$
$$\partial_{\overline{v}}\left(\overline{\ell}\,\overline{b}_{,\overline{s}}\right) = 4\overline{\ell}k^{1-2\iota}\overline{\gamma}_{,\overline{v}}\overline{\gamma}_{,\overline{x}},$$

which is equation (3.4.2); and

$$-k^2\overline{a}_{,\overline{vv}} + k^2\frac{\overline{a}_{,\overline{v}}{}^2}{2\overline{a}} = 4\overline{a}k^{2-2\iota}\overline{\gamma}_{,\overline{v}}{}^2$$
$$-k^2\overline{\ell}_{,\overline{vv}} = 2\overline{\ell}k^{2-2\iota}\overline{\gamma}_{,\overline{v}}{}^2$$
$$-\overline{\delta\ell}_{,\overline{vv}} = 2\overline{\ell}k^{1-2\iota}\overline{\gamma}_{,\overline{v}}{}^2,$$

which is equation (3.4.3). This completes the demonstration. QED.

Proposition 2.4.1 in Chapter 2 shows that, given the quantities

$$\overline{\delta\ell}, \quad \overline{\delta\ell}_{,\overline{s}}, \quad \overline{b}_{,\overline{s}}.$$

on a line $\{\overline{s} = 0, \overline{v} = \overline{v}_0\}$ for some $\overline{v}_0 \geq 0$, there is a neighbourhood of this line on which the system (2.4.2 – 2.4.1) will have a unique solution.

We now wish to study the solutions to the system (3.4.1 – 3.4.3), and in particular to show that it admits solutions on sufficiently large regions and with sufficiently well-behaved bounds to serve as initial data for the equations of motion (3.3.5 – 3.3.8). We shall do this in Chapter 5. First we pause to build up some analytic machinery for that chapter as well as the work in partial differential equations we shall do in Chapter 6.

### 4.1. Introduction

In this chapter we collect some algebraic and analytic results for use in the final two chapters. These fall into three categories: first, two simple algebraic results; next, more-or-less standard $L^\infty$ bounds on solutions to ordinary differential equations, which will be used to prove the existence of initial data satisfying appropriate bounds; finally, modifications of standard Poincaré and Sobolev estimates, together with related results, which will be needed in the analysis of the system (3.3.5 – 3.3.8).

This chapter is independent of the rest of the thesis. In particular, the results we give are in spaces independent from those on which we solve (0.2.2 – 0.2.5).

### 4.2. Algebraic results

We have the following lemmata.

4.2.1. LEMMA. Let $a \in \mathbf{R}^n$, and let $g$ be a function on a neighbourhood of $a$ which satisfies $g(a) \neq 0$. Let $p \geq 1$ and suppose that $J$ is any multiindex for which $\partial^J g(a)$ exists. Let $\mathcal{K}$ denote the set of all collections of multiindices $\{K_k\}$ whose sum equals $J$. There is a collection of combinatorial constants $\{C^p_{\{K_k\}} \mid \{K_k\} \in \mathcal{K}\}$ such that at $a$

$$\partial^J \frac{1}{g^p} = \sum_{\{K_k\}\in\mathcal{K}} C^p_{\{K_k\}} \frac{\prod\limits_{K\in\{K_k\}} \partial^K g}{g^{|\{K_k\}|+p}},$$

where $|\{K_k\}|$ denotes the cardinality of $\{K_k\}$.

Proof. This may be seen by induction. If $|J| = 1$, then it suffices to take $C^p_{\{J\}} = -1$. Suppose the above formula holds for all $J$ with $|J| \leq j$, some $j \geq 1$. Then differentiating gives

$$\partial_i \partial^J \frac{1}{g^p} = \sum_{\{K_k\}\in\mathcal{K}} C^p_{\{K_k\}} \left\{ \frac{\sum_{K\in\{K_k\}} \partial_i \partial^K g \prod\limits_{K'\in\{K_k\}\setminus K} \partial^{K'} g}{g^{|\{K_k\}|+p}} - (|\{K_k\}+1|) \frac{\partial_i g \prod\limits_{K\in\{K_k\}} \partial^K g}{g^{|\{K_k\}|+1+p}} \right\},$$

which is of the correct form. QED.

Let $\|M\|_{HS} = \left(\sum_{cd} |M_{cd}|^2\right)^{1/2}$ denote the Hilbert-Schmidt norm of a matrix $M$. (Note that $M$ need not be square.)

4.2.2. LEMMA. The Hilbert-Schmidt norm has the following properties:

(a) If $V$ is a covector in some Euclidean space and $|V|$ denotes the Euclidean norm, then $\|V_i V_j\|_{HS} = |V|^2$.

(b) If $A$ and $B$ are two square matrices, then $\operatorname{Tr} AB \leq \|A\|_{HS} \|B\|_{HS}$.

Proof. (a) We have

$$\|V_i V_j\|^2_{HS} = \sum_{i,j} V_i^2 V_j^2 = \left(\sum_i V_i^2\right)\left(\sum_j V_j^2\right) = |V|^2 \cdot |V|^2 = |V|^4.$$

(b) Since the Hilbert-Schmidt norm on square matrices can be written as $\|A\|^2_{HS} = \operatorname{Tr} A^2$, and $(A, B) \mapsto \operatorname{Tr} AB$ is an inner product on square matrices, this follows from the Cauchy-Schwartz inequality. QED.

4.2.3. LEMMA. Let $f$ be a real-valued $C^\infty$ function on some convex open set $O \subset \mathbf{R}^m$. Then there is a $C^\infty$ map $F : O \times O \to \mathbf{R}^m$ such that for all $\mathbf{x}_1, \mathbf{x}_2 \in O$,

$$f(\mathbf{x}_1) - f(\mathbf{x}_2) = (\mathbf{x}_1 - \mathbf{x}_2) \cdot F(\mathbf{x}_1, \mathbf{x}_2).$$

Proof. This is elementary: define

$$F(\mathbf{x}_1, \mathbf{x}_2) = \int_0^1 \nabla f(\mathbf{x}_2 + t(\mathbf{x}_1 - \mathbf{x}_2))\, dt;$$

then clearly $F : O \times O \to \mathbf{R}^m$ is $C^\infty$, and moreover

$$f(\mathbf{x}_1 - \mathbf{x}_2) = \int_0^1 \frac{d}{dt} f(\mathbf{x}_2 + t(\mathbf{x}_1 - \mathbf{x}_2))\, dt = \int_0^1 \nabla f(\mathbf{x}_2 + t(\mathbf{x}_1 - \mathbf{x}_2)) \cdot (\mathbf{x}_1 - \mathbf{x}_2)\, dt = (\mathbf{x}_1 - \mathbf{x}_2) \cdot F(\mathbf{x}_1, \mathbf{x}_2),$$

as claimed. QED.

4.2.4. LEMMA. Let $O \subset \mathbf{R}^m$ be open, let $F : O \to \mathbf{R}^p$ be $C^\infty$, and suppose that $f_1, \cdots, f_m : O' \to \mathbf{R}^1$, $O' \subset \mathbf{R}^q$ open, are also $C^\infty$. Let $K$ be a multiindex in $\mathbf{R}^q$. Then $\partial^K F(f_1(\mathbf{x}), \cdots, f_m(\mathbf{x}))$ is a sum of combinatorial constants multiplying expressions of the form

$$(\partial^J F)(f_1(\mathbf{x}), \cdots, f_m(\mathbf{x})) \prod \partial^{K_k} f_k(\mathbf{x}), \tag{4.2.1}$$

where $K_k$ are multiindices on $\mathbf{R}^q$ and $J$ is in $\mathbf{R}^m$, satisfying $|J| \le |K|$ and $\sum |K_k| = |K|$.

Proof. We proceed by induction. If $|K| = 1$, say $K = \partial_i$, then we have by the chain rule

$$\partial_i F(f_1(\mathbf{x}), \cdots, f_m(\mathbf{x})) = \sum_j (\partial_j F)(f_1(\mathbf{x}), \cdots, f_m(\mathbf{x})) \partial_i f_j(\mathbf{x}),$$

which is of the desired form. Now suppose that the result holds for all multiindices $K$ with $|K| \le n$, $n \ge 1$. If we differentiate (4.2.1) with respect to $\partial_i$, we obtain by the product and chain rules

$$\sum_j (\partial_j \partial^J F)(f_1(\mathbf{x}), \cdots, f_m(\mathbf{x})) \partial_i f_j(\mathbf{x}) \prod \partial^{K_k} f_k(\mathbf{x}) \\ + (\partial^J F)(f_1(\mathbf{x}), \cdots, f_m(\mathbf{x})) \sum \partial_i \partial^{K^*} f_*(\mathbf{x}) \prod_{K_k \ne K^*} \partial^{K_k} f_k(\mathbf{x}),$$

which is again of the correct form. QED.

## 4.3. $L^\infty$ bounds for ordinary differential equations

We recall that Proposition 2.4.1 shows that the constraint equations (3.4.1 – 3.4.3) can be solved locally as ordinary differential equations for the initial data $a$, $a_{,s}$, and $b_{,s}$. The following results will be applied in the next chapter to derive $L^\infty$ bounds on these initial data, as well as the initial data for $\gamma$. Those bounds will also allow us to provide lower bounds on the interval of existence of $a$, $a_{,s}$, and $b_{,s}$.*

* We note that the following results could be sharpened considerably in many places by replacing quantities like $X\|f\|_{L^\infty([0,X])}$ with $\|f\|_{L^1([0,X])}$. On the other hand, the bounds in the form we give them are suitable for our purposes, and to directly apply results with $L^1$ norms in place of $L^\infty$ norms in Chapter 5 would require us to introduce yet another series of norms on the initial data.

Let $M_{m\times m}(\mathbf{R})$, where $m$ is a positive integer, denote the set of $m\times m$ matrices over $\mathbf{R}$.

4.3.1. PROPOSITION. Let $[0,X]$ be some compact interval in $\mathbf{R}^1$, and suppose that $\mathbf{x}:[0,X]\to\mathbf{R}^m$, $\mathbf{M}:[0,X]\to M_{m\times m}(\mathbf{R})$, and $\mathbf{b}:[0,X]\to\mathbf{R}^m$ are $C^\infty$ functions on $[0,X]$ which satisfy

$$\dot{\mathbf{x}}=\mathbf{M}\mathbf{x}+\mathbf{b}$$

on $(0,X)$. Then on $[0,X]$, $\mathbf{x}$ must satisfy

$$|\mathbf{x}|(t)\le e^{X\|\|\mathbf{M}(t)\|_{HS}\|_{L^\infty}}\left(|\mathbf{x}|(0)+X\,\||\mathbf{b}|(t)\|_{L^\infty}\right),$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbf{R}^m$, and $\|\cdot\|_{L^\infty}$ denotes the $L^\infty$ norm on $[0,X]$.

Proof. For simplicity we write everything in index notation; upper and lower indices are equivalent since we are working with a Euclidean metric. The differential equation becomes

$$\dot{\mathbf{x}}_i=\mathbf{M}_{ij}\mathbf{x}_j+\mathbf{b}_i.$$

Let $\epsilon>0$, and define $e=|\mathbf{x}|^2+\epsilon=\mathbf{x}_i\mathbf{x}_i+\epsilon$; then

$$\dot{e}=2\mathbf{x}_i\dot{\mathbf{x}}_i=2\mathbf{x}_i\mathbf{M}_{ij}\mathbf{x}_j+2\mathbf{b}_i\mathbf{x}_i,$$

from which we see, by Lemma 4.2.2, that we have on $[0,X]$

$$\dot{e}\le 2\|\mathbf{M}\|_{HS}|\mathbf{x}|^2+2|\mathbf{b}||\mathbf{x}|=2\|\mathbf{M}\|_{HS}e+2|\mathbf{b}|e^{1/2}.$$

Dividing through by $e^{1/2}$, and multiplying by the integrating factor $\exp\left[\left(-\int_0^t\|\mathbf{M}(t')\|_{HS}\,dt'\right)\right]$, we see that this is equivalent to

$$\frac{d}{dt}\left\{\left[\exp\left[\left(-\int_0^t\|\mathbf{M}(t')\|_{HS}\,dt'\right)\right]\right]e^{1/2}\right\}\le|\mathbf{b}|(t)e^{-\int_0^t\|\mathbf{M}(t')\|_{HS}\,dt'},$$

whence we obtain

$$\begin{aligned}e^{1/2}&\le e^{1/2}(0)e^{\int_0^t\|\mathbf{M}(t')\|_{HS}\,dt'}+\int_0^t e^{\int_{t'}^t\|\mathbf{M}(t'')\|_{HS}\,dt''}|\mathbf{b}(t')|\,dt'\\&\le e^{1/2}(0)e^{X\|\|M(t)\|_{HS}\|_{L^\infty}}+X\||\mathbf{b}|(t)\|_{L^\infty}e^{X\|\|M(t)\|_{HS}\|_{L^\infty}}\\&=e^{X\|\|M(t)\|_{HS}\|_{L^\infty}}\left(e^{1/2}(0)+X\||\mathbf{b}|(t)\|_{L^\infty}\right),\end{aligned}$$

from which the result follows by taking $\epsilon\to0$. QED.

We obtain the following three corollaries.

4.3.1. COROLLARY. Let $f$, $g$, $F$ be $C^\infty$ functions on an interval $[0,X]$, and suppose that $f$ satisfies

$$f'+gf=F$$

on $[0,X]$. Then on $[0,X]$, $f$ satisfies

$$|f|\le\left[|f(0)|+X\|F\|_{L^\infty}\right]e^{X\|g\|_{L^\infty}},$$

where $\|\cdot\|_{L^\infty}$ denotes the $L^\infty$ norm on $[0, X]$.

Proof. This is just Proposition 4.3.1 in the case $m = 1$. QED.

4.3.2. COROLLARY. Let $f$, $h$, $g$, $F$ be $C^\infty$ functions on an interval $[0, X]$, and suppose that $f$ satisfies

$$f'' + hf' + gf = F \tag{4.3.1}$$

on $[0, X]$. Then on $[0, X]$, $f$ satisfies

$$\left[|f|^2(t) + |f'|^2(t)\right]^{1/2} \leq \left(\left[|f|^2(0) + |f'|^2(0)\right]^{1/2} + X\|F\|_{L^\infty}\right) e^{X[1+\|h\|_{L^\infty}+\|g\|_{L^\infty}]},$$

where $\|\cdot\|_{L^\infty}$ denotes the $L^\infty$ norm on $[0, X]$.

Proof. Note that the equation (4.3.1) is equivalent to the system

$$f' = u$$
$$u' = -gf - hu + F,$$

which is of the form of that in Proposition 4.3.1 with

$$\mathbf{x} = \begin{pmatrix} f \\ u \end{pmatrix}, \qquad \mathbf{M} = \begin{pmatrix} 0 & 1 \\ -g & -h \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 0 \\ F \end{pmatrix};$$

since for any $t \in [0, X]$

$$|\mathbf{x}(t)| = \left[|f|^2(t) + |f'|^2(t)\right]^{1/2}, \qquad \|\mathbf{M}(t)\|_{HS} \leq 1 + |g(t)| + |h(t)|, \qquad |\mathbf{b}(t)| = |F(t)|,$$

the result follows from Proposition 4.3.1. QED.

4.3.3. COROLLARY. In Corollary 4.3.2, suppose that $h = 0$ and $f(X) = f'(X) = 0$. Then on $[0, X]$ we have

$$|f(t)|, |f'(t)| \leq X\|F\|_{L^\infty} e^{X[1+\|g\|_{L^\infty}]}.$$

Proof. This follows by replacing $x$ by $X - x$ in the differential equation satisfied by $f$. QED.

Clearly, many terms like $X\|g\|_{L^\infty}$ in the foregoing results could be replaced with the $L^1$ norms which they bound. In the applications we shall make of these results below, though, it is more natural to deal with the $L^\infty$ norms, which is why we have stated the foregoing as we have.

We conclude this section with one result on $L^\infty$ Sobolev spaces. Let $U$ be any open set in some linear submanifold of $\mathbf{R}^p$, and let $J$ denote a multiindex in derivatives tangent to $U$. We define the Sobolev norm

$$\|f\|_{W^{m,\infty}(U)} = \sum_{|J|\leq m} \left\|\partial^J f\right\|_{L^\infty(U)} \tag{4.3.2}$$

(this is equivalent to the usual definition with a maximum instead of a sum), let $W^{m,\infty}(U)$ denote the set of all functions on $U$ for which the above is defined and finite, and note the following lemma.

4.3.1. LEMMA. For any $m \geq 0$ there exist constants $C_m$ and $C^i_m$ such that, for any $f, g \in W^{m,\infty}(U)$,

$$\|fg\|_{W^{m,\infty}(U)} \leq C_m \|f\|_{W^{m,\infty}(U)} \|g\|_{W^{m,\infty}(U)},$$

and for any $f \in W^{m,\infty}(U)$ for which $\|1/f\|_{L^\infty(U)}$ is finite,

$$\left\|\frac{1}{f}\right\|_{W^{m,\infty}(U)} \leq C^i_m \left(1 + \|f\|_{W^{m,\infty}(U)}\right)^m \cdot \left\|\frac{1}{f}\right\|^{m+1}_{L^\infty(U)}.$$

Proof. The first inequality is a trivial application of the product rule

$$\partial^I(fg) = \sum_{J \leq I} \binom{I}{J} \partial^{I-J} f \partial^J g,$$

where $I - J$ is the componentwise difference and $J \leq I$ if and only if $I - J$ has nonnegative entries. The second inequality follows from Lemma 4.2.1: if $|I| \leq m$, then

$$\begin{aligned}
\partial^I \frac{1}{f} &= \sum_{\{K_k\} \in \mathcal{K}} C^p_{\{K_k\}} \frac{\prod_{K \in \{K_k\}} \partial^K f}{f^{|\{K_k\}|+1}} \\
&\leq \sum_{\{K_k\} \in \mathcal{K}} C^p_{\{K_k\}} \left(1 + \|f\|_{W^{m,\infty}(U)}\right)^{|\{K_k\}|} \cdot \left\|\frac{1}{f}\right\|^{|\{K_k\}|+1} \\
&\leq C^i_m \left(1 + \|f\|_{W^{m,\infty}(U)}\right)^m \cdot \left\|\frac{1}{f}\right\|^{m+1},
\end{aligned}$$

as claimed. QED.

**4.4. Poincaré- and Sobolev-type inequalities.**

As usual, when deriving energy bounds we shall need to avail ourselves of Poincaré and Sobolev inequalities on spacelike surfaces of constant $\tau$. However, these surfaces shrink to a line in the limit as $\tau \to 0^+$, which means that the standard versions of the Poincaré and Sobolev inequalities cannot be applied directly as the 'constants' could potentially go to infinity. Thus in this section we derive appropriate replacements which can be used in our context.

The general definitions of Sobolev norms in 0.9 give rise to the following special cases. If $X \subset \mathbf{R}^n$ is any open subset of an affine submanifold of $\mathbf{R}^p$, or any such set with some or all of its boundary points included,, and $m \geq 1$, we have, according to (0.9.2),

$$\|f\|^2_{H^m_\circ(X)} = \sum_{1 \leq |I| \leq m} \|\partial^I f\|^2_{L^2(X)}, \tag{4.4.1}$$

where $I$ indicates a multiindex in the directions tangential to $X$. Further, if $L > 0$, we define $\Omega_L = \mathbf{R}^1 \times [0, L]$ and $\underline{\partial}\Omega_L = \mathbf{R}^1 \times \{0\}$. Let $m \geq 1$. Then we have

$$\|f\|^2_{H^1(\underline{\partial}\Omega_L)} = \int_{-\infty}^{+\infty} |f(x,0)|^2\, dx + \int_{-\infty}^{+\infty} |\partial_x f(x,0)|^2\, dx. \tag{4.4.2}$$

Here these definitions are for all functions $f$ for which the right-hand sides exist.

We shall denote a generic point in $\Omega_L$ by $(x, y)$. We have the following two results, which adapt the Poincaré inequality and a Sobolev inequality, respectively, to our situation.

4.4.1. PROPOSITION. Let $L > 0$, and let $f \in H^1(\Omega_L) \cap L^2(\underline{\partial}\Omega_L)$. Then

$$\|f\|_{L^2(\Omega_L)} \le \sqrt{2L}\left[\sqrt{L}\,\|\partial_y f\|_{L^2(\Omega_L)} + \|f\|_{L^2(\underline{\partial}\Omega_L)}\right].$$

Proof. It suffices to show this for $f \in C^\infty(\Omega_L)$, and that case is elementary:

$$\begin{aligned}\|f\|^2_{L^2(\Omega_L)} &= \int_{-\infty}^{+\infty}\int_0^L |f|^2\,dy\,dx \le 2\int_{-\infty}^{+\infty}\int_0^L\left[\left|\int_0^y \partial_y f(x,y')\,dy'\right|^2 + |f(x,0)|^2\right]dy\,dx \\ &\le 2\int_{-\infty}^{+\infty}\int_0^L |f(x,0)|^2 + L\int_0^L |\partial_y f|^2\,dy'\,dx \le 2L\left[L|\partial_y f|^2_{L^2(\Omega_L)} + |f(x,0)|^2_{L^2(\underline{\partial}\Omega_L)}\right],\end{aligned}$$

from which the result follows immediately. QED.

4.4.2. PROPOSITION. Let $L \le \sqrt{2}$, and let $m \ge 2$. There is a constant $C$, independent of $L$, such that for all $f \in H^m(\Omega_L) \cap H^1(\underline{\partial}\Omega_L)$

$$\|f\|_{L^\infty(\Omega_L)} \le C\left[\|f\|_{H^m_\circ(\Omega_L)} + \|f\|_{H^1(\underline{\partial}\Omega_L)}\right]. \tag{4.4.3}$$

Proof. It suffices to show this for $f \in C^\infty(\Omega_L)$ having support which is compact in the first variable. Let $L \le \sqrt{2}$. We note the following elementary version of the Poincaré inequality: if $g \in C^\infty([0, L])$, then for any $x \in [0, L]$,

$$|g(x)| = \left|\int_0^x g'(u)\,du + g(0)\right| \le |g(0)| + \int_0^L |g'(u)|\,du \le |g(0)| + L^{1/2}\|g'\|_{L^2([0,L])},$$

by the Cauchy-Schwartz inequality; in other words, there is a constant $C$, independent of $L$ (more precisely, depending only on an upper bound for $L$, here $\sqrt{2}$), such that for $g \in C^\infty([0, L])$ we have

$$\|g\|^2_{L^\infty([0,L])} \le C\left[\|g'\|^2_{L^2([0,L])} + |g(0)|^2\right].$$

Now let $f \in C^\infty(\Omega_L)$ have support compact in the first variable, and for any $x \in \mathbf{R}^1$ define $f_x \in C^\infty([0, L])$ by $f_x(y) = f(x, y)$. Then clearly for all $x \in \mathbf{R}^1$ we have (here $f_x$ is a function of one variable, and $f'_x$ is its derivative, $f'_x(y) = \partial_y f(x, y)$)

$$\|f_x(y)\|^2_{L^\infty([0,L])} \le C\left[\|f'_x\|^2_{L^2([0,L])} + |f(x,0)|^2\right].$$

Further,

$$\|f(x,y)\|^2_{L^\infty(\Omega_L)} \le \left\|\|f_x(y)\|^2_{L^\infty([0,L])}\right\|_{L^\infty(\mathbf{R}^1)} \le C\left\|\left[\|f'_x\|^2_{L^2([0,L])} + |f(x,0)|^2\right]\right\|_{L^\infty(\mathbf{R}^1)};$$

now if we let $C'$ denote the Sobolev embedding constant on $\mathbf{R}^1$, we have

$$\||f(x,0)|^2\|_{L^\infty(\mathbf{R}^1)} = \|f(x,0)\|^2_{L^\infty(\mathbf{R}^1)} \le C'^2\left[\|f(x,0)\|^2_{L^2(\mathbf{R}^1)} + \|\partial_x f(x,0)\|^2_{L^2(\mathbf{R}^1)}\right]$$

and, since $f$ has compact support in $x$,

$$\Big\| \|f_x'\|^2_{L^2([0,L])} \Big\|_{L^\infty(\mathbf{R}^1)} \le \int_{-\infty}^{\infty} dx \left| \partial_x \int_0^L |\partial_y f|^2 \, dy \right| = \int_{-\infty}^{\infty} \int_0^L 2|\partial_y f \partial_x \partial_y f| \, dy \, dx$$

$$\le \int_{-\infty}^{\infty} \int_0^L |\partial_y f|^2 + |\partial_x \partial_y f|^2 \, dy \, dx \le \|f\|^2_{H^m_\circ(\Omega_L)},$$

so finally

$$\|f(x,y)\|^2_{L^\infty(\Omega_L)} \le 2C(C'+1)^2 \left[ \|f\|^2_{H^m_\circ(\Omega_L)} + \|f\|^2_{H^1(\underline{\partial}\Omega_L)} \right],$$

from which the result follows. QED.

We shall denote the constant in this proposition by $C_0$ when necessary.

We now wish to prove bounds on norms of products. We begin with the following extension lemma. We let $C_c^\infty(\Omega_L)$ denote the set of functions on $\Omega_L$ which are $C^\infty$ and have compact support.

4.4.1. Lemma. Let $L \le \sqrt{2}$, let $m \ge 0$, and let $\phi \in C^\infty(\mathbf{R}^1)$ have support contained in $[-1,3]$ and satisfy $\phi|_{[-1/2,2]} = 1$. Then there is an extension map $e : C_c^\infty(\Omega_L) \to H^m_0(\mathbf{R}^1 \times [-1,3])$ such that

(i) $e(f)|_{\Omega_L} = f$,

(ii) $\|e(f)\|_{H^m(\mathbf{R}^1 \times [-1,3])} \le C^e \left[ \|f\|_{H^m(\Omega_L)} + \sum_{\ell=0}^m \left\| \partial_y^\ell f \right\|_{H^{m-\ell}(\underline{\partial}\Omega_L)} \right]$,

where $\partial_y$ denotes the one-sided derivative into $\Omega_L$, and $C^e$ is a constant depending only on $m$ and $\phi$ (in particular, $C^e$ is independent of $L$ and the size of the support of $f$).

Proof. The main idea is to extend $\partial_y^m f$ by 0 and then integrate $m$ times in $y$, multiplying by a cutoff at the very end. We first show how to do this in one dimension. Fix some $f \in L^2([-1,3])$, and let $a_0$, $a_1$, $\cdots$, $a_m$ be a sequence of real numbers. We first define, for $x \in [-1,3]$,

$$I(f,a)(x) = a + \int_0^x f(t) \, dt.$$

Now since $[-1,3]$ has finite measure, we have $f \in L^1([-1,3])$ as well, so that $I(f,a)(x)$ is differentiable almost everywhere and

$$\frac{d}{dx} I(f,a)(x) = f(x)$$

as functions in $L^1$. Thus

$$\|I(f,a)\|_{H^1_\circ([-1,3])} = \|f\|_{L^2([-1,3])}.$$

We now claim that

$$\|I(f,a)\|_{L^2([-1,3])} \le 4(|a| + \|f\|_{L^2([-1,3])}).$$

It suffices to show this for $f \in C^\infty([-1,3])$. For such $f$, we have clearly

$$\|I(f,a)\|_{L^2([-1,3])} \le 2|a| + \left\| \int_0^x f(t) \, dt \right\|_{L^2([-1,3])},$$

$$\left\| \int_0^x f(t) \, dt \right\|^2_{L^2([-1,3])} \le \int_{-1}^3 \left[ \int_0^x f(t) \, dt \right]^2 dx \le \int_{-1}^3 |x| \left| \int_0^x |f(t)|^2 \, dt \right| dx$$

$$\le 3 \int_{-1}^3 \int_{-1}^3 |f(t)|^2 \, dt \, dx = 12 \|f\|^2_{L^2([-1,3])},$$

so

$$\|I(f,a)\|_{L^2([-1,3])} \le 4\left(|a| + \|f\|_{L^2([-1,3])}\right),$$

as claimed. Thus we have

$$\|I(f,a)\|_{H^1([-1,3])} \le 5(|a| + \|f\|_{L^2([-1,3])}).$$

We now define $I_k^\circ(f)$ inductively by

$$I_1^\circ(f) = I(f,a_0), \qquad I_{k+1}^\circ(f) = I(I_k^\circ(f), a_k).$$

We claim that $\|I_k^\circ(f)\|_{L^2([-1,3])} \le 4^k \left(\sum_{\ell=0}^{k-1} |a_\ell| + \|f\|_{L^2([-1,3])}\right)$. This may be shown by induction:

$$\begin{aligned} \|I_1^\circ(f)\|_{L^2([-1,3])} &= \|I(f,a_0)\|_{L^2([-1,3])} \le 4\left(|a_0| + \|f\|_{L^2([-1,3])}\right), \\ \|I_{k+1}^\circ(f)\|_{L^2([-1,3])} &= \|I(I_k^\circ(f),a_k)\|_{L^2([-1,3])} \le 4\left(|a_k| + \|I_k^\circ(f)\|_{L^2([-1,3])}\right) \\ &\le 4\left(|a_k| + 4^k\left[\sum_{\ell=0}^{k-1}|a_\ell| + \|f\|_{L^2([-1,3])}\right]\right) \le 4^{k+1}\left(\sum_{\ell=0}^{k}|a_\ell| + \|f\|_{L^2([-1,3])}\right), \end{aligned}$$

establishing the claim. Now we note moreover that

$$\frac{d}{dx} I_{k+1}^\circ(f) = I_k^\circ(f); \tag{4.4.4}$$

thus

$$\|I_k^\circ(f)\|_{H^k([-1,3])} \le k4^k \left(\sum_{\ell=0}^{k}|a_\ell| + \|f\|_{L^2([-1,3])}\right),$$

and similarly there must be some constant $C$ depending only on $\phi$ and $k$ such that

$$\|\phi I_k^\circ(f)\|_{H^k([-1,3])} \le k4^k C \left(\sum_{\ell=0}^{k}|a_\ell| + \|f\|_{L^2([-1,3])}\right). \tag{4.4.5}$$

We now define the extension map $e$. It suffices to work with $f \in C^\infty(\Omega_L)$ with support compact in the first variable. Pick such an $f$, and define first

$$F(x,y) = \begin{cases} \partial_y^m f(x,y), & y \in [0,L] \\ 0, & \text{otherwise} \end{cases}$$

clearly $F(x,\cdot) \in L^2([-1,3])$ for every $x$. We then define

$$e(f)(x,y) = \phi(y) I_m^\circ(F(x,\cdot))(y),$$

where the sequence for each $I_m^\circ$ is $a_\ell = (\partial_y^\ell f)(x,0)$. Clearly $e(f)$ is zero outside $\mathbf{R}^1 \times [-1,3]$. It is also clearly $C^\infty$ in $x$. Further, the bound (4.4.5) will hold with $k = m$ and $f = F(x,\cdot)$. Thus, noting that all of this holds with $f$ replaced by $\partial_x^i f$ for any $i$, and that differentiating by $x$ commutes with $e$, we have

$$\begin{aligned} \|e(f)\|_{H^m(\mathbf{R}^1\times[-1,3])} &\le \sum_{j+k\le m} \|\partial_x^j \partial_y^k e(f)\|_{L^2(\mathbf{R}^1\times[-1,3])} \\ &= \sum_{j+k\le m} \|\partial_y^k e(\partial_x^j f)\|_{L^2(\mathbf{R}^1\times[-1,3])} \\ &\le \sum_{k=0}^{m}\sum_{j=0}^{m-k} k4^k C \left\| \sum_{\ell=0}^{k} |\partial_x^j \partial_y^\ell f(x,0)| + \|\partial_x^j f(x,\cdot)\|_{L^2([-1,3])} \right\|_{L^2(\mathbf{R}^1)} \\ &\le C'\left[\|f\|_{H^m(\Omega_L)} + \sum_{\ell=0}^{m} \|(\partial_y^\ell f)(x,0)\|_{H^{m-\ell}(\underline{\partial}\Omega_L)}\right], \end{aligned}$$

completing the proof. QED.

It is worth noting that the $H^m$ norm on the right-hand side of the inequality in (ii) can be replaced with a norm only on the $x$ derivatives of $f$. Further, by continuity, the map $e$ can clearly be extended to the set of all functions $f \in H^m(\Omega_L)$ which satisfy also $\partial_y^\ell f \in H^{m-\ell}(\underline{\partial}\Omega_L)$, $\ell = 0, \cdots, m$ (in a trace sense, for example).

This allows us to derive the following lemma; see [12], Lemma 6.16.

4.4.2. LEMMA. Let $m \geq 2$, $L \leq \sqrt{2}$. Let $f_1, \cdots, f_k \in H^m(\Omega_L)$ satisfy $\partial_y^\ell f_i \in H^{m-\ell}(\underline{\partial}\Omega_L)$, $\ell = 0, \cdots, m$, $i = 1, \cdots, k$, and let $I_1, \cdots, I_k$ be multiindices with $|I_1 + \cdots + I_k| \leq m$. Then there is a constant $C$ such that

$$\left\|\partial^{I_1} f_1 \cdots \partial^{I_k} f_k\right\|_{L^2(\Omega_L)} \leq C\,(C^e)^k \prod_{i=1}^k \left( \sum_{\ell=0}^m \|\partial_y^\ell f_i\|_{H^{m-\ell}(\underline{\partial}\Omega_L)} + \|f_i\|_{H^m(\Omega_L)} \right).$$

Proof. This follows from the result just cited, together with the embedding in Lemma 4.4.1 and the Sobolev inequality on $\mathbf{R}^2$: specifically, for $i = 1, \cdots, k$, let $\tilde{f}_i$ be the extension by 0 of $e(f_i)$ to all of $\mathbf{R}^2$; then by [12], Lemma 6.16, and the Sobolev inequality on $\mathbf{R}^2$ we have

$$\left\|\partial^{I_1} \tilde{f}_1 \cdots \partial^{I_k} \tilde{f}_k\right\|_{L^2(\mathbf{R}^2)} \leq C \prod_{i=1}^k \|\tilde{f}_i\|_{H^m(\mathbf{R}^2)}.$$

But by the lemma $\|\tilde{f}_i\|_{H^m(\mathbf{R}^2)} \leq C^e(\sum_{\ell=0}^m \|\partial_y^\ell f_i\|_{H^{m-\ell}(\underline{\partial}\Omega_L)} + \|f_i\|_{H^m(\Omega_L)})$, so that we have

$$\left\|\partial^{I_1} f_1 \cdots \partial^{I_k} f_k\right\| \leq \left\|\partial^{I_1} \tilde{f}_1 \cdots \partial^{I_k} \tilde{f}_k\right\|_{L^2(\mathbf{R}^2)} \leq C\,(C^e)^m \prod_{i=1}^k \left( \sum_{\ell=0}^m \|\partial_y^\ell f_i\|_{H^{m-\ell}(\underline{\partial}\Omega_L)} + \|f_i\|_{H^m(\Omega_L)} \right),$$

as claimed. QED.

The constant can evidently be taken to depend only on $m$ (by taking a maximum over multiindices); we denote it by $C_M$ when convenient.

The form of the next result follows as an easy corollary, but we give a separate derivation which is more careful with the constant.

4.4.3. LEMMA. Let $m \geq 2$, $L \leq \sqrt{2}$. There is a constant $C \geq 1$ such that if $f_1, f_2, \cdots, f_k \in H^m(\Omega_L)$ satisfy $\partial_y^\ell f_i \in H^{m-\ell}(\underline{\partial}\Omega_L)$, $\ell = 0, \cdots, m$, $i = 1, \cdots, k$,

$$\|f_1 \cdots f_k\|_{H^m(\Omega_L)} \leq C^k \prod_{i=1}^k \left( \sum_{\ell=0}^m \|\partial_y^\ell f_i\|_{H^{m-\ell}(\underline{\partial}\Omega_L)} + \|f_i\|_{H^m(\Omega_L)} \right).$$

Proof. As in the lemma, the key point here is that the constant $C$ does not depend on $L$. Let $f, g \in H_0^m(\mathbf{R}^2)$; then there is a constant $C'$ such that

$$\|f \cdot g\|_{H^m(\mathbf{R}^2)} \leq C' \|f\|_{H^m(\mathbf{R}^2)} \|g\|_{H^m(\mathbf{R}^2)}.$$

We may assume $C' \geq 1$ without loss of generality. Now by the lemma, we may extend the $f_i$ to functions $e(f_i)$ in $H_0^m(\mathbf{R}^1 \times [-1, 3])$; extending by 0 outside of $\mathbf{R}^1 \times [-1, 3]$, we obtain functions $\tilde{f}_i$ on $\mathbf{R}^2$, which still satisfy

$$\|\tilde{f}_i\|_{H^m(\mathbf{R}^2)} \leq C^e \left( \sum_{\ell=0}^m \|\partial_y^\ell f_i\|_{H^{m-\ell}(\underline{\partial}\Omega_L)} + \|f_i\|_{H^m(\Omega_L)} \right).$$

Thus finally

$$\|f_1 f_2 \cdots f_k\|_{H^m(\Omega_L)} \leq \|\tilde{f}_1 \cdot \tilde{f}_2 \cdots \tilde{f}_k\|_{H^m(\mathbf{R}^2)} \leq {C'}^{k-1} \|\tilde{f}_1\|_{H^m(\mathbf{R}^2)} \|\tilde{f}_2\|_{H^m(\mathbf{R}^2)} \cdots \|\tilde{f}_k\|_{H^m(\mathbf{R}^2)}$$
$$\leq {C'}^k (C^e)^k \prod_{i=1}^{k} (\sum_{\ell=0}^{m} \|\partial_y^\ell f_i\|_{H^{m-\ell}(\underline{\partial}\Omega_L)} + \|f_i\|_{H^m(\Omega_L)}),$$

so taking $C = \max\{C' C^e, 1\}$ (and recalling that $C^e$ is independent of $L$) gives the result. QED.

We denote the constant $C$ by $C^M$ or $C^M_m$ when convenient.

# 5. EXISTENCE OF INITIAL DATA

## 5.1. Introduction

In this chapter we shall construct initial data for the system (3.3.5 – 3.3.8),

$$\Bigg[\left(-2\partial_{\overline{s}}\partial_{\overline{v}}+\partial_{\overline{x}}^2\right)+\frac{1}{k}\left(2\frac{\overline{b}}{\overline{a}}\partial_{\overline{s}}\partial_{\overline{x}}-\overline{c}\partial_{\overline{s}}^2-\overline{\delta^{-1}a}\partial_{\overline{x}}^2-\left(\partial_{\overline{s}}\overline{c}-\frac{1}{\overline{a}}\partial_{\overline{x}}\overline{b}+\frac{\partial_{\overline{v}}\overline{\delta\ell}}{\overline{\ell}}\right)\partial_{\overline{s}}+\left(\frac{1}{\overline{a}}\partial_{\overline{s}}\overline{b}-\frac{\overline{\ell}\partial_{\overline{x}}\overline{\delta\ell}}{\overline{a}^2}\right)\partial_{\overline{x}}-\frac{\partial_{\overline{s}}\overline{\delta\ell}}{\overline{\ell}}\partial_{\overline{v}}\right)$$

$$+\frac{1}{k^2}\left(\frac{\overline{b}^2}{\overline{a}}\partial_{\overline{s}}^2-\left(\overline{c}\frac{\partial_{\overline{s}}\overline{\delta\ell}}{\overline{\ell}}-2\frac{\overline{b}}{\overline{a}}\partial_{\overline{s}}\overline{b}\right)\partial_{\overline{s}}-\frac{\overline{b}\overline{\ell}\partial_{\overline{s}}\overline{\delta\ell}}{\overline{a}^2}\partial_{\overline{x}}\right)-\frac{1}{k^3}\frac{\overline{b}^2\overline{\ell}\partial_{\overline{s}}\overline{\delta\ell}}{\overline{a}^2}\partial_{\overline{s}}\Bigg]\overline{\gamma}=0 \tag{3.3.5}$$

$$\partial_{\overline{s}}^2\overline{a}=\frac{(\partial_{\overline{s}}\overline{a})^2}{2\overline{a}}-4\overline{a}k^{-2\iota}(\partial_{\overline{s}}\overline{\gamma})^2,\qquad \partial_{\overline{s}}^2\overline{\ell}=-2\overline{\ell}k^{-2\iota}(\partial_{\overline{s}}\overline{\gamma})^2,\qquad \partial_{\overline{s}}^2\overline{\delta\ell}=-2\overline{\ell}k^{1-2\iota}(\partial_{\overline{s}}\overline{\gamma})^2 \tag{3.3.6}$$

$$\partial_{\overline{s}}^2\overline{b}=\frac{1}{2\overline{a}}(\partial_{\overline{s}}\overline{a})(\partial_{\overline{s}}\overline{b})-4k^{-2\iota}\partial_{\overline{s}}\overline{\gamma}(\overline{b}\partial_{\overline{s}}\overline{\gamma}+k\partial_{\overline{x}}\overline{\gamma})=\frac{1}{\overline{\ell}}k^{-1}(\partial_{\overline{s}}\overline{\delta\ell})(\partial_{\overline{s}}\overline{b})-4k^{1-2\iota}\partial_{\overline{s}}\overline{\gamma}(\partial_{\overline{x}}\overline{\gamma}+k^{-1}\overline{b}\partial_{\overline{s}}\overline{\gamma}) \tag{3.3.7}$$

$$\partial_{\overline{s}}^2\overline{c}=k^{-1}\frac{(\partial_{\overline{s}}\overline{b})^2}{2\overline{a}}-2k^{1-2\iota}\partial_{\overline{s}}\overline{\gamma}\left(2\partial_{\overline{v}}\overline{\gamma}+2k^{-1}\frac{\overline{b}}{\overline{a}}\partial_{\overline{x}}\overline{\gamma}+k^{-1}\left(\overline{c}+k^{-1}\frac{\overline{b}^2}{\overline{a}}\right)\partial_{\overline{s}}\overline{\gamma}\right)-k^{1-2\iota}\frac{2}{\overline{a}}(\partial_{\overline{x}}\overline{\gamma})^2 \tag{3.3.8}$$

which satisfy the following two conditions:

(1) The solution obtained from the initial data, when substituted into the metric (0.2.1) via (cf. (0.3.35))

$$a=[1+k^{-1}\overline{\delta\ell}]^2,\qquad b=k^{-1/2}\overline{b},\qquad c=\overline{c},\qquad \gamma=k^{-1/2}\overline{\gamma}, \tag{5.1.1}$$

will give a solution to the Einstein vacuum equations;

(2) The initial data, together with their transverse derivatives on $\Sigma_0$ and $U_0$, satisfy the bounds (0.3.31 – 0.3.33) on $\Sigma_0$ and $U_0$. More specifically, this requires us to find bounds* on the quantities

$$\partial^J\partial_{\overline{s}}^\ell\overline{\delta\ell},\qquad \partial^J\partial_{\overline{s}}^\ell\overline{b},\qquad \partial^J\partial_{\overline{s}}^\ell\overline{c} \tag{5.1.2}$$

on $\Sigma_0$, and bounds on

$$\partial^I\partial_{\overline{s}}^\ell\overline{\gamma},\qquad \partial^J\partial_{\overline{s}}^\ell\partial_\tau\overline{\gamma}, \tag{5.1.3}$$

on $\Sigma_0\cup U_0$, where in all cases $I$ and $J$ are multiindices in $\overline{x}$ and $\overline{v}$ and $\ell\in\{0,1\}$, and we recall (see equation (0.3.19)) that $\tau=(\overline{s}+\overline{v})/\sqrt{2}$.

Satisfying condition (1) comes down to finding initial data which satisfy the constraint equations (0.2.24 – 0.2.26) and the gauge conditions, see Corollary 2.4.1. Let us lay out the process systematically. It is clear that sufficient initial data for the system (3.3.5 – 3.3.8) by itself, in isolation from condition (1), is given by

$$\partial_{\overline{s}}^\ell\overline{\delta\ell}|_{\overline{s}=0},\qquad \partial_{\overline{s}}^\ell\overline{b}|_{\overline{s}=0},\qquad \partial_{\overline{s}}^\ell\overline{c}|_{\overline{s}=0}, \tag{5.1.4}$$

$$\overline{\gamma}|_{\overline{s}=0},\qquad \overline{\gamma}|_{\overline{v}=0}, \tag{5.1.5}$$

where $\ell=0,1$, and where the functions in (5.1.5) satisfy the consistency conditions (1.3.2). The gauge condition (2.4.4) fixes three of the six functions in (5.1.4):

$$\overline{b}|_{\overline{s}=0}=\overline{c}|_{\overline{s}=0}=\partial_{\overline{s}}\overline{c}|_{\overline{s}=0}=0. \tag{5.1.6}$$

* These bounds will be with respect to various different norms. We shall obtain data which are compactly supported, in the which case the norms of the initial data in all of these spaces are bounded by $L^\infty$ norms.

Given $\overline{\gamma}|_{\overline{s}=0}$, the constraint equations, written in the form (0.2.24 – 0.2.26)

$$\frac{\partial^2 \overline{\delta\ell}}{\partial \overline{v}^2} = -2(1+k^{-1}\overline{\delta\ell})k^{1-2\iota}\left(\partial_{\overline{v}}\overline{\gamma}\right)^2 \tag{0.2.24}$$

$$\partial_{\overline{v}}\left([1+k^{-1}\overline{\delta\ell}]\,\partial_{\overline{s}}\overline{b}\right) = 4(1+k^{-1}\overline{\delta\ell})k^{1-2\iota}\partial_{\overline{v}}\overline{\gamma}\partial_{\overline{x}}\overline{\gamma} \tag{0.2.25}$$

$$2(1+k^{-1}\overline{\delta\ell})\cdot\frac{\partial^2\overline{\delta\ell}}{\partial\overline{v}\partial\overline{s}} = (1+k^{-1}\overline{\delta\ell})\partial_{\overline{x}}\left([1+k^{-1}\overline{\delta\ell}]^{-1}\partial_{\overline{s}}\overline{b}\right) + \frac{1}{2k}\left(\partial_{\overline{s}}\overline{b}\right)^2 + 2k^{1-2\iota}\left(\partial_{\overline{x}}\overline{\gamma}\right)^2, \tag{0.2.26}$$

give ordinary differential equations in $\overline{v}$ whose solution will give the remaining three functions in (5.1.4),

$$\overline{\delta\ell}|_{\overline{s}=0}, \qquad \partial_{\overline{s}}\overline{\delta\ell}|_{\overline{s}=0}, \qquad \partial_{\overline{s}}\overline{b}|_{\overline{s}=0}, \tag{5.1.7}$$

given, for some $\overline{v}_0 \geq 0$, the values (see Proposition 2.4.1)

$$\overline{\delta\ell}|_{\overline{s}=0,\,\overline{v}=\overline{v}_0}, \qquad \partial_{\overline{v}}\overline{\delta\ell}|_{\overline{s}=0,\,\overline{v}=\overline{v}_0}, \qquad \partial_{\overline{s}}\overline{\delta\ell}|_{\overline{s}=0,\,\overline{v}=\overline{v}_0}, \qquad \partial_{\overline{s}}\overline{b}|_{\overline{s}=0,\,\overline{v}=\overline{v}_0}.$$

To sum up, we have the following lemma.

5.1.1. Lemma. Suppose given the quantities

$$\overline{\gamma}|_{\overline{s}=0}, \qquad \overline{\gamma}|_{\overline{v}=0}, \qquad \overline{\delta\ell}|_{\overline{s}=0,\,\overline{v}=\overline{v}_0}, \qquad \partial_{\overline{v}}\overline{\delta\ell}|_{\overline{s}=0,\,\overline{v}=\overline{v}_0}, \qquad \partial_{\overline{s}}\overline{\delta\ell}|_{\overline{s}=0,\,\overline{v}=\overline{v}_0}, \qquad \partial_{\overline{s}}\overline{b}|_{\overline{s}=0,\,\overline{v}=\overline{v}_0}, \tag{5.1.8}$$

where $\overline{v}_0$ is such that $\{(0,\overline{x},\overline{v}_0)\,|\,\overline{x}\in\mathbf{R}\}\subset\Sigma_0$, and $\overline{\gamma}|_{\overline{s}=0}$ and $\overline{\gamma}|_{\overline{v}=0}$ satisfy the consistency conditions (1.3.2). Then on some neighbourhood of $\{(0,\overline{x},\overline{v}_0)\in\Sigma_0\,|\,\overline{x}\in\mathbf{R}\}$ in $\Sigma_0$ there is a unique set of initial data (5.1.4), (5.1.5) taking the values indicated in (5.1.8) and satisfying the constraint equations (0.2.24 – 0.2.26). Any solution to (3.3.5 – 3.3.8) corresponding to such initial data will give, via (5.1.1) and (0.2.1), a solution to the Einstein vacuum equations.

Proof. The first statement follows from the foregoing, and the second follows from Corollary 2.4.1.QED.

This takes care of condition (1).

Condition (2) is much trickier. Obtaining bounds on the initial data (5.1.4 – 5.1.5) and their derivatives *tangent* to the initial hypersurfaces is not difficult. Note however that obtaining bounds on the quantities in (5.1.2 – 5.1.3) – which are necessary to allow us to close our estimates with respect to the energies $\overline{E}_n[\gamma]$ and $E_n[h]$ (see equation (0.3.30) and equation (6.2.26)) – will require, on both hypersurfaces, bounds on derivatives with respect to

$$\zeta = \frac{1}{\sqrt{2}}(\overline{s}-\overline{v}),$$

and hence will require that we bound derivatives with respect to $\overline{s}$ on $\Sigma_0$ and with respect to $\overline{v}$ on $U_0$, for all four functions $\overline{\delta\ell}$, $\overline{b}$, $\overline{c}$, $\overline{\gamma}$. (See also the conditions (0.3.31 – 0.3.33) and the definitions (0.3.26 – 0.3.27).) For $\overline{\delta\ell}$, $\overline{b}$, and $\overline{c}$, this can be done using the Riccati equations (3.3.6 – 3.3.8) and the constraint equations (0.2.24 – 0.2.26). For $\overline{\gamma}$, note that the wave equation (3.3.5) on $\Sigma_0$ simplifies by (5.1.6) to

$$\left[-2\partial_{\overline{s}}\partial_{\overline{v}} + \partial_{\overline{x}}^2 + k^{-1}\left(-\overline{\delta^{-1}}\overline{a}\partial_{\overline{x}}^2 - \frac{\partial_{\overline{v}}\overline{\delta\ell}}{\overline{\ell}}\partial_{\overline{s}} + \left(\frac{1}{\overline{a}}\partial_{\overline{s}}\overline{b} - \frac{\overline{\ell}\partial_{\overline{x}}\overline{\delta\ell}}{\overline{a}^2}\right)\partial_{\overline{x}} - \frac{\partial_{\overline{s}}\overline{\delta\ell}}{\overline{\ell}}\partial_{\overline{v}}\right)\right]\overline{\gamma} = 0; \tag{5.1.9}$$

given the quantities in (5.1.7), as well as $\overline{\gamma}|_{\overline{s}=0}$, this is an ordinary differential equation in $\overline{v}$ for $\partial_{\overline{s}}\overline{\gamma}|_{\overline{s}=0}$, and hence we can obtain $\partial_{\overline{s}}\overline{\gamma}|_{\overline{s}=0}$ given

$$\partial_{\overline{s}}\overline{\gamma}|_{\overline{s}=0,\,\overline{v}=\overline{v}_0}$$

for $\overline{v}_0$ as before. Similarly, as we shall discuss in more detail later (see Proposition 5.2.2), differentiating (3.3.5 – 3.3.8) with respect to $\overline{s}$ allows us to obtain

$$\partial_{\overline{s}}^{\ell}\overline{\delta\ell}|_{\overline{s}=0}, \qquad \partial_{\overline{s}}^{\ell}\overline{b}|_{\overline{s}=0}, \qquad \partial_{\overline{s}}^{\ell}\overline{c}|_{\overline{s}=0}, \qquad \partial_{\overline{s}}^{\ell}\overline{\gamma}|_{\overline{s}=0} \tag{5.1.10}$$

for $\ell = 0, 1, \cdots, m$ given

$$\partial_{\overline{s}}^{\ell}\overline{\delta\ell}|_{\overline{s}=0,\,\overline{v}=\overline{v}_0}, \qquad \partial_{\overline{s}}^{\ell}\overline{b}|_{\overline{s}=0,\,\overline{v}=\overline{v}_0}, \qquad \partial_{\overline{s}}^{\ell}\overline{c}|_{\overline{s}=0,\,\overline{v}=\overline{v}_0}, \qquad \partial_{\overline{s}}^{\ell}\overline{\gamma}|_{\overline{s}=0,\,\overline{v}=\overline{v}_0},$$

for $\ell = 0, 1, \cdots, m$, and moreover to obtain $L^\infty$ bounds for the quantities in (5.1.10) in terms of $L^\infty$ bounds on $\overline{\gamma}|_{\overline{s}=0}$, $\partial_{\overline{x}}\overline{\gamma}|_{\overline{s}=0}$, and $\partial_{\overline{v}}\overline{\gamma}|_{\overline{s}=0}$, all of which are known completely once $\overline{\gamma}|_{\overline{s}=0}$ is specified. (There is a loss of derivative in these bounds but this does not cause any difficulties here as $\overline{\gamma}|_{\overline{s}=0}$ is entirely specified.) A similar method (which is however rather more involved, due to some nontrivial coupling; see Proposition 5.3.2) works to determine the transverse derivatives $\partial_{\overline{v}}^{\ell}\overline{\gamma}|_{\overline{v}=0}$, etc.. Further differentiating with respect to tangential derivatives ($\partial_{\overline{x}}$ and $\partial_{\overline{v}}$ on $\Sigma_0$, $\partial_{\overline{x}}$ and $\partial_{\overline{s}}$ on $U_0$) allows us to obtain bounds on the tangential derivatives of the quantities in (5.1.10). This is all spelled out in great detail in Section 5.2 and Section 5.3.

On $\Sigma_0$ there is an extra issue to be dealt with, which does not come up on $U_0$. Note (compare our discussion at the end of Section 0.2) that to determine the quantities in (5.1.10) we must integrate equations (0.2.24 – 0.2.26) and (5.1.9) and its $\overline{s}$ derivatives *with respect to* $\overline{v}$ – and $\overline{v}$ ranges over a interval $[0, kT\sqrt{2}]$ with length of size $k$. In order for the $\overline{s}$ derivatives of $\overline{\gamma}$ to have bounds in $L^2(\Sigma_0)$ which are independent of $k$ – as required by (0.3.31) – we must be able to prove that this integration does not lead to factors of $k$. The simplest way to ensure this is suggested by the following proposition.*

5.1.1. PROPOSITION. Let $C > 0$ be some constant independent of $k$, let $\delta\overline{v}_1, \delta\overline{v}_2 \in [0, C]$, and suppose that the set

$$\Sigma^* = \{(0, \overline{x}, \overline{v}) \in \Sigma_0 \,|\, \overline{v} \in (\delta\overline{v}_1, kT\sqrt{2} - \delta\overline{v}_2)\} \tag{5.1.11}$$

satisfies

$$\Sigma^* \cap \operatorname{supp}\overline{\gamma}|_{\Sigma_0} = \emptyset. \tag{5.1.12}$$

If $\overline{v}_0 \in (\delta\overline{v}_1, kT\sqrt{2} - \delta\overline{v}_2)$ and $\overline{\delta\ell}$, $\partial_{\overline{s}}\overline{\delta\ell}$, $\partial_{\overline{s}}\overline{b}$ is the solution to (0.2.24 – 0.2.26) on $\Sigma_0$ satisfying

$$\overline{\delta\ell}|_{\overline{s}=0,\,\overline{v}=\overline{v}_0} = 0, \qquad \partial_{\overline{v}}\overline{\delta\ell}|_{\overline{s}=0,\,\overline{v}=\overline{v}_0} = 0, \qquad \partial_{\overline{s}}\overline{\delta\ell}|_{\overline{s}=0,\,\overline{v}=\overline{v}_0} = 0, \qquad \partial_{\overline{s}}\overline{b}|_{\overline{s}=0,\,\overline{v}=\overline{v}_0} = 0,$$

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* See also the footnote at the end of Section 0.2, and the discussion in Section 0.5 – we believe it may be possible to deal with this issue by other methods, such as, for example, by using a special class of functions for $\overline{\gamma}$ which will make all terms of higher order in $k$ vanish identically which might appear upon integrating (5.1.9) and its $\overline{s}$ derivatives. That is however beyond the scope of the current work.

then

$$\overline{\delta\ell}|_{\overline{s}=0} = 0, \qquad \partial_{\overline{v}}\overline{\delta\ell}|_{\overline{s}=0} = 0, \qquad \partial_{\overline{s}}\overline{\delta\ell}|_{\overline{s}=0} = 0, \qquad \partial_{\overline{s}}\overline{b}|_{\overline{s}=0} = 0$$

on $\Sigma^*$.

Proof. Note that, by equation (5.1.12), on $\Sigma^*$ the constraint equations (0.2.24 – 0.2.26) become

$$\frac{\partial^2\overline{\delta\ell}}{\partial\overline{v}^2} = 0 \tag{5.1.13}$$

$$\partial_{\overline{v}}\left([1+k^{-1}\overline{\delta\ell}]\,\partial_{\overline{s}}\overline{b}\right) = 0 \tag{5.1.14}$$

$$2(1+k^{-1}\overline{\delta\ell})\cdot\frac{\partial^2\overline{\delta\ell}}{\partial\overline{v}\partial\overline{s}} = (1+k^{-1}\overline{\delta\ell})\partial_{\overline{x}}\left([1+k^{-1}\overline{\delta\ell}]^{-1}\partial_{\overline{s}}\overline{b}\right) + \frac{1}{2k}\left(\partial_{\overline{s}}\overline{b}\right)^2. \tag{5.1.15}$$

(5.1.13 – 5.1.14) together with the conditions $\overline{\delta\ell}|_{\overline{s}=0,\,\overline{v}=\overline{v}_0} = \partial_{\overline{v}}\overline{\delta\ell}|_{\overline{s}=0,\,\overline{v}=\overline{v}_0} = \partial_{\overline{s}}\overline{b}|_{\overline{s}=0,\,\overline{v}=\overline{v}_0} = 0$ clearly give $\overline{\delta\ell} = \partial_{\overline{s}}\overline{b} = 0$ on $\Sigma^*$, whence (5.1.15) becomes

$$2\frac{\partial^2\overline{\delta\ell}}{\partial\overline{v}\partial\overline{s}} = 0,$$

and $\partial_{\overline{s}}\overline{\delta\ell}|_{\overline{s}=0,\,\overline{v}=\overline{v}_0} = 0$ implies $\partial_{\overline{s}}\overline{\delta\ell} = 0$ on $\Sigma^*$ also. QED.

Because of this proposition, we shall assume for the rest of this chapter that all our choices of $\overline{\gamma}|_{\overline{s}=0}$ are of the following form. Let $\delta\overline{v}_1$, $\delta\overline{v}_2 \in \mathbf{R}$, $\delta\overline{v}_1$, $\delta\overline{v}_2 \in (0,1)$ be two fixed numbers,* independent of $k$, and assume that $k$ is large enough that $kT/\sqrt{2} \in (\delta\overline{v}_1, kT\sqrt{2}-\delta\overline{v}_2)$. Let $\varpi_1$, $\varpi_2$ be $C^\infty$ functions on $\mathbf{R}^2$ with support contained in

$$[0,1]\times[0,\delta\overline{v}_1], \qquad [0,1]\times[0,\delta\overline{v}_2],$$

respectively, and which, together with all of their derivatives, have $L^\infty$ bounds on $\mathbf{R}^2$ which are independent of $k$. (This condition will be satisfied, for example, if $\varpi_1$ and $\varpi_2$ are fixed functions independent of $k$.) We assume in particular that they satisfy

$$\|\partial_{\overline{v}}\varpi_i\|_{L^\infty} \le \frac{1}{2}, \|\partial_{\overline{x}}\varpi_i\|_{L^\infty} \le \frac{1}{2}. \tag{5.1.16}$$

Now define $\varpi_0(\overline{x},\overline{v})$ on $\Sigma_0$ by

$$\varpi_0(\overline{x},\overline{v}) = \begin{cases} \varpi_1(\overline{x},\overline{v}), & \overline{v}\in[0,\delta\overline{v}_1] \\ 0, & \overline{v}\in[\delta\overline{v}_1, kT\sqrt{2}-\delta\overline{v}_2] \\ \varpi_2(\overline{x},\overline{v}-(kT\sqrt{2}-\delta\overline{v}_2)), & \overline{v}\in[kT\sqrt{2}-\delta\overline{v}_2, kT\sqrt{2}] \end{cases}; \tag{5.1.17}$$

note that $\varpi_0$, together with all of its derivatives, has an $L^\infty$ bound on $\Sigma_0$ which is independent of $k$, and also satisfies (5.1.16) (since the supports of the two 'blips' comprising $\varpi_0$ are disjoint and (5.1.16) involves an $L^\infty$ norm). We now specify

$$\overline{\gamma}(0,\overline{x},\overline{v}) = o\cdot\varpi_0(\overline{x},\overline{v}), \tag{5.1.18}$$

where $o \le 1$ is a scaling parameter (also independent of $k$) which we shall set later, and

$$\overline{\delta\ell}|_{\overline{s}=0,\,\overline{v}=kT/\sqrt{2}} = 0, \qquad \partial_{\overline{v}}\overline{\delta\ell}|_{\overline{s}=0,\,\overline{v}=kT/\sqrt{2}} = 0, \qquad \partial_{\overline{s}}\overline{\delta\ell}|_{\overline{s}=0,\,\overline{v}=kT/\sqrt{2}} = 0, \qquad \partial_{\overline{s}}\overline{b}|_{\overline{s}=0,\,\overline{v}=kT/\sqrt{2}} = 0. \tag{5.1.19}$$

* The restriction $\delta\overline{v}_1$, $\delta\overline{v}_2 < 1$ is purely for technical convenience.

We define also the function $\overline{\gamma}_0$ on $\Sigma_0$ by

$$\overline{\gamma}_0(\overline{x},\overline{v}) = o\cdot\varpi_0(\overline{x},\overline{v}), \tag{5.1.20}$$

so that (5.1.18) may be written $\overline{\gamma}|_{\Sigma_0} = \overline{\gamma}_0$. With these choices we have specified everything in (5.1.8) except for $\overline{\gamma}|_{\overline{v}=0}$. We will choose this (see Section 5.3) so that for $\ell = 0,\cdots,m$, $m$ some positive integer, we have

$$\partial_{\overline{s}}^{\ell}\overline{\gamma}|_{\overline{s}=0,\,\overline{v}=kT/\sqrt{2}} = 0; \tag{5.1.21}$$

logic similar to that used in Proposition 5.1.1 will then allow us to conclude that $\partial_{\overline{s}}^{\ell}\overline{\gamma}|_{\overline{s}=0}$ must vanish when $\overline{v}\in(\delta\overline{v}_1, kT\sqrt{2}-\delta\overline{v}_2)$. (Note that (5.1.21) is *not* part of the specification of the initial data – the transverse derivatives $\partial_{\overline{s}}^{\ell}\overline{\gamma}|_{\overline{s}=0}$ are quantities *derived* from the other choices of initial data. What we are saying here is that we can force (5.1.21) by an appropriate choice of $\overline{\gamma}|_{\overline{v}=0}$, which *is* part of the initial data.) This will complete our specification of the initial data, and we shall assume for the rest of the chapter that all initial data is chosen in this way.

On $\Sigma_0$, the problem, as thus phrased, has a formal reflection symmetry in the line $\overline{v} = kT/\sqrt{2}$, in that we shall obtain the quantities in (5.1.2 – 5.1.3) for $\overline{v}\in[0,kT/\sqrt{2}]$ by integrating *backwards* from $kT/\sqrt{2}$ to 0, and for $\overline{v}\in[kT/\sqrt{2},kT\sqrt{2}]$ by integrating *forwards* from $kT/\sqrt{2}$ to $kT\sqrt{2}$, and these problems are mapped into each other by exchanging $\varpi_1$ and $\varpi_2$ and mapping

$$\overline{v}\mapsto kT\sqrt{2}-\overline{v}. \tag{5.1.22}$$

Note that equation (0.2.24) is preserved by (5.1.22); the other equations we solve are preserved only up to signs, but since we are only interested in obtaining bounds, signs will not matter. It thus suffices to consider just one half of the problem. Since the half with $\overline{v}\in[0,kT/\sqrt{2}]$ is the only one which matters for determining $\overline{\gamma}|_{\overline{v}=0}$, we shall focus on it in the following, with the understanding that the other case follows by the same logic. Thus in the next two sections, unless otherwise noted, we assume $\overline{v}\in[0,kT/\sqrt{2}]$.

## 5.2. Initial data on $\overline{s}=0$

In this section (only) we let $I$ denote a multiindex in $\overline{x}$ and $\overline{v}$.

We have the conditions

$$\overline{\delta\ell}|_{\overline{s}=0,\overline{v}=kT/\sqrt{2}} = \overline{\delta\ell}_{,\overline{v}}|_{\overline{s}=0,\overline{v}=kT/\sqrt{2}} = \overline{\delta\ell}_{,\overline{s}}|_{\overline{s}=0,\overline{v}=kT/\sqrt{2}} = \overline{b}_{,\overline{s}}|_{\overline{s}=0,\overline{v}=kT/\sqrt{2}} = 0, \tag{5.2.1}$$

which imply (for $k$ sufficiently large) the conditions

$$\overline{\delta\ell}|_{\overline{s}=0,\overline{v}=1} = \overline{\delta\ell}_{,\overline{v}}|_{\overline{s}=0,\overline{v}=1} = \overline{\delta\ell}_{,\overline{s}}|_{\overline{s}=0,\overline{v}=1} = \overline{b}_{,\overline{s}}|_{\overline{s}=0,\overline{v}=1} = 0. \tag{5.2.2}$$

In all of the results in this section we shall assume that (5.2.1), and hence (5.2.2), holds.

We define the set

$$\Sigma_0^{\frac{1}{2}} = \{(0,\overline{x},\overline{v})\in\Sigma_0 \,|\, \overline{v}\le kT/\sqrt{2}\}.$$

As observed at the end of Section 5.1, the results we prove below on $\Sigma_0^{\frac{1}{2}}$ will also hold on the complement $\Sigma_0\backslash\Sigma_0^{\frac{1}{2}}$.

In accordance with our definition of $W^{m,\infty}$ (see (4.3.2)), we have, for any (say $C^m$) function $f$ defined on $\Sigma_0^{\frac{1}{2}}$,

$$\|f\|_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})} = \sum_{j_1+j_2\le m} \sup_{(0,\overline{x},\overline{v})\in\Sigma_0^{\frac{1}{2}}} \left|\partial_{\overline{x}}^{j_1}\partial_{\overline{v}}^{j_2} f(0,\overline{x},\overline{v})\right|.$$

We now have the following results. Recall that on $\Sigma_0^{\frac{1}{2}}$ we have $\overline{\gamma}=\overline{\gamma}_0$.

5.2.1. Proposition. Let $m$ be a fixed positive integer. There is a combinatorial constant $C_m^u>0$ such that any solution to (0.2.24 – 0.2.26) with initial data given by (5.1.18) and (5.2.2) satisfies on $\overline{s}=0$ the bounds

$$0\ge\overline{\delta\ell}\ge -k^{1-2\iota}\|\partial_{\overline{v}}\overline{\gamma}_0\|_{L^\infty}^2, \tag{5.2.3}$$

$$\|\overline{\delta\ell}\|_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})} \le C_m^u k^{1-2\iota}\|\overline{\gamma}_{,\overline{v}}\|^2_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})}\left[1+\|\overline{\gamma}_{,\overline{v}}\|^2_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})}\right]^m, \tag{5.2.4}$$

$$\left\|\frac{1}{\overline{\ell}}\right\|_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})} \le C_m^u\frac{\left(2+C_m^u k^{-2\iota}\left\|\overline{\gamma}_{,\overline{v}}\right\|^2_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})}(1+\left\|\overline{\gamma}_{,\overline{v}}\right\|^2_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})})^m\right)^m}{\left(1-k^{-2\iota}\left\|\overline{\gamma}_{,\overline{v}}\right\|^2_{L^\infty(\Sigma_0^{\frac{1}{2}})}\right)^{m+1}}, \tag{5.2.5}$$

$$\begin{aligned}\left\|\overline{b}_{,\overline{s}}\right\|_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})} \le C_m^u k^{1-2\iota}&\left[\frac{1+C_m^u k^{-2\iota}\left\|\overline{\gamma}_{,\overline{v}}\right\|^2_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})}\left(1+\left\|\overline{\gamma}_{,\overline{v}}\right\|^2_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})}\right)^m}{1-k^{-2\iota}\left\|\overline{\gamma}_{,\overline{v}}\right\|^2_{L^\infty(\Sigma_0^{\frac{1}{2}})}}\right]^{m+1}\\ &\cdot\left\|\overline{\gamma}_{,\overline{v}}\right\|_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})}\left\|\overline{\gamma}_{,\overline{x}}\right\|_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})},\end{aligned} \tag{5.2.6}$$

$$\begin{aligned}\left\|\overline{\ell}_{,\overline{s}}\right\|_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})} \le C_m^u k^{1-2\iota}&\left[\frac{2+C_{m+1}^u k^{-2\iota}\left\|\overline{\gamma}_{,\overline{v}}\right\|^2_{W^{m+1,\infty}(\Sigma_0^{\frac{1}{2}})}(1+\left\|\overline{\gamma}_{,\overline{v}}\right\|^2_{W^{m+1,\infty}(\Sigma_0^{\frac{1}{2}})})^{m+1}}{1-k^{-2\iota}\left\|\overline{\gamma}_{,\overline{v}}\right\|^2_{L^\infty(\Sigma_0^{\frac{1}{2}})}}\right]^{3m+6}\\ \cdot\left(\left\|\overline{\gamma}_{,\overline{v}}\right\|_{W^{m+1,\infty}(\Sigma_0^{\frac{1}{2}})}\left\|\overline{\gamma}_{,\overline{x}}\right\|_{W^{m+1,\infty}(\Sigma_0^{\frac{1}{2}})}\right.&\left.+\left\|\overline{\gamma}_{,\overline{v}}\right\|^2_{W^{m+1,\infty}(\Sigma_0^{\frac{1}{2}})}\left\|\overline{\gamma}_{,\overline{x}}\right\|^2_{W^{m+1,\infty}(\Sigma_0^{\frac{1}{2}})}+\left\|\overline{\gamma}_{,\overline{x}}\right\|^2_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})}\right).\end{aligned} \tag{5.2.7}$$

Proof. We shall derive the result for separate constants in each place, after which the result as stated follows by letting $C_m^u$ be the maximum of all of the constants. Since all results in this proposition are to hold on $\Sigma_0^{\frac{1}{2}}$, we drop the $\overline{s}=0$ subscript for notational simplicity. It suffices to prove the result for $\overline{v}\in[0,1]$; the result will then follow for $\overline{v}\in[0,kT/\sqrt{2}]$ by Proposition 5.1.1.

Consider (5.2.3). Recall that we have the conditions (see (5.2.2))

$$\overline{\delta\ell}|_{\overline{v}=1}=\partial_{\overline{v}}\overline{\delta\ell}|_{\overline{v}=1}=0, \qquad \text{so} \qquad \overline{\ell}|_{\overline{v}=1}=1,\ \partial_{\overline{v}}\overline{\ell}|_{\overline{v}=1}=0.$$

Thus there is some $\overline{v}_0\in[0,1]$ such that on $[\overline{v}_0,1]$, $\overline{\ell}\ge 1/2>0$, and on this interval we have

$$\overline{\delta\ell}_{,\overline{v}\overline{v}}=-2\overline{\ell}k^{1-2\iota}\overline{\gamma}_{,\overline{v}}{}^2\le 0,$$

which, since $\overline{\delta\ell}_{,\overline{v}}|_{\overline{v}=1}=0$, implies that

$$\overline{\delta\ell}_{,\overline{v}}=-\int_{\overline{v}}^1\overline{\delta\ell}_{,\overline{v}\overline{v}}\,d\overline{v}'\ge 0 \tag{5.2.8}$$

on $[\overline{v}_0, 1]$, so $\overline{\ell}$ is nondecreasing and hence must satisfy $\overline{\ell} \le \overline{\ell}|_{\overline{v}=1} = 1$ on $[\overline{v}_0, 1]$. Since, by (5.1.16), ${\overline{\gamma}_{,\overline{v}}}^2 \le 1/4$, we obtain on $[\overline{v}_0, 1]$ that

$$\overline{\delta\ell}_{,\overline{v}\overline{v}} \ge -\frac{1}{2}.$$

Since $1 - \overline{v}_0 \le 1$ and $\overline{\delta\ell}_{,\overline{v}}|_{\overline{v}=1} = 0$, this implies that on $[\overline{v}_0, 1]$

$$\overline{\delta\ell}_{,\overline{v}} \le \frac{1}{2},$$

whence since also $\overline{\delta\ell}|_{\overline{v}=1} = 0$, on $[\overline{v}_0, 1]$

$$\overline{\delta\ell} \ge -\frac{1}{2}.$$

Thus finally (since $k \ge 1$) $\overline{\ell} = 1 + k^{-1}\overline{\delta\ell} \ge 1/2$ on $[\overline{v}_0, 1]$ for any $\overline{v}_0 \ge 0$, and hence $\overline{\ell} \ge 1/2$ on $[0, 1]$. From (5.2.8) we then see that $\overline{\delta\ell} \le 0$ on $[0, 1]$.

Substituting $\overline{\ell} \le 1$ into (0.2.24), we obtain

$$\begin{aligned}
\overline{\delta\ell}_{,\overline{v}\overline{v}} &\ge -2k^{1-2\iota}\|\overline{\gamma}_{,\overline{v}}\|^2_{L^\infty(\Sigma_0^{\frac{1}{2}})}, \\
\overline{\delta\ell}_{,\overline{v}} &= -\int_{\overline{v}}^1 \overline{\delta\ell}_{,\overline{v}\overline{v}}\, d\overline{v}' \le 2(1-\overline{v})k^{1-2\iota}\|\overline{\gamma}_{,\overline{v}}\|^2_{L^\infty(\Sigma_0^{\frac{1}{2}})}, \\
\overline{\delta\ell} &= -\int_{\overline{v}}^1 \overline{\delta\ell}_{,\overline{v}}\, d\overline{v}' \ge -2\left(\frac{1}{2} - \overline{v} + \frac{1}{2}\overline{v}^2\right) k^{1-2\iota}\|\overline{\gamma}_{,\overline{v}}\|^2_{L^\infty(\Sigma_0^{\frac{1}{2}})} = -(1-\overline{v})^2 k^{1-2\iota}\|\overline{\gamma}_{,\overline{v}}\|^2_{L^\infty(\Sigma_0^{\frac{1}{2}})},
\end{aligned}$$

so since $\overline{v} \in [0, 1]$ equation (5.2.3) follows. Note that we also have

$$\left\|\frac{1}{\overline{\ell}}\right\|_{L^\infty(\Sigma_0^{\frac{1}{2}})} \le \frac{1}{1 - k^{-2\iota}\|\overline{\gamma}_{,\overline{v}}\|^2_{L^\infty(\Sigma_0^{\frac{1}{2}})}},$$

which will be used in proving (5.2.5), (5.2.6) and (5.2.7).

The proof of (5.2.4) is by induction on $m$. For $m = 0$ it follows from (5.2.3) with $C_0^u = 1$. Now suppose it holds up to some $m \ge 0$, let $I$ be a multiindex in $\overline{x}$ and $\overline{v}$ with $|I| = m + 1$, and differentiate equation (0.2.24) using $\partial^I$:

$$\partial_{\overline{v}}^2 \partial^I \overline{\delta\ell} = -2k^{1-2\iota}\partial^I \overline{\delta\ell}{\overline{\gamma}_{,\overline{v}}}^2 - 2k^{1-2\iota} \sum_{|J|\ge 1,\, J \le I} \binom{I}{J} \partial^{I-J}\overline{\delta\ell}\, \partial^J \left[{\overline{\gamma}_{,\overline{v}}}^2\right].$$

By induction and Lemma 4.3.1, and since $k \ge 1$, $\iota \ge 1/2$, every term in the sum is bounded by

$$\begin{aligned}
&C'\|\overline{\gamma}_{,\overline{v}}\|^2_{W^{|I-J|,\infty}(\Sigma_0^{\frac{1}{2}})} \left[1 + \|\overline{\gamma}_{,\overline{v}}\|^2_{W^{|I-J|,\infty}(\Sigma_0^{\frac{1}{2}})}\right]^{|I-J|} \|\overline{\gamma}_{,\overline{v}}\|^2_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})} \\
&\qquad \le C'\|\overline{\gamma}_{,\overline{v}}\|^2_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})} \left[1 + \|\overline{\gamma}_{,\overline{v}}\|^2_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})}\right]^m,
\end{aligned}$$

since $|I - J| \le m - 1$. Thus we may write

$$\partial_{\overline{v}}^2 \partial^I \overline{\delta\ell} = -2k^{1-2\iota}\partial^I \overline{\delta\ell}{\overline{\gamma}_{,\overline{v}}}^2 - 2k^{1-2\iota} F$$

where

$$\|F\|_{L^\infty(\Sigma_0^{\frac{1}{2}})} \le C\|\overline{\gamma}_{,\overline{v}}\|^2_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})}\left[1 + \|\overline{\gamma}_{,\overline{v}}\|^2_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})}\right]^m$$

for some constant $C$. By Corollary 4.3.3, then, we have (since $\overline{\delta\ell}|_{\Sigma_0^{\frac{1}{2}}}$ is supported on $\{(0,\overline{x},\overline{v}) \in \Sigma_0^{\frac{1}{2}} \mid \overline{x},\overline{v} \in [0,1]\}$)

$$\begin{aligned}\|\partial^I\overline{\delta\ell}\|_{L^\infty(\Sigma_0^{\frac{1}{2}})} &\le 2k^{1-2\iota}\|F\|_{L^\infty(\Sigma_0^{\frac{1}{2}})} e^{1+2k^{1-2\iota}\|\overline{\gamma}_{,\overline{v}}\|^2_{L^\infty(\Sigma_0^{\frac{1}{2}})}} \\ &\le 2k^{1-2\iota}C\|\overline{\gamma}_{,\overline{v}}\|^2_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})}\left[1 + \|\overline{\gamma}_{,\overline{v}}\|^2_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})}\right]^m e^2,\end{aligned}$$

using $\|\overline{\gamma}_{,\overline{v}}\|_{L^\infty(\Sigma_0^{\frac{1}{2}})} \le 1/2$ and $k^{1-2\iota} \le 1$. This gives the desired result.

(5.2.5) follows from this and Lemma 4.2.1:

$$\left\|\frac{1}{\overline{\ell}}\right\|_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})} \le C^i_m\left(2 + k^{-1}\left\|\overline{\delta\ell}\right\|_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})}\right)^m \cdot \frac{1}{\left(1 - k^{-2\iota}\|\overline{\gamma}_{,\overline{v}}\|^2_{L^\infty(\Sigma_0^{\frac{1}{2}})}\right)^{m+1}},$$

so the result follows by substituting in (5.2.4).

To prove (5.2.6), we recall equation (0.2.25):

$$\partial_{\overline{v}}\left(\overline{\ell}\,\overline{b}_{,\overline{s}}\right) = 4\overline{\ell}k^{1-2\iota}\overline{\gamma}_{,\overline{v}}\overline{\gamma}_{,\overline{x}};$$

integrating and using $\overline{b}_{,\overline{s}}|_{\overline{v}=1} = 0$, this gives

$$\overline{b}_{,\overline{s}} = -\frac{4}{\overline{\ell}}k^{1-2\iota}\int_{\overline{v}}^1 (1 + k^{-1}\overline{\delta\ell})\overline{\gamma}_{,\overline{v}}\overline{\gamma}_{,\overline{x}}\,d\overline{v}'.$$

We note first that

$$-\partial_{\overline{v}}\int_{\overline{v}}^1 (1 + k^{-1}\overline{\delta\ell})\overline{\gamma}_{,\overline{v}}\overline{\gamma}_{,\overline{x}}\,d\overline{v}' = (1 + k^{-1}\overline{\delta\ell})\overline{\gamma}_{,\overline{v}}\overline{\gamma}_{,\overline{x}} = -\int_{\overline{v}}^1 \partial_{\overline{v}}\left[(1 + k^{-1}\overline{\delta\ell})\overline{\gamma}_{,\overline{v}}\overline{\gamma}_{,\overline{x}}\right]\,d\overline{v}',$$

and similarly that $\partial_{\overline{x}}$ commutes with the integral. Since moreover for any function $f$ on $[0,1]$ and any $a, b \in [0,1]$ we have

$$\left|\int_a^b f(t)\,dt\right| \le \|f\|_{L^\infty([0,1])},$$

this fact together with Lemma 4.3.1 and equations (5.2.3), (5.2.4) allows us to conclude that there is a constant $C$ such that

$$\begin{aligned}\left\|\overline{b}_{,\overline{s}}\right\|_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})} &\le Ck^{1-2\iota}\left\|\frac{1}{\overline{\ell}}\right\|_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})}\left[1 + k^{-1}\left\|\overline{\delta\ell}\right\|_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})}\right]\left\|\overline{\gamma}_{,\overline{v}}\right\|_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})}\left\|\overline{\gamma}_{,\overline{x}}\right\|_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})} \\ &\le Ck^{1-2\iota}\left[\frac{1 + C^u_m k^{-2\iota}\left\|\overline{\gamma}_{,\overline{v}}\right\|^2_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})}\left(1 + \left\|\overline{\gamma}_{,\overline{v}}\right\|^2_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})}\right)}{1 - k^{-2\iota}\left\|\overline{\gamma}_{,\overline{v}}\right\|^2_{L^\infty(\Sigma_0^{\frac{1}{2}})}}\right]^{m+1}\left\|\overline{\gamma}_{,\overline{v}}\right\|_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})}\left\|\overline{\gamma}_{,\overline{x}}\right\|_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})},\end{aligned}$$

as claimed.

(5.2.7) may be proved similarly. We have the equation (0.2.26):

$$2\overline{\ell}\cdot\overline{\delta\ell}_{,\overline{v}\overline{s}}-\overline{\ell}\partial_{\overline{x}}\left(\overline{\ell}^{-1}\overline{b}_{,\overline{s}}\right)-\frac{1}{2k}\overline{b}_{,\overline{s}}{}^{2}=2k^{1-2\iota}\overline{\gamma}_{,\overline{x}}{}^{2};$$

integrating, we obtain

$$\overline{\delta\ell}_{,\overline{s}}=-\frac{1}{2}\int_{\overline{v}}^{1}\partial_{\overline{x}}\left(\frac{\overline{b}_{,\overline{s}}}{\overline{\ell}}\right)+\frac{1}{2\overline{\ell}}k^{-1}\overline{b}_{,\overline{s}}{}^{2}+\frac{2}{\overline{\ell}}k^{1-2\iota}\overline{\gamma}_{,\overline{x}}{}^{2}\,d\overline{v}'.$$

Note that there is a loss of derivative in the first term in the integral, inasmuch as one derivative of $\overline{\delta\ell}$ depends on two derivatives of $\overline{b}$; this will not however impact us since our initial data $\overline{\gamma}_0$ is $C^\infty$. Since $k\geq 1$, we may bound the second term in $W^{m,\infty}(\Sigma_0^{\frac{1}{2}})$ by its norm in $W^{m+1,\infty}(\Sigma_0^{\frac{1}{2}})$; thus there is a constant $C$ such that

$$\begin{aligned}
&\|\overline{\delta\ell}_{,\overline{s}}\|_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})}\\
&\leq C\left\{\left(\frac{2+C^u_{m+1}k^{-2\iota}\left\|\overline{\gamma}_{,\overline{v}}\right\|^2_{W^{m+1,\infty}(\Sigma_0^{\frac{1}{2}})}(1+\left\|\overline{\gamma}_{,\overline{v}}\right\|^2_{W^{m+1,\infty}(\Sigma_0^{\frac{1}{2}})})^{m+1}}{1-k^{-2\iota}\left\|\overline{\gamma}_{,\overline{v}}\right\|^2_{L^\infty(\Sigma_0^{\frac{1}{2}})}}\right)^{m+2}\right.\\
&\cdot\left[C^u_{m+1}k^{1-2\iota}\left(\frac{2+C^u_{m+1}k^{-2\iota}\left\|\overline{\gamma}_{,\overline{v}}\right\|^2_{W^{m+1,\infty}(\Sigma_0^{\frac{1}{2}})}(1+\left\|\overline{\gamma}_{,\overline{v}}\right\|^2_{W^{m+1,\infty}(\Sigma_0^{\frac{1}{2}})})^{m+1}}{1-k^{-2\iota}\left\|\overline{\gamma}_{,\overline{v}}\right\|^2_{L^\infty(\Sigma_0^{\frac{1}{2}})}}\right)^{m+2}\right.\\
&\qquad\cdot\left\|\overline{\gamma}_{,\overline{v}}\right\|_{W^{m+1,\infty}(\Sigma_0^{\frac{1}{2}})}\left\|\overline{\gamma}_{,\overline{x}}\right\|_{W^{m+1,\infty}(\Sigma_0^{\frac{1}{2}})}\\
&+C^u_{m+1}k^{1-4\iota}\left(\frac{2+C^u_{m+1}k^{-2\iota}\left\|\overline{\gamma}_{,\overline{v}}\right\|^2_{W^{m+1,\infty}(\Sigma_0^{\frac{1}{2}})}(1+\left\|\overline{\gamma}_{,\overline{v}}\right\|^2_{W^{m+1,\infty}(\Sigma_0^{\frac{1}{2}})})^{m+1}}{1-k^{-2\iota}\left\|\overline{\gamma}_{,\overline{v}}\right\|^2_{L^\infty(\Sigma_0^{\frac{1}{2}})}}\right)^{2m+4}\\
&\qquad\left.\cdot\left\|\overline{\gamma}_{,\overline{v}}\right\|^2_{W^{m+1,\infty}(\Sigma_0^{\frac{1}{2}})}\left\|\overline{\gamma}_{,\overline{x}}\right\|^2_{W^{m+1,\infty}(\Sigma_0^{\frac{1}{2}})}\right]\\
&\left.+Ck^{1-2\iota}\left(\frac{2+C^u_{m}k^{-2\iota}\left\|\overline{\gamma}_{,\overline{x}}\right\|^2_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})}(1+\left\|\overline{\gamma}_{,\overline{v}}\right\|^2_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})})^{m}}{1-k^{-2\iota}\left\|\overline{\gamma}_{,\overline{v}}\right\|^2_{L^\infty(\Sigma_0^{\frac{1}{2}})}}\right)^{m+1}\left\|\overline{\gamma}_{,\overline{x}}\right\|^2_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})}\right\}\\
&\leq Ck^{1-2\iota}\left(\frac{2+C^u_{m+1}k^{-2\iota}\left\|\overline{\gamma}_{,\overline{v}}\right\|^2_{W^{m+1,\infty}(\Sigma_0^{\frac{1}{2}})}(1+\left\|\overline{\gamma}_{,\overline{v}}\right\|^2_{W^{m+1,\infty}(\Sigma_0^{\frac{1}{2}})})^{m+1}}{1-k^{-2\iota}\left\|\overline{\gamma}_{,\overline{v}}\right\|^2_{L^\infty(\Sigma_0^{\frac{1}{2}})}}\right)^{3m+6}\\
&\cdot\left(\left\|\overline{\gamma}_{,\overline{v}}\right\|_{W^{m+1,\infty}(\Sigma_0^{\frac{1}{2}})}\left\|\overline{\gamma}_{,\overline{x}}\right\|_{W^{m+1,\infty}(\Sigma_0^{\frac{1}{2}})}+\left\|\overline{\gamma}_{,\overline{v}}\right\|^2_{W^{m+1,\infty}(\Sigma_0^{\frac{1}{2}})}\left\|\overline{\gamma}_{,\overline{x}}\right\|^2_{W^{m+1,\infty}(\Sigma_0^{\frac{1}{2}})}+\left\|\overline{\gamma}_{,\overline{x}}\right\|^2_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})}\right),
\end{aligned}$$

as claimed. QED.

This gives the following corollary.

5.2.1. Corollary. Consider any solution to (0.2.24 – 0.2.26) with initial data given by (5.1.18) and (5.2.2). For every $\epsilon>0$ there is a $\delta>0$, independent of $o$ and $k$, such that

$$\left\|\overline{\gamma}_{,\overline{v}}\right\|_{W^{m+1,\infty}(\Sigma_0^{\frac{1}{2}})},\left\|\overline{\gamma}_{,\overline{x}}\right\|_{W^{m+1,\infty}(\Sigma_0^{\frac{1}{2}})}<\delta\quad\text{implies}\quad\|\overline{\delta\ell}\|_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})},\|\overline{b}_{,\overline{s}}\|_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})},\|\overline{\delta\ell}_{,\overline{s}}\|_{W^{m,\infty}(\Sigma_0^{\frac{1}{2}})}<\epsilon.$$

Moreover, $\overline{\delta\ell}$, $\overline{b}_{,\overline{s}}$ and $\overline{\delta\ell}_{,\overline{s}}$ all vanish identically for $\overline{v}>1$.

Proof. Since

$$\left\|\overline{\gamma}_{,\overline{v}}\right\|_{L^\infty(\Sigma_0^{\frac12})}, \left\|\overline{\gamma}_{,\overline{v}}\right\|_{W^{m,\infty}(\Sigma_0^{\frac12})} \le \left\|\overline{\gamma}_{,\overline{v}}\right\|_{W^{m+1,\infty}(\Sigma_0^{\frac12})}$$

and

$$\left\|\overline{\gamma}_{,\overline{x}}\right\|_{W^{m,\infty}(\Sigma_0^{\frac12})} \le \left\|\overline{\gamma}_{,\overline{x}}\right\|_{W^{m+1,\infty}(\Sigma_0^{\frac12})},$$

the first statement follows from continuity of the upper bounds in Proposition 5.2.1 and the observation that these bounds vanish when $\left\|\overline{\gamma}_{,\overline{v}}\right\|_{W^{m+1,\infty}(\Sigma_0^{\frac12})} = \left\|\overline{\gamma}_{,\overline{x}}\right\|_{W^{m+1,\infty}(\Sigma_0^{\frac12})} = 0$. The second statement follows from Proposition 5.1.1. QED.

We note that the conditions on $\left\|\overline{\gamma}_{,\overline{v}}\right\|_{W^{m+1,\infty}(\Sigma_0^{\frac12})}$ and $\left\|\overline{\gamma}_{,\overline{x}}\right\|_{W^{m+1,\infty}(\Sigma_0^{\frac12})}$ in Corollary 5.2.1 and the similar results below can be reformulated as a smallness condition on the parameter $o$, see (5.1.18).

Our next aim is to produce a generalisation of this corollary to higher $\overline{s}$ derivatives of $\overline{\delta\ell}_{,\overline{s}}$, $\overline{b}_{,\overline{s}}$ and $\overline{c}_{,\overline{s}}$, as well as $\overline{s}$ derivatives of $\overline{\gamma}$. We first pause to consider exactly how these higher $\overline{s}$ derivatives are to be obtained. Our final aim is of course to solve the system (3.3.5 − 3.3.8), at least for $\overline{s}<T'$. Thus we seek functions $\overline{a}$, $\overline{b}$, $\overline{c}$ and $\overline{\gamma}$ for $\overline{s}\ge 0$ in some neighbourhood of zero which satisfy the Riccati equations

$$\begin{aligned}
&\partial_{\overline{s}}^2\overline{\delta\ell} = -2\overline{\ell}k^{1-2\iota}(\partial_{\overline{s}}\overline{\gamma})^2, \qquad \partial_{\overline{s}}^2\overline{b} = \frac{1}{\overline{\ell}}k^{-1}(\partial_{\overline{s}}\overline{\delta\ell})(\partial_{\overline{s}}\overline{b}) - 4k^{1-2\iota}\partial_{\overline{s}}\overline{\gamma}(\partial_{\overline{x}}\overline{\gamma}+k^{-1}\overline{b}\partial_{\overline{s}}\overline{\gamma}),\\
&\partial_{\overline{s}}^2\overline{c} = k^{-1}\frac{(\partial_{\overline{s}}\overline{b})^2}{2\overline{a}} - 2k^{1-2\iota}\partial_{\overline{s}}\overline{\gamma}\left(2\partial_{\overline{v}}\overline{\gamma}+2k^{-1}\frac{\overline{b}}{\overline{a}}\partial_{\overline{x}}\overline{\gamma}+k^{-1}\left(\overline{c}+k^{-1}\frac{\overline{b}^2}{\overline{a}}\right)\partial_{\overline{s}}\overline{\gamma}\right) - k^{1-2\iota}\frac{2}{\overline{a}}(\partial_{\overline{x}}\overline{\gamma})^2,
\end{aligned} \tag{5.2.9}$$

and the wave equation

$$\begin{aligned}
&\left[\left(-2\partial_{\overline{s}}\partial_{\overline{v}}+\partial_{\overline{x}}^2\right)+\frac{1}{k}\left(2\frac{\overline{b}}{\overline{a}}\partial_{\overline{s}}\partial_{\overline{x}} - \overline{c}\partial_{\overline{s}}^2 - \overline{\delta^{-1}a}\partial_{\overline{x}}^2 - \left(\partial_{\overline{s}}\overline{c} - \frac{1}{\overline{a}}\partial_{\overline{x}}\overline{b} + \frac{\partial_{\overline{v}}\overline{\delta\ell}}{\overline{\ell}}\right)\partial_{\overline{s}} + \left(\frac{1}{\overline{a}}\partial_{\overline{s}}\overline{b} - \frac{\overline{\ell}\partial_{\overline{x}}\overline{\delta\ell}}{\overline{a}^2}\right)\partial_{\overline{x}} - \frac{\partial_{\overline{s}}\overline{\delta\ell}}{\overline{\ell}}\partial_{\overline{v}}\right)\right.\\
&\qquad\left. +k^{-2}\left(\frac{\overline{b}^2}{\overline{a}}\partial_{\overline{s}}^2 - \left(\overline{c}\frac{\partial_{\overline{s}}\overline{\delta\ell}}{\overline{\ell}} - 2\frac{\overline{b}}{\overline{a}}\partial_{\overline{s}}\overline{b}\right)\partial_{\overline{s}} - \frac{\overline{b}\overline{\ell}\partial_{\overline{s}}\overline{\delta\ell}}{\overline{a}^2}\partial_{\overline{x}}\right) - k^{-3}\frac{\overline{b}^2\overline{\ell}\partial_{\overline{s}}\overline{\delta\ell}}{\overline{a}^2}\partial_{\overline{s}}\right]\overline{\gamma} = 0.
\end{aligned} \tag{5.2.10}$$

Now on $\overline{s}=0$ we have $\overline{c}=0$, so given $\overline{\gamma}|_{\overline{s}=0}$ the wave equation (5.2.10) becomes an ordinary differential equation (in $\overline{v}$) for $\partial_{\overline{s}}\overline{\gamma}|_{\overline{s}=0}$, from which we can determine $\partial_{\overline{s}}\overline{\gamma}|_{\Sigma_0^{\frac12}}$ (and $\partial_{\overline{s}}\overline{\gamma}|_{\Sigma_0\setminus\Sigma_0^{\frac12}}$) given $\overline{\gamma}|_{\overline{s}=0,\,\overline{v}=kT/\sqrt{2}}$. The Riccati equations (5.2.9) *at* $\overline{s}=0$ then allow us to obtain $\partial_{\overline{s}}^2\overline{\omega}$, $\overline{\omega}\in\{\overline{\delta\ell},\overline{b},\overline{c}\}$ on $\Sigma_0^{\frac12}$; differentiating the wave equation with respect to $\overline{s}$ *at* $\overline{s}=0$, we can then obtain $\partial_{\overline{s}}^2\overline{\gamma}|_{\Sigma_0^{\frac12}}$ given $\partial_{\overline{s}}^2\overline{\gamma}|_{\overline{s}=0,\,\overline{v}=\overline{v}_0}$, and so forth. In particular, we are able to obtain a priori expressions for all $\overline{s}$ derivatives of $\overline{\delta\ell}$, $\overline{b}$, $\overline{c}$, and $\overline{\gamma}$, knowing only that they satisfy the Riccati and wave equations with, respectively, $\overline{\gamma}$ replaced by *any* function satisfying the initial conditions on $\overline{\gamma}$, and $\overline{\delta\ell}$, $\overline{b}$, $\overline{c}$ replaced by *any* set of functions satisfying the initial conditions on $\overline{\delta\ell}$, $\overline{b}$, and $\overline{c}$.

5.2.2. Proposition. Consider any solution to (0.2.24 – 0.2.26) with initial data given by (5.1.18) and (5.2.2). Let $m\ge 0$, and suppose that

$$\partial_{\overline{s}}^\ell\overline{\gamma}|_{\overline{s}=0,\overline{v}=kT/\sqrt{2}} = 0, \quad \ell=0,\cdots,m. \tag{5.2.11}$$

Then the following two statements hold:

(i) For every $\epsilon > 0$ there is a $\delta > 0$, independent of $o$ and $k$, such that

$$\left\|\overline{\gamma}_{,\overline{v}}\right\|_{W^{m+1,\infty}(\Sigma_0^{\frac{1}{2}})}, \left\|\overline{\gamma}_{,\overline{x}}\right\|_{W^{m+1,\infty}(\Sigma_0^{\frac{1}{2}})} < \delta \quad \text{implies} \quad \|\partial^I \partial_{\overline{s}}^{\ell} \overline{\omega}\|_{L^\infty(\Sigma_0^{\frac{1}{2}})} < \epsilon,$$

and moreover $\partial^I \partial_{\overline{s}}^{\ell} \overline{\omega}$ vanishes for $\overline{v} > 1$, where $\overline{\omega} \in \overline{\Omega}_0 = \{\overline{\delta\ell}, \overline{b}, \overline{c}\}$, $|I| + 2\ell \leq m$;

(ii) For every $\epsilon > 0$ there is a $\delta > 0$, independent of $o$ and $k$, such that

$$\left\|\overline{\gamma}_{,\overline{v}}\right\|_{W^{m+1,\infty}(\Sigma_0^{\frac{1}{2}})}, \left\|\overline{\gamma}_{,\overline{x}}\right\|_{W^{m+1,\infty}(\Sigma_0^{\frac{1}{2}})} < \delta \quad \text{implies} \quad \|\partial^I \partial_{\overline{s}}^{\ell} \overline{\gamma}\|_{L^\infty(\Sigma_0^{\frac{1}{2}})} < \epsilon,$$

and moreover $\partial^I \partial_{\overline{s}}^{\ell} \overline{\gamma}$ vanishes for $\overline{v} > 1$, where $|I| + 2\ell \leq m$.

Proof. Fix some $m \geq 0$, and let $\overline{\Omega}_0 = \{\overline{\delta\ell}, \overline{b}, \overline{c}\}$ for convenience. The proof is by induction on $\ell$. For $\ell = 0$, (i) and (ii) follow from the previous corollary and $\overline{\gamma}|_{\overline{s}=0} = \overline{\gamma}_0$ since we have $|I| \leq m$ in this case. (i) also holds for $\ell = 1$ and all $I$ satisfying $|I| \leq m$, again by the previous corollary and the fact that $\overline{c}_{,\overline{s}} = 0$ on $\Sigma_0^{\frac{1}{2}}$. Now suppose that (i) and (ii) hold for $|I| + 2\ell \leq m$ and $\ell \leq \ell_0$, $\ell_0 \geq 0$. We shall show that they hold also for $\ell = \ell_0 + 1$. Let $\ell = \ell_0 + 1$, and let $I$ be such that $|I| + 2\ell \leq m$. Note that this implies that $|I| \leq m - 2$. We show (i) first. If $\ell = 1$ then (i) is already known to be true, so we may assume that $\ell \geq 2$ in this case. Now $\overline{\delta\ell}$, $\overline{b}$ and $\overline{c}$ satisfy on $\overline{s} = 0$ the Riccati equations

$$\partial_{\overline{s}}^2 \overline{\delta\ell} = -2\overline{\ell} k^{1-2\iota} (\partial_{\overline{s}} \overline{\gamma})^2, \qquad \partial_{\overline{s}}^2 \overline{b} = \frac{1}{\overline{\ell}} k^{-1} (\partial_{\overline{s}} \overline{\delta\ell})(\partial_{\overline{s}} \overline{b}) - 4k^{1-2\iota} \partial_{\overline{s}} \overline{\gamma} (\partial_{\overline{x}} \overline{\gamma} + k^{-1} \overline{b} \partial_{\overline{s}} \overline{\gamma}),$$
$$\partial_{\overline{s}}^2 \overline{c} = k^{-1} \frac{(\partial_{\overline{s}} \overline{b})^2}{2\overline{a}} - 2k^{1-2\iota} \partial_{\overline{s}} \overline{\gamma} \left( 2\partial_{\overline{v}} \overline{\gamma} + 2k^{-1} \frac{\overline{b}}{\overline{a}} \partial_{\overline{x}} \overline{\gamma} + k^{-1} \left( \overline{c} + k^{-1} \frac{\overline{b}^2}{\overline{a}} \right) \partial_{\overline{s}} \overline{\gamma} \right) - k^{1-2\iota} \frac{2}{\overline{a}} (\partial_{\overline{x}} \overline{\gamma})^2;$$

differentiating with respect to $\partial^I \partial_{\overline{s}}^{\ell-2}$, we see by the product rule that there exist polynomials $P_1$, $P_2$, $P_3$ in $\{\partial^{I'} \partial_{\overline{s}}^{\ell'} \overline{\omega} \,|\, |I'| \leq |I|, \ell' \leq \ell - 1\}$ and $\{\partial^{I'} \partial_{\overline{s}}^{\ell'} \overline{\gamma} \,|\, |I'| + 2\ell' \leq |I| + 2\ell - 2\}$ with no constant term such that

$$\partial^I \partial_{\overline{s}}^{\ell} \overline{\delta\ell} = P_1, \qquad \partial^I \partial_{\overline{s}}^{\ell} \overline{b} = P_2, \qquad \partial^I \partial_{\overline{s}}^{\ell} \overline{c} = P_3.$$

Note that $|I| + 2\ell - 2 \leq m - 1$, so that all of the arguments in the above polynomials are bounded in $L^\infty(\Sigma_0^{\frac{1}{2}})$ by the induction step, and vanish for $\overline{v} > 1$; thus by continuity of polynomials and the fact that the polynomials have zero constant term, we have that if $\epsilon > 0$ there must be a $\delta > 0$ such that $\left\|\overline{\gamma}_{,\overline{v}}\right\|_{W^{m+1,\infty}(\Sigma_0^{\frac{1}{2}})}, \left\|\overline{\gamma}_{,\overline{x}}\right\|_{W^{m+1,\infty}(\Sigma_0^{\frac{1}{2}})} < \delta$ implies $\|\partial^I \partial_{\overline{s}}^{\ell} \overline{\omega}\|_{L^\infty(\Sigma_0^{\frac{1}{2}})} < \epsilon$, and that $\partial^I \partial_{\overline{s}}^{\ell} \overline{\omega}$ must vanish for $\overline{v} > 1$, for all $\overline{\omega} \in \overline{\Omega}_0$, showing (i).

(ii) can be shown in a similar fashion. Here we must include the case $\ell = 1$. Recall that $\overline{\gamma}$ must satisfy the wave equation (3.3.5) for $\overline{s} \leq T'$:

$$\Bigg[ (-2\partial_{\overline{s}} \partial_{\overline{v}} + \partial_{\overline{x}}^2) + \frac{1}{k} \left( 2\frac{\overline{b}}{\overline{a}} \partial_{\overline{s}} \partial_{\overline{x}} - \overline{c} \partial_{\overline{s}}^2 - \overline{\delta^{-1} a} \partial_{\overline{x}}^2 - \left( \partial_{\overline{s}} \overline{c} - \frac{1}{\overline{a}} \partial_{\overline{x}} \overline{b} + \frac{\partial_{\overline{v}} \overline{\delta\ell}}{\overline{\ell}} \right) \partial_{\overline{s}} + \left( \frac{1}{\overline{a}} \partial_{\overline{s}} \overline{b} - \frac{\overline{\ell} \partial_{\overline{x}} \overline{\delta\ell}}{\overline{a}^2} \right) \partial_{\overline{x}} - \frac{\partial_{\overline{s}} \overline{\delta\ell}}{\overline{\ell}} \partial_{\overline{v}} \right)$$
$$+ k^{-2} \left( \frac{\overline{b}^2}{\overline{a}} \partial_{\overline{s}}^2 - \left( \overline{c} \frac{\partial_{\overline{s}} \overline{\delta\ell}}{\overline{\ell}} - 2\frac{\overline{b}}{\overline{a}} \partial_{\overline{s}} \overline{b} \right) \partial_{\overline{s}} - \frac{\overline{b\ell} \partial_{\overline{s}} \overline{\delta\ell}}{\overline{a}^2} \partial_{\overline{x}} \right) - k^{-3} \frac{\overline{b}^2 \overline{\ell} \partial_{\overline{s}} \overline{\delta\ell}}{\overline{a}^2} \partial_{\overline{s}} \Bigg] \overline{\gamma} = 0.$$

We may rewrite this schematically as

$$-2\partial_{\overline{v}}\partial_{\overline{s}}\overline{\gamma} = 2k^{-1}\frac{\overline{b}}{\overline{a}}\partial_{\overline{x}}\partial_{\overline{s}}\overline{\gamma} - k^{-1}\overline{c}\partial^2_{\overline{s}}\overline{\gamma} + k^{-2}\frac{\overline{b}^2}{\overline{a}}\partial^2_{\overline{s}}\overline{\gamma} + P_1\partial^2_{\overline{x}}\overline{\gamma} + P_2^i\partial_i\overline{\gamma}, \tag{5.2.12}$$

where $P_1$ and $P_2^i$ are polynomials in $\overline{\Omega}' = \overline{\Omega}_0 \cup \partial_{\overline{s}}\overline{\Omega}_0 \cup \{\partial_{\overline{x}}\overline{\delta\ell}, \partial_{\overline{v}}\overline{\delta\ell}, \partial_{\overline{x}}\overline{b}, \overline{\delta^{-1}a}\}$ with coefficients which are constants or nonpositive powers of $k$ and with no constant terms, and $\partial_i \in \{\partial_{\overline{s}}, \partial_{\overline{x}}, \partial_{\overline{v}}\}$. Differentiating this with $\partial^I\partial_{\overline{s}}^{\ell-1}$, we obtain

$$-2\partial_{\overline{v}}\partial^I\partial^\ell_{\overline{s}}\overline{\gamma} = 2k^{-1}\frac{\overline{b}}{\overline{a}}\partial_{\overline{x}}\partial^I\partial^\ell_{\overline{s}}\overline{\gamma} + k^{-1}\left[k^{-1}\frac{\overline{b}^2}{\overline{a}} - \overline{c}\right]\partial^I\partial^{\ell+1}_{\overline{s}}\overline{\gamma} + P_2^0\partial^I\partial^\ell_{s}\overline{\gamma} + \mathcal{P}, \tag{5.2.13}$$

where $\mathcal{P}$ is a polynomial in $\partial^I\partial^{\ell-1}_{\overline{s}}\left[\overline{\Omega}' \cup \{\partial^2_{\overline{x}}\overline{\gamma}, \partial_{\overline{x}}\overline{\gamma}, \partial_{\overline{v}}\overline{\gamma}\}\right]$ with no constant term. Since $\ell - 1 = \ell_0$ and $|I| + 2(\ell-1) = |I| + 2\ell - 2 \le m-2$, we see that all of the independent variables in $\mathcal{P}$ are bounded and vanish for $\overline{v} > 1$, by the induction step. Moreover, on $\Sigma_0^{\frac{1}{2}}$ we have $\overline{b} = \overline{c} = 0$, so that equation (5.2.13) reduces to

$$-2\partial_{\overline{v}}\partial^I\partial^\ell_{\overline{s}}\overline{\gamma} = P_2^0\partial^I\partial^\ell_{s}\overline{\gamma} + \mathcal{P}.$$

Since $\partial^\ell_{\overline{s}}\overline{\gamma}|_{\overline{s}=0,\,\overline{v}=kT/\sqrt{2}} = 0$, and thus $\partial^\ell_{\overline{s}}\overline{\gamma}|_{\overline{s}=0,\,\overline{v}=1} = 0$, we may integrate this to obtain

$$\partial^I\partial^\ell_{\overline{s}}\overline{\gamma} = \frac{1}{2}\int_{\overline{v}}^1 \mathcal{P}e^{-\frac{1}{2}\int_{\overline{v}'}^{\overline{v}} P_2^0\,d\overline{v}'}\,d\overline{v}';$$

noting that $P_2^0$ and $\mathcal{P}$ both vanish for $\overline{v} > 1$ by the induction step, we see that $\partial^I\partial^\ell_{\overline{s}}\overline{\gamma}$ does also and

$$\|\partial^I\partial^\ell_{\overline{s}}\overline{\gamma}\|_{L^\infty(\Sigma_0^{\frac{1}{2}})} \le \frac{1}{2}\|\mathcal{P}\|_{L^\infty(\Sigma_0^{\frac{1}{2}})}e^{\frac{1}{2}\|P_2^0\|_{L^\infty(\Sigma_0^{\frac{1}{2}})}};$$

since $\mathcal{P}$ and $P_2^0$ go to zero uniformly as $\left\|\overline{\gamma}_{,\overline{v}}\right\|_{W^{m+1,\infty}(\Sigma_0^{\frac{1}{2}})}$ and $\left\|\overline{\gamma}_{,\overline{x}}\right\|_{W^{m+1,\infty}(\Sigma_0^{\frac{1}{2}})}$ do (by induction and continuity of polynomials), so does $\|\partial^I\partial^\ell_{\overline{s}}\overline{\gamma}\|_{L^\infty(\Sigma_0^{\frac{1}{2}})}$. This completes the proof. QED.

We note one last time that all of the results in this section are still true if we replace $\overline{v}$ by $kT\sqrt{2} - \overline{v}$, and hence hold on $\Sigma_0\backslash\Sigma_0^{\frac{1}{2}}$, and hence on all of $\Sigma_0$.

The specification of the initial data on $U_0$ is more complicated, since we need our choice to be such that (5.2.11) holds. We consider this now.

### 5.3. Initial data on $\overline{v} = 0$

In this section (only) we let $I$ denote a multiindex in $\overline{x}$ and $\overline{s}$. We also assume that on $\overline{s} = 0$ the quantities

$$\partial^\ell_{\overline{s}}\overline{a}, \qquad \partial^\ell_{\overline{s}}\overline{b}, \qquad \partial^\ell_{\overline{s}}\overline{c}, \qquad \partial^\ell_{\overline{s}}\overline{\gamma}$$

have been solved for as described in the previous section, from initial data given by (5.1.18) and (5.2.2) (see Proposition 5.2.2).

We now describe the initial data for $\overline{\gamma}$ on $\overline{v} = 0$. Let $\partial^\ell_{\overline{s}}\overline{\gamma}(0,\overline{x},0)$ denote the derivatives of $\partial^\ell_{\overline{s}}\overline{\gamma}$ at $\overline{s} = \overline{v} = 0$ as solved for in Proposition 5.2.2, assuming as there that $\partial^\ell_{\overline{s}}\overline{\gamma}|_{\overline{s}=0,\overline{v}=kT/\sqrt{2}}$ have been set to zero, for $\ell$ up to some $m$. Recall (see above equation (0.3.37)) that we have $\chi \in C^\infty(\mathbf{R}^1)$ which has support contained in $[-2,2]$, and which satisfies $\chi|_{[-1,1]} = 1$. We define

$$U(\overline{s},\overline{x},0) = \chi(2\overline{s}) \sum_{\ell=0}^{m} \frac{\overline{s}^\ell}{\ell!} \partial_{\overline{s}}^\ell \overline{\gamma}(0,\overline{x},0). \tag{5.3.1}$$

Note that for $\overline{s} \in [1/2,1]$, derivatives of $U$ with respect to $\overline{s}$ will add extra powers of 2, but since we are only interested in taking finitely many derivatives such powers only contribute overall constant factors, which we may absorb without particular comment. Clearly

$$\partial_{\overline{s}}^\ell U(0,\overline{x},0) = \partial_{\overline{s}}^\ell \overline{\gamma}(0,\overline{x},0). \tag{5.3.2}$$

We define $\overline{\gamma}$ on $\overline{v}=0$ by

$$\overline{\gamma}|_{\overline{v}=0}(\overline{s},\overline{x},0) = U(\overline{s},\overline{x},0);$$

by solving the equations for $\partial_{\overline{s}}^\ell \overline{\gamma}|_{\Sigma_0}$ obtained in Proposition 5.2.2 backwards, it is evident that this choice will give $\partial_{\overline{s}}^\ell \overline{\gamma}|_{\overline{s}=0,\overline{v}=1} = 0$. This choice of data clearly satisfies the first of the consistency equations (1.3.2).

As on $\Sigma_0$, we shall also need the transverse derivatives of $\overline{\gamma}$ along $U_0$, which in this case are the $\overline{v}$ derivatives. Given $\overline{\gamma}|_{\overline{v}=0}$ as above, we may solve the first two Riccati equations (3.3.6 – 3.3.7) to obtain $\overline{\delta\ell}$ and $\overline{b}$ along $U_0$; however, the equations for $\partial_{\overline{v}}\overline{\gamma}$, $\partial_{\overline{v}}\overline{\delta\ell}$, and $\overline{c}$ all couple together, and the process of obtaining appropriate bounds therefore requires a more careful treatment than on $\Sigma_0$. We shall prove the following analogue to Proposition 5.2.2. We set $U_0^0 = \{(\overline{s},\overline{x},0) \in \mathbf{R}^3 \,|\, \overline{s} \in [0,1]\}$. Note that $U$ is supported on $\{(\overline{s},\overline{x},0) \in U_0^0 \,|\, \overline{x} \in [0,1]\}$.

We have the following straightforward proposition.

5.3.1. Proposition. For every $\epsilon > 0$ there is a $\delta > 0$, independent of $o$ and $k$, such that

$$\left\|\overline{\gamma}_{,\overline{v}}\right\|_{W^{2m+1,\infty}(\Sigma_0^{\frac{1}{2}})}, \left\|\overline{\gamma}_{,\overline{x}}\right\|_{W^{2m+1,\infty}(\Sigma_0^{\frac{1}{2}})} < \delta \quad \text{implies} \quad \|U\|_{W^{m,\infty}(U_0^0)} < \epsilon.$$

Proof. Let $\epsilon > 0$. Note that the condition $\|U\|_{W^{m,\infty}(U_0^0)} < \epsilon$ is equivalent (dividing $\epsilon$ by a combinatorial constant) to the condition that for all $I$ with $|I| \le m$,

$$\|\partial^I U\|_{L^\infty(U_0^0)} < \epsilon. \tag{5.3.3}$$

Let

$$U^* = \sum_{\ell=0}^{m} \frac{\overline{s}^\ell}{\ell!} \partial_{\overline{s}}^\ell \overline{\gamma}(0,\overline{x},0),$$

and note that Proposition 5.2.2 implies that there is a $\delta > 0$ such that for all $I$ with $|I| \le m$,

$$\left\|\overline{\gamma}_{,\overline{v}}\right\|_{W^{2m+1,\infty}(\Sigma_0^{\frac{1}{2}})}, \left\|\overline{\gamma}_{,\overline{x}}\right\|_{W^{2m+1,\infty}(\Sigma_0^{\frac{1}{2}})} < \delta \quad \text{implies} \quad \|\partial^I U^*\|_{L^\infty(U_0^0)} < \epsilon. \tag{5.3.4}$$

The extra factor of 2 is necessary since $I$ can contain up to $m$ derivatives in $\overline{s}$. Since for any $I$ with $|I| \le m$ we must have (by Lemma 4.3.1) that, for some constant $C$ depending on $\chi$ (through $C_\chi$, see (0.3.37))

$$\|\partial U\|_{L^\infty(U_0^0)} \le C\|\partial^I U^*\|_{L^\infty(U_0^0)},$$

the result now follows. QED.

5.3.2. PROPOSITION. For every $\epsilon > 0$ there is a $\delta > 0$, independent of $o$ and $k$, such that the following two statements hold:

(i)

$$\left\|\overline{\gamma}_{,\overline{v}}\right\|_{W^{2m+5,\infty}(\Sigma_0^{\frac{1}{2}})}, \left\|\overline{\gamma}_{,\overline{x}}\right\|_{W^{2m+5,\infty}(\Sigma_0^{\frac{1}{2}})} < \delta \quad \text{implies} \quad \|\partial^I \partial_{\overline{v}}^{\ell} \overline{\omega}\|_{L^\infty(U_0^0)} < \epsilon,$$

where $\overline{\omega} \in \{\overline{\delta\ell}, \overline{b}, \overline{c}\}$, $|I| + 2\ell \le m$ ($2\ell \le m-2$ if $\overline{\omega} = \overline{c}$);

(ii)

$$\left\|\overline{\gamma}_{,\overline{v}}\right\|_{W^{2m+5,\infty}(\Sigma_0^{\frac{1}{2}})}, \left\|\overline{\gamma}_{,\overline{x}}\right\|_{W^{2m+5,\infty}(\Sigma_0^{\frac{1}{2}})} < \delta \quad \text{implies} \quad \|\partial^I \partial_{\overline{v}}^{\ell} \overline{\gamma}\|_{L^\infty(U_0^0)} < \epsilon,$$

where $|I| + 2\ell \le m$.

Proof. Fix $m \ge 1$; we will proceed by induction on $\ell$. Suppose $\ell = 0$. Part (ii) holds for $\ell = 0$ by Proposition 5.3.1. Let as above $\overline{\Omega}_0 = \{\overline{\delta\ell}, \overline{b}, \overline{c}\}$, and recall that for $\overline{\omega} \in \overline{\Omega}_0$ we have the Riccati equations (3.3.6 – 3.3.8)

$$\partial_{\overline{s}}^2 \overline{\delta\ell} = -2\overline{\ell} k^{1-2\iota} (\partial_{\overline{s}} \overline{\gamma})^2, \qquad \partial_{\overline{s}}^2 \overline{b} = \frac{1}{\overline{\ell}} k^{-1} (\partial_{\overline{s}} \overline{\delta\ell})(\partial_{\overline{s}} \overline{b}) - 4k^{1-2\iota} \partial_{\overline{s}} \overline{\gamma} (\partial_{\overline{x}} \overline{\gamma} + k^{-1} \overline{b} \partial_{\overline{s}} \overline{\gamma}),$$

$$\partial_{\overline{s}}^2 \overline{c} = k^{-1} \frac{(\partial_{\overline{s}} \overline{b})^2}{2\overline{a}} - 2k^{1-2\iota} \partial_{\overline{s}} \overline{\gamma} \left( 2\partial_{\overline{v}} \overline{\gamma} + 2k^{-1} \frac{\overline{b}}{\overline{a}} \partial_{\overline{x}} \overline{\gamma} + k^{-1} \left( \overline{c} + k^{-1} \frac{\overline{b}^2}{\overline{a}} \right) \partial_{\overline{s}} \overline{\gamma} \right) - k^{1-2\iota} \frac{2}{\overline{a}} (\partial_{\overline{x}} \overline{\gamma})^2.$$

These equations may be rearranged as follows:

$$\partial_{\overline{s}}^2 \overline{\delta\ell} + 2k^{-2\iota} (\partial_{\overline{s}} \overline{\gamma})^2 \overline{\delta\ell} = -2k^{1-2\iota} (\partial_{\overline{s}} \overline{\gamma})^2, \tag{5.3.5}$$

$$\partial_{\overline{s}}^2 \overline{b} - k^{-1} \frac{\partial_{\overline{s}} \overline{\delta\ell}}{\overline{\ell}} \partial_{\overline{s}} \overline{b} + 4k^{-2\iota} (\partial_{\overline{s}} \overline{\gamma})^2 \overline{b} = -4k^{1-2\iota} \partial_{\overline{s}} \overline{\gamma} \partial_{\overline{x}} \overline{\gamma}, \tag{5.3.6}$$

$$\partial_{\overline{s}}^2 \overline{c} + 2k^{-2\iota} (\partial_{\overline{s}} \overline{\gamma})^2 \overline{c} = -4k^{1-2\iota} \partial_{\overline{s}} \overline{\gamma} \partial_{\overline{v}} \overline{\gamma} + R_{0,0}(\overline{a}, \overline{\ell}^{-1}, \overline{b}, \partial_{\overline{s}} \overline{b}, \partial_{\overline{s}} \overline{\gamma}, \partial_{\overline{x}} \overline{\gamma}), \tag{5.3.7}$$

where $R$ is a polynomial with coefficients made up of constants and nonpositive powers of $k$, and with no constant term. Since $\partial_{\overline{s}}^2 \overline{c}$ depends on $\partial_{\overline{v}} \overline{\gamma}$, which we have not yet bounded, we leave $\overline{c}$ for the moment and verify (i) for $\overline{\delta\ell}$ and $\overline{b}$. By Corollary 4.3.2, we may write for $\ell \in \{0, 1\}$, bounding $\overline{\delta\ell}$, $\partial_{\overline{s}} \overline{\delta\ell}$, and $\partial_{\overline{s}} \overline{b}$ on $\overline{s} = \overline{v} = 0$ by their bounds on $\Sigma_0$, and since on $U_0^0$ we have $\overline{\gamma} = U$,

$$\|\partial_{\overline{s}}^{\ell} \overline{\delta\ell}\|_{L^\infty(U_0^0)} \le \left( \left[ \|\overline{\delta\ell}\|_{L^\infty(\Sigma_0)}^2 + \|\partial_{\overline{s}} \overline{\delta\ell}\|_{L^\infty(\Sigma_0)}^2 \right]^{1/2} + 2k^{1-2\iota} \|\partial_{\overline{s}} U\|_{L^\infty(U_0^0)}^2 \right) e^{1 + 2k^{-2\iota} \|\partial_{\overline{s}} U\|_{L^\infty(U_0^0)}^2},$$

$$\|\partial_{\overline{s}}^{\ell} \overline{b}\|_{L^\infty(U_0^0)} \le \left( \|\partial_{\overline{s}} \overline{b}\|_{L^\infty(\Sigma_0)} + 4k^{1-2\iota} \|\partial_{\overline{s}} U\|_{L^\infty(U_0^0)} \|\partial_{\overline{x}} U\|_{L^\infty(U_0^0)} \right) e^{1 + k^{-1} \frac{\partial_{\overline{s}} \overline{\delta\ell}}{\overline{\ell}} + 4k^{-2\iota} \|\partial_{\overline{s}} U\|_{L^\infty(U_0^0)}^2},$$

so that by Corollary 5.2.1, Proposition 5.2.1, and Proposition 5.3.1, part (i) holds for $\partial^I = \partial_{\overline{s}}^j$, $j = 0, 1$ (assuming $m \ge 1$). Part (i) for arbitrary $I$ then holds by induction, as follows. To simplify our work we write $\overline{\omega}_0 = \overline{\delta\ell}$, $\overline{\omega}_1 = \overline{b}$. Then the first two equations (5.3.5 – 5.3.6) are of the form

$$\partial_{\overline{s}}^2 \overline{\omega}_i + P_{1,i} \partial_{\overline{s}} \overline{\omega}_i + P_{2,i} \overline{\omega}_i = P_{0,i}, \tag{5.3.8}$$

where $P_{j,i}$, $j \neq 0$ are polynomials in $\{\overline{\omega}_\ell, \partial_{\overline{s}}\overline{\omega}_\ell, \partial_{\overline{s}}\overline{\gamma} \,|\, \ell < i\}$ while $P_{0,i}$ is a polynomial in (letting $\partial_0 = \partial_{\overline{s}}$, $\partial_1 = \partial_{\overline{x}}$, $\partial_2 = \partial_{\overline{v}}$)

$$\overline{\Omega}_{P,i} = \{\overline{\omega}_\ell, \partial_{\overline{s}}\overline{\omega}_\ell, \partial_j\overline{\gamma} \,|\, \ell < i,\, j \leq i\}$$

(note that this contains the previous set), with coefficients which are made of constants and nonpositive powers of $k$, and zero constant term. Suppose now that (i) holds for $\ell = 0$ and derivative operators $\partial^I$, $\partial_{\overline{s}}\partial^I$ with $|I| \leq m_0$, some $m_0 \geq 0$;* note that this form of the inductive hypothesis corresponds to the base case just proved. Since we only need to prove (i) for $I$ satisfying $|I| \leq m$, if $m_0 = m$ then we are done. Otherwise, let $J$ be such that $\partial^J = \partial^I\partial_j$ for some $j \in \{0, 1\}$, $|J| \leq m$. Applying $\partial^J$ to (5.3.8), we see that after moving lower-order derivatives of $\overline{\omega}_i$ to the right-hand side we have

$$\partial_{\overline{s}}^2\partial^J\overline{\omega}_i + P_{1,i}\partial_{\overline{s}}\partial^J\overline{\omega}_i + P_{2,i}\partial^J\overline{\omega}_i = \mathcal{P},$$

where $\mathcal{P}$ is a polynomial in $\partial^J\overline{\Omega}_{P,i} \cup \{\partial^{I'}\overline{\omega}_i, \partial^{I'}\partial_{\overline{s}}\overline{\omega}_i \,|\, |I'| \leq |I|\}$ with coefficients made of constants and nonpositive powers of $k$ and no constant term. $\mathcal{P}$ will then go to zero with $\left\|\overline{\gamma}_{,\overline{v}}\right\|_{W^{2m+5,\infty}(\Sigma_0^{\frac{1}{2}})}$, $\left\|\overline{\gamma}_{,\overline{x}}\right\|_{W^{2m+5,\infty}(\Sigma_0^{\frac{1}{2}})}$ for $i = 0$, by the inductive step and Proposition 5.3.1, since in this case we have

$$\partial^J\overline{\Omega}_{P,i} = \{\partial^J\partial_{\overline{s}}\overline{\gamma}\},$$

and $|J| + 1 \leq m + 1$ so Proposition 5.3.1 is applicable; since $P_{1,0}$ and $P_{2,0}$ are polynomials with no constant term in the same set (in this case), they will also go to zero with $\left\|\overline{\gamma}_{,\overline{v}}\right\|_{W^{2m+5,\infty}(\Sigma_0^{\frac{1}{2}})}$, $\left\|\overline{\gamma}_{,\overline{x}}\right\|_{W^{2m+5,\infty}(\Sigma_0^{\frac{1}{2}})}$. Thus by Corollary 4.3.1 again, we have that $\partial^J\overline{\omega}_0$, $\partial^J\partial_{\overline{s}}\overline{\omega}_0$ go to zero with $\left\|\overline{\gamma}_{,\overline{v}}\right\|_{W^{2m+5,\infty}(\Sigma_0^{\frac{1}{2}})}$, $\left\|\overline{\gamma}_{,\overline{x}}\right\|_{W^{2m+5,\infty}(\Sigma_0^{\frac{1}{2}})}$. Since these are the only elements of $\overline{\Omega}_{P,1}\backslash\overline{\Omega}_{P,0}$, the same is true for $\partial^J\overline{\omega}_1$ and $\partial^J\partial_{\overline{s}}\overline{\omega}_1$. Thus the inductive step is seen to hold for $\overline{\delta\ell}$ and $\overline{b}$. Note that for these two quantities we actually have that (i) holds for derivatives of the form $\partial_{\overline{s}}\partial^J$ with $|J| \leq m + 1$ (because we are assuming $\overline{\gamma}_{,\overline{v}}$ and $\overline{\gamma}_{,\overline{x}}$ bounded in $W^{2m+5,\infty}$).

To sum up, then, we have shown that part (i) holds also for $\overline{\omega} \in \{\overline{\delta\ell}, \overline{b}\}$ if we require $\ell = 0$.

We now consider the case $\ell = 1$. We shall bound the quantities $\overline{c}$, $\partial_{\overline{s}}\overline{c}$, $\partial_{\overline{v}}\overline{\delta\ell}$, and $\partial_{\overline{v}}\overline{\gamma}$ simultaneously. To bound $\partial_{\overline{v}}\overline{\delta\ell}$ we use the constraint equation $R_{11} = 2{\gamma_{,x}}^2$. In a moment we shall also need $R_{12} = 2\gamma_{,x}\gamma_{,v}$, so we write both out together. From the expression for the Ricci tensor derived in Section 2.2, these equations are

$$ac_{,s}a_{,s} - (b^2 - ac)a_{,ss} + b_{,s}a_{,x} - a(2b_{,xs} - 2a_{,vs}) + \frac{b^2{a_{,s}}^2}{2a} - \frac{1}{2}c{a_{,s}}^2 - a_{,s}a_{,v} - a{b_{,s}}^2 = 4a{\gamma_{,x}}^2,$$

$$\begin{aligned} &- b{b_{,s}}^2 - (b^2 - ac)b_{,ss} + b(a_{,vs} - b_{,xs}) + \frac{b^2a_{,s}b_{,s}}{2a} - \frac{ba_{,v}a_{,s}}{2a} + \frac{bb_{,s}a_{,x}}{2a} \\ &\qquad + a(b_{,vs} - c_{,xs}) + ba_{,v}c_{,s} - \frac{1}{2}ca_{,s}b_{,s} + \frac{1}{2}a_{,s}c_{,x} + \frac{1}{2}a_{,v}b_{,s} = 4a\gamma_{,x}\gamma_{,v}. \end{aligned}$$

* For clarity, we remind the reader that $I$ is a multiindex in $\overline{s}$ and $\overline{x}$; thus $\partial_{\overline{s}}\partial^I$ can also be written as $\partial^J$ for some multiindex $J$.

In terms of the scaled coordinates and scaled quantities, these become

$$\overline{ac}_{,\overline{s}}\overline{a}_{,\overline{s}}+\overline{aca}_{,\overline{ss}}+\overline{b}_{,\overline{s}}\overline{a}_{,\overline{x}}-\overline{a}(2\overline{b}_{,\overline{xs}}-2\overline{\delta a}_{,\overline{vs}})-\frac{1}{2}\overline{ca}_{,\overline{s}}{}^{2}-\overline{a}_{,\overline{s}}\overline{\delta a}_{,\overline{v}} + k^{-1}\left(\frac{\overline{b}^{2}\overline{a}_{,\overline{s}}{}^{2}}{2\overline{a}}-\overline{ab}_{,\overline{s}}{}^{2}-\overline{b}^{2}\overline{a}_{,\overline{ss}}\right)=4\overline{a}k^{1-2\iota}\overline{\gamma}_{,\overline{x}}{}^{2}, \tag{5.3.9}$$

$$\overline{a}\left(\overline{b}_{,\overline{vs}}-\overline{c}_{,\overline{xs}}\right)+k^{-1}\left[\overline{b}(\overline{\delta a}_{,\overline{vs}}-\overline{b}_{,\overline{xs}})+\overline{ac}\overline{b}_{,\overline{ss}}+\overline{b\delta a}_{,\overline{v}}\overline{c}_{,\overline{s}}+\frac{1}{2}\overline{\delta a}_{,\overline{s}}\overline{c}_{,\overline{x}}+\frac{1}{2}\overline{\delta a}_{,\overline{v}}\overline{b}_{,\overline{s}}\right] + k^{-2}\left[-\overline{bb}_{,\overline{s}}{}^{2}-\overline{b}^{2}\overline{b}_{,\overline{ss}}-\frac{\overline{b\delta a}_{,\overline{v}}\overline{\delta a}_{,\overline{s}}}{2\overline{a}}+\frac{\overline{bb}_{,s}\overline{\delta a}_{,\overline{x}}}{2\overline{a}}-\frac{1}{2}\overline{c\delta a}_{,\overline{s}}\overline{b}_{,\overline{s}}\right]+k^{-3}\frac{\overline{b}^{2}\overline{\delta a}_{,\overline{s}}\overline{b}_{,\overline{s}}}{2\overline{a}}=4\overline{a}k^{1-2\iota}\overline{\gamma}_{,\overline{x}}\overline{\gamma}_{,\overline{v}}. \tag{5.3.10}$$

Equation (5.3.9) may be solved for $\overline{\delta a}_{,\overline{vs}}$; doing this, and writing in terms of $\overline{\delta\ell}$, we obtain

$$\partial_{\overline{s}}\partial_{\overline{v}}\overline{\delta\ell}=P_{1}\partial_{\overline{v}}\overline{\delta\ell}+P_{0,1}c+P_{0,2}\overline{c}_{,\overline{s}}+P_{0,0}, \tag{5.3.11}$$

where $P_1$ and $P_{0,i}$ are polynomials in $\overline{\delta\ell}$, $1/\overline{\ell}$, $\overline{b}$, $\overline{\delta\ell}_{,\overline{s}}$, $\overline{b}_{,\overline{s}}$, $\overline{\delta\ell}_{,\overline{x}}$, $\overline{\delta\ell}_{,\overline{ss}}$, $\overline{b}_{,\overline{xs}}$, and $\overline{\gamma}_{,\overline{x}}$ and which have coefficients as usual and no constant terms. Note that the quantities on which $P_1$ and $P_{0,i}$ depend are quantities which we have already bounded; we shall term such quantities *known quantities* for convenience. Similarly, for $\overline{c}$ we have the equation (5.3.7):

$$\partial_{\overline{s}}^{2}\overline{c}+2k^{-2\iota}(\partial_{\overline{s}}\overline{\gamma})^{2}\overline{c}=-4k^{1-2\iota}\partial_{\overline{s}}\overline{\gamma}\partial_{\overline{v}}\overline{\gamma}+R_{0,0}(\overline{a},\overline{\ell}^{-1},\overline{b},\partial_{\overline{s}}\overline{b},\partial_{\overline{s}}\overline{\gamma},\partial_{\overline{x}}\overline{\gamma}),$$

where we note that the coefficients of $\overline{c}$ and $\partial_{\overline{v}}\overline{\gamma}$, as well as all of the arguments of $R_{0,0}$, are again known quantities. Finally, recall the version of the wave equation we used in the proof of Proposition 5.2.2 above (equation (5.2.12)):

$$-2\partial_{\overline{v}}\partial_{\overline{s}}\overline{\gamma}=2k^{-1}\frac{\overline{b}}{\overline{a}}\partial_{\overline{x}}\partial_{\overline{s}}\overline{\gamma}-k^{-1}\overline{c}\partial_{\overline{s}}^{2}\overline{\gamma}+k^{-2}\frac{\overline{b}^{2}}{\overline{a}}\partial_{\overline{s}}^{2}\overline{\gamma}+P_{1}\partial_{\overline{x}}^{2}\overline{\gamma}+P_{2}^{i}\partial_{i}\overline{\gamma}; \tag{5.3.12}$$

where $P_1$ and $P_2^i$ are polynomials in $\overline{\Omega}'=\overline{\Omega}_0\cup\partial_{\overline{s}}\overline{\Omega}_0\cup\{\partial_{\overline{x}}\overline{\delta\ell},\partial_{\overline{v}}\overline{\delta\ell},\partial_{\overline{x}}\overline{b},\overline{\delta^{-1}a}\}$ with coefficients made of constants and nonpositive powers of $k$ and no constant terms. More carefully, $\partial_{\overline{v}}\overline{\delta\ell}$ appears only to linear order in the term involving $\partial_{\overline{s}}\overline{\gamma}$, so that we may rewrite this as

$$\partial_{\overline{v}}\partial_{\overline{s}}\overline{\gamma}=Q_{1}\partial_{\overline{v}}\overline{\gamma}+Q_{0,1}\partial_{\overline{v}}\overline{\delta\ell}+Q_{0,2}\overline{c}+Q_{0,3}\partial_{\overline{s}}\overline{c}+Q_{0,0}, \tag{5.3.13}$$

where $Q_1$, $Q_{0,1}$ and $Q_{0,0}$ are polynomials in known quantities. Combining equations (5.3.11 – 5.3.13), we obtain the four by four system

$$\begin{aligned}
\partial_{\overline{s}}\partial_{\overline{v}}\overline{\delta\ell}&=P_{1}\partial_{\overline{v}}\overline{\delta\ell}+P_{0,1}\overline{c}+P_{0,2}\overline{c}_{,\overline{s}}+P_{0,0}\\
\partial_{\overline{s}}\overline{c}&=\overline{c}_{,\overline{s}}\\
\partial_{\overline{s}}\overline{c}_{,\overline{s}}&=-2k^{-2\iota}(\partial_{\overline{s}}\overline{\gamma})^{2}\overline{c}-4k^{1-2\iota}\partial_{\overline{s}}\overline{\gamma}\partial_{\overline{v}}\overline{\gamma}+R_{0,0}\\
\partial_{\overline{s}}\partial_{\overline{v}}\overline{\gamma}&=Q_{0,1}\partial_{\overline{v}}\overline{\delta\ell}+Q_{1}\partial_{\overline{v}}\overline{\gamma}+Q_{0,0}
\end{aligned}$$

which may be rewritten in matrix form as

$$\partial_{\overline{s}} \begin{pmatrix} \partial_{\overline{v}}\overline{\delta\ell} \\ \overline{c} \\ \overline{c}_{,\overline{s}} \\ \partial_{\overline{v}}\overline{\gamma} \end{pmatrix} = \begin{pmatrix} P_1 & P_{0,1} & P_{0,2} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2k^{-2\iota}(\partial_{\overline{s}}\overline{\gamma})^2 & -4k^{1-2\iota}\partial_{\overline{s}}\overline{\gamma} & \\ Q_{0,1} & Q_{0,2} & Q_{0,3} & Q_1 \end{pmatrix} \begin{pmatrix} \partial_{\overline{v}}\overline{\delta\ell} \\ \overline{c} \\ \overline{c}_{,\overline{s}} \\ \partial_{\overline{v}}\overline{\gamma} \end{pmatrix} + \begin{pmatrix} P_{0,0} \\ 0 \\ R_{0,0} \\ Q_{0,0} \end{pmatrix}. \tag{5.3.14}$$

Since at $\overline{s} = \overline{v} = 0$ we have that $\overline{c} = \overline{c}_{,\overline{s}} = \partial_{\overline{v}}\overline{\gamma} = 0$, while $\partial_{\overline{v}}\overline{\delta\ell}$ goes to zero with $\|\overline{\gamma}_{,\overline{v}}\|_{W^{2m+5,\infty}(\Sigma_0^{\frac{1}{2}})}$ and $\|\overline{\gamma}_{,\overline{x}}\|_{W^{2m+5,\infty}(\Sigma_0^{\frac{1}{2}})}$, Proposition 4.3.1 shows that $\partial_{\overline{v}}\overline{\delta\ell}$, $\overline{c}$, $\overline{c}_{,\overline{s}}$, and $\partial_{\overline{v}}\overline{\gamma}$ must go to zero with $\|\overline{\gamma}_{,\overline{v}}\|_{W^{2m+5,\infty}(\Sigma_0^{\frac{1}{2}})}$, $\|\overline{\gamma}_{,\overline{x}}\|_{W^{2m+5,\infty}(\Sigma_0^{\frac{1}{2}})}$ also. Thus (i) holds with $\ell = 0$, $I = \partial_{\overline{s}}^j$ $(j = 0, 1)$, $\overline{\omega} = \overline{c}$, and also with $\ell = 1$, $I = (0, 0)$, $\overline{\omega} = \overline{\delta\ell}$, while (ii) holds with $I = (0, 0)$ and $\ell = 1$. Now noting that the quantities on the right-hand side of system (5.3.14) involving $\overline{\gamma}$ have at most two derivatives, and those involving $\overline{\delta\ell}$ and $\overline{b}$ have at most one, we see that we may differentiate it by $\partial^J$ for any $J$ satisfying $|J| \leq m$, and the resulting dependent variables will still go to zero with $\|\overline{\gamma}_{,\overline{v}}\|_{W^{2m+5,\infty}(\Sigma_0^{\frac{1}{2}})}$, $\|\overline{\gamma}_{,\overline{x}}\|_{W^{2m+5,\infty}(\Sigma_0^{\frac{1}{2}})}$. This proves (i) for $\overline{\omega} = \overline{\delta\ell}$ and $\ell = 1$, and also for $\overline{\omega} = \overline{c}$ and $\ell = 0$.

To show (i) for $\overline{\omega} = \overline{b}$ we use equation (5.3.10). Solving for $\partial_{\overline{s}}\partial_{\overline{v}}\overline{b}$, we see that we have

$$\partial_{\overline{s}}\partial_{\overline{v}}\overline{b} = S, \tag{5.3.15}$$

where $S$ is a polynomial in known quantities and involves derivatives of metric components up to order 2. Thus we may differentiate this equation by any multiindex $J$ with $|J| \leq m-2$ and obtain another polynomial in known quantities, showing that (i) holds for $\overline{\omega} = \overline{b}$ and $\ell = 1$ also.

We may now proceed by induction. Suppose that (i) holds for $\overline{\delta\ell}$ and $\overline{b}$ for $\ell \leq \ell_0$ and for $\overline{c}$ for $\ell < \ell_0$, and that (ii) holds for $\ell \leq \ell_0$, where $\ell_0 \geq 1$. Then differentiating system (5.3.14) $\ell_0 + 1$ times with respect to $\overline{v}$, and moving the lower-order (in $\overline{v}$) terms to the forcing term, we see that (i) (for $\overline{\omega} = \overline{\delta\ell}$ or $\overline{\omega} = \overline{c}$) and (ii) hold for $J = (0, 0)$ in this case; moreover, we may further differentiate by any $J$ with $|J| \leq m - 2\ell_0 - 2$, and again move the lower-order terms to the forcing term, to conclude that (i) (for $\overline{\omega} = \overline{\delta\ell}$ or $\overline{\omega} = \overline{c}$) and (ii) hold for this value of $\ell$ as well (with $\ell = \ell_0$ if $\overline{\omega} = \overline{c}$). Differentiating (5.3.15) in the same fashion allows us to conclude that (i) holds for $\overline{\omega} = \overline{b}$ as well. This completes the proof. QED.

### 5.4. Summary

To sum up, then, the initial data for the system (3.3.5 – 3.3.8) corresponding to the choice of $\overline{\gamma}|_{\overline{s}=0}$ in equation (5.1.18) is obtained as follows. Fix some $n \geq 4$ (the same as that in Chapter 6, immediately below). Set

$$\overline{b}|_{\overline{s}=0} = \overline{c}|_{\overline{s}=0} = \overline{c}_{,\overline{s}}|_{\overline{s}=0} = 0, \tag{5.4.1}$$

as required by the gauge choice (see Proposition 1.2.1 and the preceding discussion); set (see equation (5.1.18))

$$\overline{\gamma}(0, \overline{x}, \overline{v}) = o \cdot \varpi_0(\overline{x}, \overline{v}), \tag{5.4.2}$$

where $\varpi_0$ is given by (see equation (5.1.17))

$$\varpi_0(\overline{x}, \overline{v}) = \begin{cases} \varpi_1(\overline{x}, \overline{v}), & \overline{v} \in [0, \delta\overline{v}_1] \\ 0, & \overline{v} \in [\delta\overline{v}_1, kT\sqrt{2} - \delta\overline{v}_2] \\ \varpi_2(\overline{x}, \overline{v} - (kT\sqrt{2} - \delta\overline{v}_2)), & \overline{v} \in [kT\sqrt{2} - \delta\overline{v}_2, kT\sqrt{2}] \end{cases}, \tag{5.1.17}$$

and $\varpi_1$, $\varpi_2$ are $C^\infty$ functions on $\mathbf{R}^2$ with support contained in $[0,1]\times[0,\delta\overline{v}_1]$ and $[0,1]\times[0,\delta\overline{v}_2]$, respectively, all of whose derivatives have $L^\infty$ bounds independent of $k$, and which also satisfy (see equation (5.1.16))

$$\|\partial_{\overline{v}}\varpi_i\|_{L^\infty} \leq \frac{1}{2}, \|\partial_{\overline{x}}\varpi_i\|_{L^\infty} \leq \frac{1}{2}. \tag{5.1.16}$$

Now solve the constraint equations

$$\frac{\partial^2\overline{\delta\ell}}{\partial\overline{v}^2} = -2(1+k^{-1}\overline{\delta\ell})k^{1-2\iota}\left(\partial_{\overline{v}}\overline{\gamma}\right)^2 \tag{0.2.24}$$

$$\partial_{\overline{v}}\left([1+k^{-1}\overline{\delta\ell}]\,\partial_{\overline{s}}\overline{b}\right) = 4(1+k^{-1}\overline{\delta\ell})k^{1-2\iota}\partial_{\overline{v}}\overline{\gamma}\partial_{\overline{x}}\overline{\gamma} \tag{0.2.25}$$

$$2(1+k^{-1}\overline{\delta\ell})\cdot\frac{\partial^2\overline{\delta\ell}}{\partial\overline{v}\partial\overline{s}} = (1+k^{-1}\overline{\delta\ell})\partial_{\overline{x}}\left([1+k^{-1}\overline{\delta\ell}]^{-1}\partial_{\overline{s}}\overline{b}\right) + \frac{1}{2k}\left(\partial_{\overline{s}}\overline{b}\right)^2 + 2k^{1-2\iota}\left(\partial_{\overline{x}}\overline{\gamma}\right)^2, \tag{0.2.26}$$

for $\overline{\delta\ell}|_{\overline{s}=0}$, $\overline{b}_{,\overline{s}}|_{\overline{s}=0}$ and $\overline{\delta\ell}_{,\overline{s}}|_{\overline{s}=0}$ with the conditions (see equation (5.1.19))

$$\overline{\delta\ell}|_{\overline{s}=0,\overline{v}=kT/\sqrt{2}} = \overline{\delta\ell}_{,\overline{v}}|_{\overline{s}=0,\overline{v}=kT/\sqrt{2}} = \overline{\delta\ell}_{,\overline{s}}|_{\overline{s}=0,\overline{v}=kT/\sqrt{2}} = \overline{b}_{,\overline{s}}|_{\overline{s}=0,\overline{v}=kT/\sqrt{2}} = 0, \tag{5.4.3}$$

and the wave equation (3.3.5) and its $\overline{s}$ derivatives, in concert with the $\overline{s}$ derivatives of the Riccati equations (3.3.6 – 3.3.8), for $\partial_{\overline{s}}^{\ell}\overline{\gamma}$ on $\overline{s}=0$ as in Proposition 5.2.2, with the conditions

$$\partial_{\overline{s}}^{\ell}\overline{\gamma}|_{\overline{s}=0,\overline{v}=kT/\sqrt{2}} = 0, \quad \ell = 1,\cdots,n+1 \tag{5.4.4}$$

(the $\overline{s}$-differentiated Riccati equations give $\partial_{\overline{s}}^{\ell}\overline{\omega}$ directly, with no need to specify any additional conditions), and finally use this to obtain (see equations (5.3.1), (5.3.2))

$$\overline{\gamma}|_{\overline{v}=0}(\overline{s},\overline{x},0) = \chi(2\overline{s})\sum_{\ell=0}^{m}\frac{\overline{s}^{\ell}}{\ell!}\partial_{\overline{s}}^{\ell}\overline{\gamma}(0,\overline{x},0). \tag{5.4.5}$$

This will complete the determination of the initial data. Differentiating the wave and Riccati equations (3.3.5 – 3.3.8) with respect to tangential derivatives (on $\Sigma_0$ and $U_0$) then allows us to bound the tangential derivatives of the quantities in (5.1.2 – 5.1.3). We then obtain the following final results.

5.4.1. Proposition. Let the initial data be as specified in (5.4.1 – 5.4.5). For every $\epsilon > 0$ there is a $\delta > 0$, independent of $o$ and $k$, such that

$$o < \delta \qquad \text{implies} \qquad \left\|\partial_{\overline{s}}^{\ell_1}\partial_{\overline{x}}^{\ell_2}\partial_{\overline{v}}^{\ell_3}\overline{\omega}\right\|_{L^\infty(\Sigma_0)} < \epsilon, \quad \left\|\partial_{\overline{s}}^{\ell_1}\partial_{\overline{x}}^{\ell_2}\partial_{\overline{v}}^{\ell_3}\overline{\gamma}\right\|_{L^\infty(\Sigma_0)} < \epsilon, \tag{5.4.6}$$
$$\left\|\partial_{\overline{s}}^{\ell_1}\partial_{\overline{x}}^{\ell_2}\partial_{\overline{v}}^{\ell_3}\overline{\omega}\right\|_{L^\infty(U_0^0)} < \epsilon, \quad \left\|\partial_{\overline{s}}^{\ell_1}\partial_{\overline{x}}^{\ell_2}\partial_{\overline{v}}^{\ell_3}\overline{\gamma}\right\|_{L^\infty(U_0^0)} < \epsilon,$$

where $\ell_1 = 0,\cdots,n+1$, $\ell_2,\ell_3 = 0,\cdots,n$. Moreover, the quantities

$$\partial_{\overline{s}}^{\ell_1}\partial_{\overline{x}}^{\ell_2}\partial_{\overline{v}}^{\ell_3}\overline{\omega}|_{\Sigma_0\cup U_0^0}, \quad \partial_{\overline{s}}^{\ell_1}\partial_{\overline{x}}^{\ell_2}\partial_{\overline{v}}^{\ell_3}\overline{\gamma}|_{\Sigma_0\cup U_0^0}$$

are supported on the complement of (see (5.1.11))

$$\Sigma^* = \{(0,\overline{x},\overline{v}) \in \Sigma_0 \,|\, \overline{v} \in (\delta\overline{v}_1, kT\sqrt{2}-\delta\overline{v}_2)\}.$$

Finally, (0.3.31 – 0.3.33) hold.

Proof. This follows from Propositions 5.2.2 and 5.3.2 by taking $m$ sufficiently large ($m = 5(n+1)$ is certainly sufficient) and noting that there is a constant $C$, independent of $o$, such that

$$\|\overline{\gamma}_0\|_{W^{2m+5,\infty}(\Sigma_0)} + \|\overline{\gamma}_0\|_{W^{2m+5,\infty}(U_0^0)} \leq Co.$$

The second assertion follows by construction, and the third by inspection. QED.

5.4.1. Corollary. Proposition 5.4.1 remains valid if we replace the differential operator

$$\partial_{\overline{s}}^{\ell_1}\partial_{\overline{x}}^{\ell_2}\partial_{\overline{v}}^{\ell_3}$$

by

$$\partial_{\overline{s}}^{\ell}\partial_{\tau}^{\ell_1}\partial_{\xi}^{\ell_2}\partial_{\zeta}^{\ell_3}$$

where $\ell = 0, 1$, $\ell_1, \ell_2, \ell_3 = 0, \cdots, n$, and $\tau = \frac{1}{\sqrt{2}}(\overline{s} + \overline{v})$, $\xi = \overline{x}$, $\zeta = \frac{1}{\sqrt{2}}(\overline{s} - \overline{v})$.

Proof. This is clear since the derivative operators span the same space, and the coordinate transformation is linear. QED.

Finally, note that since the initial data have support which is contained in the strip $\{\overline{x} \in [0, 1]\}$ and has measure independent of $k$, Proposition 5.4.1 and Corollary 5.4.1 remain valid if we replace the $L^\infty$ norms by $L^2$ norms, and hence by arbitrary Sobolev norms (at the expense in the latter case, of course, of increasing the number of derivatives of the function $\varpi_0$ we must bound, i.e., the number $m$ in the proof of Proposition 5.4.1).

6. EXISTENCE OF SOLUTIONS

## 6.1. Introduction and summary

This chapter contains the main work in partial differential equations we perform in this thesis.

We begin in Section 6.2 by defining quantities we shall use throughout this chapter, describing again the modification of the main system (3.3.5 – 3.3.8) we shall actually solve (see our discussion after Theorem 0.3.3 above), and stating our basic bootstrap assumptions. We continue in Section 6.3 by showing how to bound a litany of quantities we shall need later, given the bootstrap assumptions and conditions on the initial data; as part of our work we introduce a particular algebra of $C^\infty$ functions* which lies at the core of our arguments in deriving the energy bounds. In Section 6.4 we use a suitable energy current from a uniformly time-like vector field to derive energy bounds for the wave $\gamma$ and the derivatives necessary to close the estimates. In Section 6.5 we apply these energy inequalities to derive extension results of sorts, and in Section 6.6 we prove existence up to a time independent of $k$. Finally, in Section 6.7 we prove properties of the solution for use in comparing it with the literature.

## 6.2. Definitions

We fix a positive integer $n \geq 4$; we shall estimate all our quantities in $H^n$. Recall that by the ansatz (0.2.14) – which is justified by the bootstrap assumptions (0.3.34) and conditions (0.3.31 – 0.3.32), see also equation (6.2.28), equation (6.2.27) and equation (6.2.29) – 'barred' quantities such as $\overline{\delta\ell}$, $\overline{b}$, $\overline{c}$, $\overline{\gamma}$ etc. are bounded – independent of $k$ – in suitable function spaces, and in particular in $L^\infty$. See the footnote following (0.2.7).

In terms of the scaled coordinates $\overline{s}\,\overline{x}\,\overline{v}$, we define coordinates $\tau\xi\zeta$ by

$$\tau = \frac{1}{\sqrt{2}}(\overline{s} + \overline{v}), \qquad \xi = \overline{x}, \qquad \zeta = \frac{1}{\sqrt{2}}(\overline{s} - \overline{v}); \tag{6.2.1}$$

were our metric $h$ the Minkowski metric these would be standard timelike-spacelike coordinates, and hence will be so also for $h$ an $L^\infty$-small perturbation of the Minkowski metric.

As discussed in Section 3.2 above, we have treated $a$, $b$ and $c$ as scalars under the coordinate scaling, so that the quantities $\overline{a}$, $\overline{b}$ and $\overline{c}$ are not the metric components in the scaled coordinates. A straightforward calculation gives in fact the following result. For convenience in analysing the wave equation, we define a scaled metric $\overline{h} = kh$, whence $\overline{h}^{-1} = k^{-1}h^{-1}$. In the $\overline{s}\,\overline{x}\,\overline{v}$ coordinate system, these have representations (for

* While we ultimately obtain existence only in the Sobolev space $H^{n-1}$, we proceed by constructing a sequence, every element of which is $C^\infty$; thus while the algebra we use could presumably be closed off inside some Sobolev space, there is no need to do so and we do not.

the first, see also equation (0.2.13))*

$$
\overline{h}_{ij} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} + \frac{1}{k}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \overline{\delta a} & \overline{b} \\ 0 & \overline{b} & \overline{c} \end{pmatrix}, \qquad (\overline{h}^{-1})^{ij} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} + \frac{1}{k}\begin{pmatrix} k^{-1}\frac{\overline{b}^2}{\overline{a}} - \overline{c} & \frac{\overline{b}}{\overline{a}} & 0 \\ \frac{\overline{b}}{\overline{a}} & \overline{\delta^{-1}a} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{6.2.2}
$$

while in the $\tau\xi\zeta$ system they have the representation

$$
\begin{aligned}
\overline{h}_{ij} &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{k}\begin{pmatrix} \frac{\overline{c}}{2} & \frac{\overline{b}}{\sqrt{2}} & -\frac{\overline{c}}{2} \\ \frac{\overline{b}}{\sqrt{2}} & \overline{\delta a} & -\frac{\overline{b}}{\sqrt{2}} \\ -\frac{\overline{c}}{2} & -\frac{\overline{b}}{\sqrt{2}} & \frac{\overline{c}}{2} \end{pmatrix}, \\
(\overline{h}^{-1})^{ij} &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{k}\begin{pmatrix} \frac{\overline{b}^2}{2k\overline{a}} - \frac{\overline{c}}{2} & \frac{\overline{b}}{\overline{a}\sqrt{2}} & \frac{\overline{b}^2}{2k\overline{a}} - \frac{\overline{c}}{2} \\ \frac{\overline{b}}{\overline{a}\sqrt{2}} & \overline{\delta^{-1}a} & \frac{\overline{b}}{\overline{a}\sqrt{2}} \\ \frac{\overline{b}^2}{2k\overline{a}} - \frac{\overline{c}}{2} & \frac{\overline{b}}{\overline{a}\sqrt{2}} & \frac{\overline{b}^2}{2k\overline{a}} - \frac{\overline{c}}{2} \end{pmatrix};
\end{aligned} \tag{6.2.3}
$$

from this we see that $\overline{h}$ is the Minkowski metric plus a correction proportional to $k^{-1}$. We let $\eta$ denote the Minkowski metric, and define $\overline{\delta h} = k(\overline{h} - \eta)$, $\overline{\delta h^{-1}} = k(\overline{h}^{-1} - \eta^{-1})$; these are given by, in the $\overline{s}\,\overline{x}\,\overline{v}$ coordinate system (by (6.2.2)),

$$
\overline{\delta h}_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \overline{\delta a} & \overline{b} \\ 0 & \overline{b} & \overline{c} \end{pmatrix}, \qquad \overline{\delta h^{-1}}^{ij} = \begin{pmatrix} k^{-1}\frac{\overline{b}^2}{\overline{a}} - \overline{c} & \frac{\overline{b}}{\overline{a}} & 0 \\ \frac{\overline{b}}{\overline{a}} & \overline{\delta^{-1}a} & 0 \\ 0 & 0 & 0 \end{pmatrix},
$$

and in the $\tau\xi\zeta$ coordinate system (by (6.2.3)),

$$
\overline{\delta h}_{ij} = \begin{pmatrix} \frac{\overline{c}}{2} & \frac{\overline{b}}{\sqrt{2}} & -\frac{\overline{c}}{2} \\ \frac{\overline{b}}{\sqrt{2}} & \overline{\delta a} & -\frac{\overline{b}}{\sqrt{2}} \\ -\frac{\overline{c}}{2} & -\frac{\overline{b}}{\sqrt{2}} & \frac{\overline{c}}{2} \end{pmatrix}, \quad \overline{\delta h^{-1}}^{ij} = \begin{pmatrix} \frac{\overline{b}^2}{2k\overline{a}} - \frac{\overline{c}}{2} & \frac{\overline{b}}{\overline{a}\sqrt{2}} & \frac{\overline{b}^2}{2k\overline{a}} - \frac{\overline{c}}{2} \\ \frac{\overline{b}}{\overline{a}\sqrt{2}} & \overline{\delta^{-1}a} & \frac{\overline{b}}{\overline{a}\sqrt{2}} \\ \frac{\overline{b}^2}{2k\overline{a}} - \frac{\overline{c}}{2} & \frac{\overline{b}}{\overline{a}\sqrt{2}} & \frac{\overline{b}^2}{2k\overline{a}} - \frac{\overline{c}}{2} \end{pmatrix}.
$$

We have so far worked on the region $\Gamma_0$ constructed in Chapter 1, namely (see (1.2.3))

$$
\Gamma_0 = \{(s,x,v) \in \mathbf{R}^3 \mid s \in [0,S),\, v \in [0,V)\}.
$$

We now restrict attention to the subset (see (0.3.20); this is actually the set $\Gamma'$ in (0.3.36))

$$
\Gamma = \{(\overline{s},\overline{x},\overline{v}) \in \mathbf{R}^3 \mid 0 \le \overline{s} \le 2T',\, -\infty < \overline{x} < \infty,\, 0 \le \overline{v} \le kT\sqrt{2},\, \tau \le kT\},
$$

where $T'$, $T > 0$ do not depend on $k$, but do depend on the size of the initial data through the parameter $\nu$ to be introduced below, see equation (6.2.27), equation (6.2.29), and equation (6.6.2). Recall (again, see (0.3.20)) that we foliate $\Gamma$ by the timelike hypersurfaces

$$
A_\sigma = \Gamma \cap \{\tau = \sigma\};
$$

* Recall our convention (see Section 0.9) that in the matrix representation of a tensor the first index represents the row and the second the column. Thus, for example, the next equation states that $\overline{h}(\partial_{\overline{s}}, \partial_{\overline{v}}) = -1$, $\overline{h}(\partial_{\overline{x}}, \partial_{\overline{v}}) = k^{-1}\overline{b}$, etc.

$\Gamma$ has the following four boundary sets (see Figure 6.2.1):

$$\Sigma_0 = \Gamma \cap \{\bar{s} = 0\}, \qquad U_0 = \Gamma \cap \{\bar{v} = 0\}, \qquad \Sigma_1 = \Gamma \cap \{\bar{s} = 2T'\}, \qquad A_{kT}.$$

For $\sigma \geq 0$, define $\underline{\upsilon} = \underline{\upsilon}(\sigma) = \sigma - T'\sqrt{2}$; then we define also the three sets

$$\begin{aligned} \Gamma_\sigma &= \Gamma \cap \{(\tau, \xi, \zeta) \in \mathbf{R}^3 \,|\, (\tau, \zeta) \in D_\sigma\}, \\ \Sigma'_\sigma &= \{(\upsilon, \xi, -\upsilon) \in \Sigma_0 \,|\, \upsilon \in [\underline{\upsilon}(\sigma), \sigma]\} \\ B_\upsilon &= \Gamma_\sigma \cap A_\upsilon, \end{aligned} \tag{6.2.4}$$

where $D_\sigma$ is the triangle with sides $\Sigma'_\sigma$ and $A_\sigma$ and in the last line $\upsilon \in [\underline{\upsilon}, \sigma]$ (see Figure 6.2.1). Note that, in the notation of Chapter 4, $B_{\underline{\upsilon}(\sigma)} = \underline{\partial} A_{\underline{\upsilon}(\sigma)}$.

We take the $L^2$ and $H^k$ norms on all of these sets using coordinate Lebesgue measure, i.e., $d\bar{x}\, d\bar{v}$, $d\bar{s}\, d\bar{x}$, $d\bar{x}\, d\bar{v}$, and $d\xi\, d\zeta$, respectively.

For technical reasons, we want $\Sigma_1$ to be a null hypersurface; to ensure this, we modify the metric $\bar{h}$, and hence the wave equation (3.3.5), as follows. Define

$$\widetilde{\overline{\delta\ell}} = \chi\left(\frac{\bar{s}}{T'}\right) \overline{\delta\ell}, \qquad \tilde{\bar{b}} = \chi\left(\frac{\bar{s}}{T'}\right) \bar{b}, \qquad \tilde{\bar{c}} = \chi\left(\frac{\bar{s}}{T'}\right) \bar{c}, \tag{6.2.5}$$

where $\chi$ is the cutoff introduced in 5.3 (we recall that $\chi$ is in $C^\infty(\mathbf{R}^1)$, with support contained in $[-2, 2]$, and satisfies $\chi = 1$ on $[-1, 1]$). Similarly, define

$$\tilde{\bar{a}} = (1 + k^{-1}\widetilde{\overline{\delta\ell}})^2, \qquad \widetilde{\overline{\delta a}} = k(\tilde{\bar{a}} - 1), \qquad \widetilde{\overline{\delta^{-1} a}} = k(\tilde{\bar{a}}^{-1} - 1). \tag{6.2.6}$$

On any Sobolev space, the map $f \mapsto f \cdot \chi(\bar{s}/T')$ will clearly have a norm bounded by some power of $1/T'$, which is independent of $k$. More carefully, let

$$C^0_\chi = \sup\{\chi^{(\ell)}(x) \,|\, x \in \mathbf{R}^1,\, \ell \in \{0, \cdots, n+1\}\}. \tag{6.2.7}$$

Then we have, letting $I$ here denote a multiindex in $\tau\xi\zeta$,

$$\sum_{|I| \leq n+1} \left\| \partial^I \left[\chi(\bar{s}/T') f\right] \right\|_{L^2(A_\sigma)} \leq C^L_n \underline{T}'^{\,-(n+1)} C^0_\chi \sum_{|I| \leq n+1} \|\partial^I f\|_{L^2(A_\sigma)}, \tag{6.2.8}$$

where $C^L_n$ is some combinatorial constant arising from the product rule. (Note that $C$ depends on $n$, but $n$ for us is fixed.) For convenience we define (cf. (0.3.37); requiring $C_\chi \geq 1$ is for technical convenience)

$$C_\chi = \max\{C^L_n C^0_\chi, 1\}. \tag{6.2.9}$$

We define further the modified metric $\tilde{h} = \eta + k^{-1}\widetilde{\delta h}$, $\widetilde{h^{-1}} = \eta^{-1} + k^{-1}\widetilde{\delta h^{-1}}$, where $\widetilde{\delta h}$ and $\widetilde{\delta h^{-1}}$ in the $\bar{s}\,\bar{x}\,\bar{v}$ coordinate system have the representations

$$\widetilde{\delta h}_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \widetilde{\overline{\delta a}} & \tilde{\bar{b}} \\ 0 & \tilde{\bar{b}} & \tilde{\bar{c}} \end{pmatrix}, \qquad \widetilde{\delta h^{-1}}^{\,ij} = \begin{pmatrix} k^{-1}\dfrac{\tilde{\bar{b}}^2}{\tilde{\bar{a}}} - \tilde{\bar{c}} & \dfrac{\tilde{\bar{b}}}{\tilde{\bar{a}}} & 0 \\ \dfrac{\tilde{\bar{b}}}{\tilde{\bar{a}}} & \widetilde{\overline{\delta a}} & 0 \\ 0 & 0 & 0 \end{pmatrix}; \tag{6.2.10}$$

informally, we replace 'barred' quantities by 'tilde-barred' quantities. This leads directly to the following modified wave equation, which is the equation we impose on $\gamma$ for the rest of the chapter:

$$
\begin{aligned}
0 &= \left(-2\partial_{\overline{s}}\partial_{\overline{v}}\overline{\gamma} + \partial_{\overline{x}}^2\overline{\gamma}\right) \\
&+ \frac{1}{k}\left(-\widetilde{\overline{\delta^{-1}a}}\,\partial_{\overline{x}}^2\overline{\gamma} - \widetilde{\overline{c}}\,\partial_{\overline{s}}^2\overline{\gamma} + 2\frac{\widetilde{\overline{b}}}{\widetilde{\overline{a}}}\partial_{\overline{s}}\partial_{\overline{x}}\overline{\gamma} - \frac{1}{2}\left(2\partial_{\overline{s}}\widetilde{\overline{c}} - \frac{2}{\widetilde{\overline{a}}}\partial_{\overline{x}}\widetilde{\overline{b}} + \frac{\partial_{\overline{v}}\widetilde{\overline{\delta a}}}{\widetilde{\overline{a}}}\right)\partial_{\overline{s}}\overline{\gamma} + \frac{1}{\widetilde{\overline{a}}}\partial_{\overline{s}}\widetilde{\overline{b}}\,\partial_{\overline{x}}\overline{\gamma} - \frac{1}{2}\frac{\partial_{\overline{x}}\widetilde{\overline{\delta a}}}{\widetilde{\overline{a}}^2}\partial_{\overline{x}}\overline{\gamma} - \frac{\partial_{\overline{s}}\widetilde{\overline{\delta a}}}{\widetilde{\overline{a}}}\partial_{\overline{v}}\overline{\gamma}\right) \\
&+ k^{-2}\left(\frac{\widetilde{\overline{b}}^2}{\widetilde{\overline{a}}}\partial_{\overline{s}}^2\overline{\gamma} - \frac{1}{2}\left(\widetilde{\overline{c}}\frac{\partial_{\overline{s}}\widetilde{\overline{\delta a}}}{\widetilde{\overline{a}}} - 4\frac{\widetilde{\overline{b}}\,\partial_{\overline{s}}\widetilde{\overline{b}}}{\widetilde{\overline{a}}} + \frac{\widetilde{\overline{b}}\,\partial_{\overline{x}}\widetilde{\overline{\delta a}}}{\widetilde{\overline{a}}^2}\right)\partial_{\overline{s}}\overline{\gamma} - \frac{1}{2}\frac{\widetilde{\overline{b}}}{\widetilde{\overline{a}}^2}\partial_{\overline{s}}\widetilde{\overline{\delta a}}\,\partial_{\overline{x}}\overline{\gamma}\right) - k^{-3}\left(\frac{1}{2}\frac{\widetilde{\overline{b}}^2}{\widetilde{\overline{a}}^2}\partial_{\overline{s}}\widetilde{\overline{\delta a}}\,\partial_{\overline{s}}\overline{\gamma}\right), \qquad (6.2.11)
\end{aligned}
$$

Note that a solution to equation (6.2.11) will satisfy (3.3.5) when $\overline{s} \le T'$. We do not modify the equations satisfied by the metric components, which are still

$$
\partial_{\overline{s}}^2\overline{\delta\ell} = -2\overline{\ell}k^{1-2\iota}(\partial_{\overline{s}}\overline{\gamma})^2, \qquad \partial_{\overline{s}}^2\overline{b} = \frac{1}{\overline{\ell}}k^{-1}(\partial_{\overline{s}}\overline{\delta\ell})(\partial_{\overline{s}}\overline{b}) - 4k^{1-2\iota}\partial_{\overline{s}}\overline{\gamma}(\partial_{\overline{x}}\overline{\gamma} + k^{-1}\overline{b}\partial_{\overline{s}}\overline{\gamma}), \quad (6.2.12) - (6.2.13)
$$

$$
\partial_{\overline{s}}^2\overline{c} = k^{-1}\frac{(\partial_{\overline{s}}\overline{b})^2}{2\overline{a}} - 2k^{1-2\iota}\partial_{\overline{s}}\overline{\gamma}\left(2\partial_{\overline{v}}\overline{\gamma} + 2k^{-1}\frac{\overline{b}}{\overline{a}}\partial_{\overline{x}}\overline{\gamma} + k^{-1}\left(\overline{c} + k^{-1}\frac{\overline{b}^2}{\overline{a}}\right)\partial_{\overline{s}}\overline{\gamma}\right) - k^{1-2\iota}\frac{2}{\overline{a}}(\partial_{\overline{x}}\overline{\gamma})^2 \quad (6.2.14)
$$

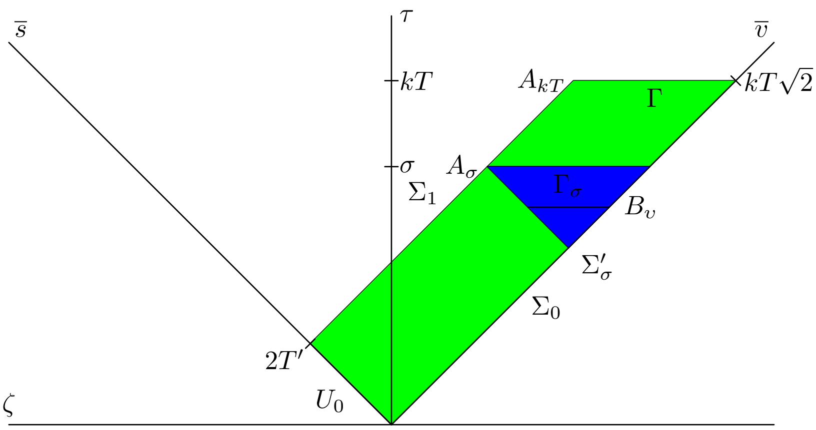


Figure 6.2.1

Recall that we have defined the following quantity $\overline{\mu}$, see equation (0.3.21):

$$
1 + \frac{1}{k}\overline{\mu} = \frac{\sqrt{1 + \frac{1}{k}(\widetilde{\overline{\delta a}} + \frac{\widetilde{\overline{c}}}{2}) + \frac{1}{2k^2}(\widetilde{\overline{c}}\widetilde{\overline{\delta a}} - \widetilde{\overline{b}}^2)}}{\sqrt{1 - \frac{\widetilde{\overline{c}}}{2k}}}; \qquad (6.2.15)
$$

note that $\overline{\mu}$ will be uniformly bounded in $k$. We also define the following quantities,* (compare (0.3.22 – 0.3.24)) which describe the deviation of the boundary integrals in the energy inequality from the corresponding integrals in the Minkowski case:

On $\Gamma$, indices $cd$ correspond to $\tau, \xi\, \zeta$:

* These quantities take the form given here with respect to the $\overline{s}\,\overline{x}\,\overline{v}$ and $\tau\,\xi\,\zeta$ coordinate systems. They do not, in general, transform as tensors to more general coordinate systems.

$$\Delta_A^{cd}(\tau,\xi,\zeta) = \left(1+\frac{1}{k}\overline{\mu}\right)\left\{-\frac{\widetilde{\overline{c}}}{4}\begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} + \frac{1}{2}\left(1-\frac{\widetilde{\overline{c}}}{2k}\right)\widetilde{\delta h^{-1}}^{cd}\right\} + \frac{1}{2}\overline{\mu}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}; \quad (6.2.16)$$

On $\Sigma_0$ and $\Sigma_1$, indices $cd$ correspond to $\overline{x}\,\overline{v}$:

$$\Delta_\Sigma^{cd} = \frac{\widetilde{\overline{\delta\ell}}}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & \frac{1}{2\widetilde{\overline{a}}}\end{pmatrix} + \frac{1}{2\sqrt{2}}\left(1+k^{-1}\widetilde{\overline{\delta\ell}}\right)\widetilde{\overline{\delta^{-1}a}}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}; \quad (6.2.17)$$

On $U_0$, indices $cd$ correspond to $\overline{s}\,\overline{x}\,\overline{v}$:

$$\Delta_U^{cd} = \widetilde{\overline{\delta\ell}}\begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & 0\\ 0 & \frac{1}{2\sqrt{2}} & 0\\ 0 & 0 & 0\end{pmatrix} + \frac{1}{2\sqrt{2}}\left(1+k^{-1}\widetilde{\overline{\delta\ell}}\right)\widetilde{\delta h^{-1}}^{cd}. \quad (6.2.18)$$

We note that $\Delta_\Sigma^{cd}$ is identically zero on $\Sigma_1$ because of the cutoff, but we carry it along sometimes anyway for symmetry.* Note also that $\Delta_U^{cd}$ is zero if $c$ or $d$ is 2, i.e., it only involves $\overline{s}$ $(c,d=0)$ and $\overline{x}$ $(c,d=1)$. We define the following norms for use with these quantities: if $M$ is a (2,0)-tensor on some Euclidean space, we define (letting $|u_i|$, $|u^i|$ denote the Euclidean norm)

$$\begin{aligned}\|M\| &= \sup\left\{\left|M^{cd}u_d\right| \,\middle|\, |u_d| = 1\right\},\\ \|M\|_{HS} &= \left(\sum_{c,d}\left(M^{cd}\right)^2\right)^{1/2}.\end{aligned} \quad (6.2.19)$$

This gives a norm on the pointwise values of the quantities $\Delta_A^{cd}$, $\Delta_\Sigma^{cd}$, and $\Delta_U^{cd}$.

The initial data for the system (6.2.11 – 6.2.14) will be that constructed in Chapter 5; see equations (5.4.1 – 5.4.5) and Proposition 5.4.1. We recall that this initial data satisfies $\overline{b} = \overline{c} = \overline{c}_{,\overline{s}} = 0$ on $\Sigma_0$, while on $\Sigma_0$ the support of $\overline{\delta\ell}$, $\partial_{\overline{s}}\overline{\delta\ell}$, $\partial_{\overline{s}}\overline{b}$, and $\partial_{\overline{s}}^{\ell}\overline{\gamma}$ $(\ell = 0,1,\cdots,n+1)$ are all contained in that of $\overline{\gamma}_0$, i.e., $\{(\overline{x},\overline{v})\in\mathbf{R}^2 \,|\, \overline{x},\overline{v}\in[0,1]\}$.

We define the following sets of dependent variables (compare (0.3.25)):

$$\begin{aligned}\overline{\Omega}_0 &= \{\overline{\delta\ell},\overline{b},\overline{c}\}, &\quad \overline{\Omega} &= \{\overline{\delta\ell},\overline{b},\overline{c},\partial_{\overline{x}}\overline{\delta\ell},\partial_{\overline{v}}\overline{\delta\ell},\partial_{\overline{x}}\overline{b}\}\\ \widetilde{\overline{\Omega}}_0 &= \{\widetilde{\overline{\delta\ell}},\widetilde{\overline{b}},\widetilde{\overline{c}}\}, &\quad \widetilde{\overline{\Omega}} &= \{\widetilde{\overline{\delta\ell}},\widetilde{\overline{b}},\widetilde{\overline{c}},\partial_{\overline{x}}\widetilde{\overline{\delta\ell}},\partial_{\overline{v}}\widetilde{\overline{\delta\ell}},\partial_{\overline{x}}\widetilde{\overline{b}}\}.\end{aligned} \quad (6.2.20)$$

The significance of these sets will become more apparent later (see, for example, Lemma 6.3.4 and ensuing discussion). For the moment we note that the (nonconstant) coefficients appearing in the wave equation are precisely the elements of $\widetilde{\overline{\Omega}}\cup\partial_{\overline{s}}\widetilde{\overline{\Omega}}_0$ $(\partial_{\overline{s}}\widetilde{\overline{\Omega}}_0 = \{\partial_{\overline{s}}\widetilde{\overline{\delta\ell}},\partial_{\overline{s}}\widetilde{\overline{b}},\partial_{\overline{s}}\widetilde{\overline{c}}\})$. When convenient, we shall also consider them as tuples, and write things like $F(\overline{\Omega})$ to denote a function which depends on the elements of $\overline{\Omega}$ (as in, for example, Proposition 6.3.3 below).

We let $I = (i_1,i_2)$ denote a multiindex, and set $|I| = i_1 + i_2$, $\partial^I = \partial_\xi^{i_1}\partial_\zeta^{i_2}$. We shall bound the initial

* We also note that it does not seem strictly necessary to cutoff $\overline{\delta\ell}$, only $\overline{c}$ and (probably) $\overline{b}$. Thus keeping $\Delta_\Sigma^{cd}$ on $\Sigma_1$ may lead to some generalisations of the succeeding results.

data by the following (cf. (0.3.26)):

$$
\begin{aligned}
\iota_{n,\ell}[h](\sigma) &= \sqrt{2} \sum_{|I|\le n-1} \sum_{\overline{\omega}\in\overline{\Omega}} \left\| \partial^I \partial_{\overline{s}}^{\ell} \overline{\omega} \right\|^2_{L^2(\Sigma'_\sigma)}, \\
\iota_n[h](\sigma) &= \sum_{\ell=0}^{1} \iota_{n,\ell}[h](\sigma), \\
\overline{I}_{\Sigma_0}[f] &= \int_{\Sigma_0} \frac{1}{\sqrt{2}} \left[ \frac{1}{2}(\partial_{\overline{x}} f)^2 + (\partial_{\overline{v}} f)^2 \right] + k^{-1} \Delta_\Sigma^{cd} \partial_c f \partial_d f \, d\overline{v}\, d\overline{x} \\
\overline{I}_{U_0}[f] &= \int_{U_0} \frac{1}{\sqrt{2}} \left[ (\partial_{\overline{s}} f)^2 + \frac{1}{2}(\partial_{\overline{x}} f)^2 \right] + k^{-1} \Delta_U^{cd} \partial_c f \partial_d f \, d\overline{s}\, d\overline{x} \\
\overline{\iota}_{X,n,\ell}[\overline{\gamma}] &= \sum_{|I|\le n-1} \overline{I}_X[\partial^I \partial_{\overline{s}}^{\ell} \overline{\gamma}] \\
\overline{\iota}_{X,n}[\overline{\gamma}] &= \sum_{\ell=0}^{1} \overline{\iota}_{X,n,\ell}[\overline{\gamma}],
\end{aligned}
\tag{6.2.21}
$$

where in the last two lines $X$ denotes one of $\Sigma_0$ and $U_0$. We also define the following quantities, which are squares of norms along lines in $\Sigma_0$, for use in applying Proposition 4.4.2 and Lemma 4.4.1:*

$$
\begin{aligned}
&\underline{I}^\circ_m[f](\sigma) = \sum_{|I|\le m} \left\| \partial^I f \right\|^2_{L^2(\underline{\partial} A_\sigma)}, &&\underline{I}^1_m[f](\sigma) = \sum_{|I|\le m} \left\| \partial^I f \right\|^2_{H^1(\underline{\partial} A_\sigma)}, \\
&\underline{\iota}[h](\sigma) = \sum_{\ell=0}^{1} \sum_{\overline{\omega}\in\overline{\Omega}} \underline{I}^\circ_{n-1}[\partial_{\overline{s}}^{\ell}\overline{\omega}](\sigma), &&\underline{\iota}^1[h](\sigma) = \sum_{\ell=0}^{1} \sum_{\overline{\omega}\in\overline{\Omega}} \underline{I}^1_{n-1}[\partial_{\overline{s}}^{\ell}\overline{\omega}](\sigma), \\
&\underline{I}[\overline{\gamma}](\sigma) = \sum_{\ell_1,\ell_2=0}^{1} \sum_{i=0}^{2} \underline{I}^\circ_{n-1}[\partial_{\overline{s}}^{\ell_1} \partial_i^{\ell_2} \overline{\gamma}](\sigma), &&\underline{I}^1[\overline{\gamma}](\sigma) = \sum_{\ell_1,\ell_2=0}^{1} \sum_{i=0}^{2} \underline{I}^1_{n-1}[\partial_{\overline{s}}^{\ell_1} \partial_i^{\ell_2} \overline{\gamma}](\sigma),
\end{aligned}
\tag{6.2.22}
$$

where here $\partial_0 = \partial_\tau$, $\partial_1 = \partial_\xi$, and $\partial_2 = \partial_\zeta$. We have clearly

$$
\underline{\iota}[h](\sigma) \le \underline{\iota}^1[h](\sigma), \qquad \underline{I}[h](\sigma) \le \underline{I}^1[h](\sigma). \tag{6.2.23}
$$

We note the following bounds, for use in applying Proposition 4.4.2:

$$
\begin{aligned}
\sum_{|I|+\ell\le 1} \sum_{m=0}^{1} \sum_{\overline{\omega}\in\overline{\Omega}} \left\| \partial^I \partial_\tau^\ell \partial_{\overline{x}}^m \overline{\omega} \right\|^2_{L^2(\underline{\partial} A_\sigma)} &\le \underline{\iota}[h](\sigma), \\
\sum_{\ell=0}^{1} \sum_{|I|\le 2} \sum_{m=0}^{1} \left\| \partial^I \partial_{\overline{s}}^\ell \partial_{\overline{x}}^m \overline{\gamma} \right\|^2_{L^2(\underline{\partial} A_\sigma)} &\le \underline{I}[\overline{\gamma}].
\end{aligned}
\tag{6.2.24}
$$

Note that, by Corollary 5.4.1, all quantities in (6.2.21 – 6.2.24) can be made as small as we like by taking the parameter $o$ in the definition of $\overline{\gamma}_0$ (see equations (5.1.18 – 5.1.18), (5.4.2)) sufficiently small.

---

* Our use of the $H^1$ norm in $\underline{I}^1_m[f]$ is dictated by the following considerations. As noted, the quantities $\underline{I}^1_m[f]$ will be used in applying Lemmas 4.4.1 – 4.4.3, so we need $\underline{I}^1_m[f]$ to bound $L^2$ norms of $\partial^I f$. For the purposes of dealing with the so-called *admissible nonlinearities* we shall introduce below, we wish $\underline{I}^1_m[f]$ itself to satisfy a multiplication inequality. The $H^1$ norm seems to be the most general choice which can simultaneously satisfy both of these requirements. Note that the extra $\overline{x}$ derivative appearing in the $H^1$ norm in the definition of $\underline{I}^1_m$ is no cause for concern since we are working only with initial data.

We define energies* on the spacelike hypersurfaces $A_\sigma$ as follows; cf. (0.3.29 – 0.3.30). Setting

$$\epsilon[f](\sigma) = \int_{A_\sigma} \frac{1}{2}\left[(\partial_\tau f)^2 + (\partial_\xi f)^2 + (\partial_\zeta f)^2\right] + k^{-1}\Delta_A^{cd}\partial_c f \partial_d f \, d\xi \, d\zeta, \tag{6.2.25}$$

we define

$$\begin{aligned} \overline{E}_n[\overline{\gamma}](\sigma) &= \sum_{|I|\le n-1} \sum_{\ell=0}^{1} \epsilon[\partial^I \partial_{\overline{s}}^{\ell} \overline{\gamma}](\sigma), \\ E_{n,\ell}[h](\sigma) &= \sum_{|I|\le n-1} \sum_{\overline{\omega}\in\overline{\Omega}} \left|\partial^I \partial_{\overline{s}}^{\ell} \overline{\omega}\right|^2_{L^2(A_\sigma)}, \\ E_n[h](\sigma) &= \sum_{\ell=0}^{1} E_{n,\ell}[h](\sigma). \end{aligned} \tag{6.2.26}$$

These give squares of (semi-) norms of $C^\infty$ functions on $A_\sigma$, and it is effectively with respect to these that we shall close the energy estimates for the system (6.2.11 – 6.2.14). Note that every element of $\overline{\Omega} \cup \partial_{\overline{s}}\overline{\Omega}$ is bounded in $H^{n-1}(A_\sigma)$ by $E_n[h](\sigma)^{1/2}$ (see Proposition 6.3.1 in the next section).

The notation $E_n[h]$ is used for convenience; clearly, what we are really bounding is the (scaled) metric components $\overline{\delta\ell}$, $\bar{b}$, and $\bar{c}$. Until we introduce admissible nonlinearities in Definition 6.3.1 below, we shall only work with a single metric. Our convention beyond that will be explained at that point.

We shall proceed via a bootstrap argument. Fix $\nu \in (0,1)$, assume that the initial data satisfy (see (0.3.31))

$$\sup_{\sigma\le kT} \iota_n[h](\sigma) \le \frac{1}{32}\nu^2, \qquad \bar{\iota}_{\Sigma_0,n}[\overline{\gamma}] + \bar{\iota}_{U_0,n}[\overline{\gamma}] \le \frac{1}{12}\nu^2, \tag{6.2.27}$$

and make the bootstrap assumption (see (0.3.34))

$$\begin{aligned} E_n[h](\sigma) &\le \nu^2, \\ \overline{E}_n[\overline{\gamma}](\sigma) &\le \nu^2 \end{aligned} \tag{6.2.28}$$

for $\sigma \in [0, \varsigma]$. At this point the dependence of $\varsigma$ on $k$ is unknown; we will show (see Theorem 6.5.1, Theorem 6.6.1) that it is bounded below by $kT$.

We assume that the initial data also satisfy (see (0.3.32 – 0.3.33))

$$\begin{aligned} \underline{\iota}^1[h](\sigma) &\le \frac{1}{1+\sqrt{2}(n+2)C_S^2 C_\chi^2}\nu^2, \\ \underline{I}^1[\overline{\gamma}](\sigma) &\le \frac{1}{1+\sqrt{2}(n+2)C_S^2 C_\chi^2}\nu^2, \end{aligned} \tag{6.2.29}$$

where $C_S$ is the Sobolev embedding constant on $\mathbf{R}^1$, and $\sigma \in [0, \varsigma]$. Note that initial data satisfying (6.2.27) and (6.2.29) exist, by Corollary 5.4.1.

---

* We term the quantity $E_n[h](\sigma)$ an energy not because of any resemblance to a physically or geometrically significant energy but because its use in proving bounds is analogous to that of the energy for the wave $\overline{\gamma}$ defined above.

The significance of the leading constant will not become apparent until we come to prove existence at the end of the chapter; until then, we only need the weaker conditions

$$\underline{\iota}^1[h](\sigma) \le \nu^2,$$
$$\underline{I}^1[\overline{\gamma}](\sigma) \le \nu^2.$$

By (6.2.23), these give

$$\begin{aligned} \underline{\iota}[h](\sigma) &\le \nu^2, \\ \underline{I}[\overline{\gamma}](\sigma) &\le \nu^2. \end{aligned} \tag{6.2.30}$$

While these last bounds shall often be used in conjunction with the bootstrap assumption (6.2.28), they are not part of it, but rather additional smallness conditions on the initial data.

### 6.3. Quantities bounded by the bootstrap

In this section we show how the results in Chapter 4 can be used to bound quantities appearing in the system (6.2.11 – 6.2.14) in terms of the energies defined in (6.2.26) and the norms on the initial data in (6.2.21) and (6.2.22). The inequalities allowing us to bound the energies in terms of integrals over the bulk will be derived in the next section.

The bounds we prove are of two kinds. The first (through Corollary 6.3.5) follow directly from the definitions of the energies, the bootstrap assumptions, and the smallness assumptions on the initial data, as given in equations (6.2.26 – 6.2.30) above, and in particular make no use of the equations of motion. These bounds are therefore independent of $T'$. In the second set we apply the equations of motion, and the resulting bounds will in general depend on $T'$.

We first make some additional definitions. The function

$$\mu_0(x) = \begin{cases} \frac{(1+x)^{-1/2}-1}{x}, & x \neq 0 \\ -\frac{1}{2}, & x = 0 \end{cases}$$

is continuous on $(-1, \infty)$; let

$$C_\mu = \sup_{x \in [-3/4, 3/4]} |\mu_0(x)| < \infty,$$

so that for $x \in [-3/4, 3/4]$ we have

$$(1+x)^{-1/2} \le 1 + C_\mu |x|.$$

(Geometrically – since $\mu_0(x)$ is just the slope of a secant line on the graph of $y = (1+x)^{-1/2}$ – it appears that $C_\mu = 4/3$, but the precise numerical value is not important.) Now define also

$$\begin{aligned} \overline{\overline{\mu}}(\nu) &= \left[2C_\mu + \frac{15}{2}\right] C_0 \nu + 15 C_\mu (C_0 \nu)^2, \\ \overline{\Delta_A}(\nu) &= \overline{\overline{\mu}}(\nu) + 98 C_0 \nu, \\ \overline{\Delta_\Sigma}(\nu) &= 10 C_0 \nu, \qquad \overline{\Delta_U}(\nu) = 17 C_0 \nu, \end{aligned}$$

where $C_0$ is the Sobolev embedding constant from Proposition 4.4.2, and assume (without loss of generality, since none of the above quantities depend on $k$)

$$k \ge 20 \max \left\{ \overline{\overline{\mu}}(\nu), \overline{\Delta_A}(\nu), \overline{\Delta_\Sigma}(\nu), \overline{\Delta_U}(\nu) \right\}. \tag{6.3.1}$$

In particular, $k \geq 12C_0\nu$.

Recall that the quantities $\overline{\mu}$, $\Delta_A$, $\Delta_\Sigma$, and $\Delta_U$ were defined in equations (6.2.15) – (6.2.18) above.

6.3.1. PROPOSITION. Suppose that $\overline{\Omega}$ is such that the bootstrap assumption (6.2.28) and the bounds (6.2.30) hold. Then (recall $n \geq 4$)

(a) $\|\partial_{\overline{s}}^\ell \overline{\omega}\|_{H^{n-1}(A_\sigma)} \leq E_n[h](\sigma)^{1/2}$ for all $\overline{\omega} \in \overline{\Omega}$, $\ell \in \{0, 1\}$.

(b) $\|\partial_i^\ell \overline{\omega}\|_{L^\infty(A_\sigma)} \leq 4C_0\nu$ for all $\overline{\omega} \in \overline{\Omega}$, where $\partial_i$ denotes any derivative in $\tau$, $\xi$, or $\zeta$ and $\ell \in \{0, 1\}$.

(c) If $k$ satisfies (6.3.1), then the following bounds hold for $\sigma \in [0, \varsigma]$:

$$\|\overline{\mu}\|_{L^\infty(A_\sigma)} \leq \overline{\overline{\mu}}, \quad \|\Delta_A\|_{L^\infty(A_\sigma)} \leq \overline{\Delta_A}(\nu), \quad \|\Delta_\Sigma\|_{L^\infty(A_\sigma \cap \Sigma_0)} \leq \overline{\Delta_\Sigma}(\nu), \quad \|\Delta_U\|_{L^\infty(A_\sigma \cap U_0)} \leq \overline{\Delta_U}(\nu),$$

where the norms on the quantities $\Delta_A$, $\Delta_\Sigma$, and $\Delta_U$ are as defined in equations (6.2.19) above.

Proof. (a) This is evident from the definition of $E_n[h](\sigma)$ (see (6.2.26)).

(b) Recall (see (6.2.24)) that $\underline{\iota}[h](\sigma)$ bounds the sum over $\overline{\Omega}$, $i$ and $\ell$ of $\|\partial_i^\ell \overline{\omega}\|^2_{H^1(\underline{\partial} A_\sigma)}$. Now let $\overline{\omega} \in \overline{\Omega} = \{\overline{\delta\ell}, \overline{b}, \overline{c}, \partial_{\overline{x}}\overline{\delta\ell}, \partial_{\overline{v}}\overline{\delta\ell}, \partial_{\overline{x}}\overline{b}\}$, and suppose first that $\ell = 0$. Then by (a) and Proposition 4.4.2, since $n \geq 4$,

$$\begin{aligned} \|\overline{\omega}\|_{L^\infty(A_\sigma)} &\leq C_0 \left[\|\overline{\omega}\|_{H^{n-2}(A_\sigma)} + \|\overline{\omega}\|_{H^1(\underline{\partial} A_\sigma)}\right] \\ &\leq C_0 \left[E_n[h](\sigma)^{1/2} + \underline{\iota}[h](\sigma)^{1/2}\right] \leq 4C_0\nu. \end{aligned}$$

Now if $\ell = 1$ and $\partial_i = \partial_\xi$ or $\partial_i = \partial_\zeta$, then

$$\begin{aligned} \|\partial_i\overline{\omega}\|_{L^\infty(A_\sigma)} &\leq C_0 \left[\|\partial_i\overline{\omega}\|_{H^{n-2}(A_\sigma)} + \|\partial_i\overline{\omega}\|_{H^1(\underline{\partial} A_\sigma)}\right] \\ &\leq C_0 \left[\|\overline{\omega}\|_{H^{n-1}(A_\sigma)} + \underline{\iota}[h](\sigma)^{1/2}\right] \leq C_0 \left[E_n[h](\sigma)^{1/2} + \underline{\iota}[h](\sigma)^{1/2}\right] \leq 4C_0\nu. \end{aligned}$$

Since $\partial_\tau = \sqrt{2}\partial_{\overline{s}} - \partial_\zeta$, the case $\partial_i = \partial_\tau$ can be handled as follows:

$$\begin{aligned} \|\partial_\tau\overline{\omega}\|_{L^\infty(A_\sigma)} &\leq C_0 \left[\|\partial_\tau\overline{\omega}\|_{H^{n-2}(A_\sigma)} + \|\partial_\tau\overline{\omega}\|_{H^1(\underline{\partial} A_\sigma)}\right] \\ &\leq C_0 \left[\sqrt{2}\|\partial_{\overline{s}}\overline{\omega}\|_{H^{n-2}(A_\sigma)} + \|\overline{\omega}\|_{H^{n-1}(A_\sigma)} + \underline{\iota}[h](\sigma)^{1/2}\right] \\ &\leq C_0 \left[3E_n[h](\sigma)^{1/2} + \underline{\iota}[h](\sigma)^{1/2}\right] \leq 4C_0\nu, \end{aligned}$$

completing the proof of this part.

(c) This part is entirely straightforward, though slightly tedious. First, we note that by (a) and equation (6.3.1) we have

$$\begin{aligned} &\|\overline{\delta\ell}\|_{L^\infty(A_\sigma)} \leq 4C_0\nu, \qquad \|\overline{b}\|_{L^\infty(A_\sigma)} \leq 4C_0\nu, \qquad \|\overline{c}\|_{L^\infty(A_\sigma)} \leq 4C_0\nu, \\ &\frac{1}{k}\|\overline{\delta\ell}\|_{L^\infty(A_\sigma)} \leq \frac{1}{12}, \qquad \frac{1}{k}\|\overline{b}\|_{L^\infty(A_\sigma)} \leq \frac{1}{12}, \qquad \frac{1}{k}\|\overline{c}\|_{L^\infty(A_\sigma)} \leq \frac{1}{12}. \end{aligned} \tag{6.3.2}$$

Moreover, we have

$$\overline{\delta a} = k(\overline{a} - 1) = \overline{\delta\ell}\left(2 + \frac{1}{k}\overline{\delta\ell}\right),$$

so

$$\|\overline{\delta a}\|_{L^\infty(A_\sigma)} \leq 12C_0\nu, \qquad \frac{1}{k}\|\overline{\delta a}\|_{L^\infty(A_\sigma)} \leq \frac{1}{4};$$

thus moreover

$$\frac{1}{\overline{a}} = \frac{1}{1 + \frac{1}{k}\overline{\delta a}} \leq \frac{1}{1 - \frac{1}{4}} = \frac{4}{3},$$

so

$$\|\overline{\delta^{-1}a}\|_{L^\infty(A_\sigma)} = k\left\|\frac{1}{\overline{a}} - 1\right\|_{L^\infty(A_\sigma)} = k\left\|\frac{\overline{a}-1}{a}\right\|_{L^\infty(A_\sigma)} \leq \frac{4}{3}\|\overline{\delta a}\|_{L^\infty(A_\sigma)} \leq 16C_0\nu,$$

$$\frac{1}{k}\|\overline{\delta^{-1}a}\|_{L^\infty(A_\sigma)} \leq \frac{1}{3}.$$

Since replacing barred quantities with tilde-barred ones amounts to multiplying $\overline{\delta\ell}$, $\overline{b}$ and $\overline{c}$ by the cutoff, which preserves the bounds in (6.3.2), the above bounds are also valid for the tilde-barred quantities. Now we note that for $x \in [-1,\infty)$ we have $(1+x)^{1/2} \leq 1 + x/2$; by the definition of $C_\mu$ and the bound $k^{-1}\|\overline{c}\|_{L^\infty(A_\sigma)} \leq 1/12$, we have

$$\left(1 - \frac{\widetilde{\overline{c}}}{2k}\right)^{-1/2} \leq 1 + C_\mu\frac{\widetilde{\overline{c}}}{2k}.$$

Pulling this together, we have the bound

$$\begin{aligned}\overline{\mu} &= k\left[\frac{\sqrt{1 + \frac{1}{k}(\widetilde{\overline{\delta a}} + \frac{\widetilde{\overline{c}}}{2}) + \frac{1}{2k^2}(\widetilde{\overline{c}}\widetilde{\overline{\delta a}} - \widetilde{\overline{b}}^2)}}{\sqrt{1 - \frac{\widetilde{\overline{c}}}{2k}}} - 1\right] \\ &\leq k\left[\left\{1 + \frac{1}{2}\left(\frac{1}{k}(12C_0\nu + 2C_0\nu) + \frac{1}{2k^2}\left(64C_0^2\nu^2\right)\right)\right\}\left\{1 + \frac{2}{k}C_\mu\mathbf{C}_0\nu\right\} - 1\right] \\ &\leq 2C_\mu C_0\nu + \frac{1}{2}(14C_0\nu + C_0\nu) + C_\mu C_0\nu(15C_0\nu) \leq \left[2C_\mu + \frac{15}{2}\right]C_0\nu + 15C_\mu(C_0\nu)^2 = \overline{\overline{\mu}}.\end{aligned}$$

Now we wish to bound the operator norms of the matrices $\Delta_A$, $\Delta_\Sigma$, and $\Delta_U$. We shall do this by bounding their Hilbert-Schmidt norms and using the bound

$$\|A\| \leq \|A\|_{HS},$$

where $\|\cdot\|$ denotes the operator norm and $\|\cdot\|_{HS}$ the Hilbert-Schmidt norm. Noting that, from the foregoing,

$$\left|\frac{\overline{b}^2}{2k\overline{a}}\right| \leq \frac{32}{3k}C_0^2\nu^2 \leq \frac{2}{9}C_0\nu, \qquad \left|\frac{\overline{c}}{2}\right| \leq 2C_0\nu, \qquad \left|\frac{\overline{b}}{\overline{a}\sqrt{2}}\right| \leq \frac{16}{3}C_0\nu,$$

and that identical bounds hold for the cutoff quantities, we see that

$$\|\overline{\delta h^{-1}}\|_{HS},\ \|\widetilde{\overline{\delta h^{-1}}}\|_{HS} \leq C_0\nu\left[4\left(\frac{2}{9}+2\right)^2 + 4\left(\frac{16}{3}\right)^2 + 16^2\right]^{1/2} \leq 47C_0\nu,$$

so that

$$\begin{aligned}\|\Delta_A\| &\leq \|\Delta_A\|_{HS} \leq \left(1 + \frac{1}{k}\overline{\overline{\mu}}\right)\left(C_0\nu\sqrt{3} + \frac{1}{2}(2)\|\widetilde{\overline{\delta h^{-1}}}\|_{HS}\right) + \frac{1}{2}\overline{\overline{\mu}}\sqrt{3} \\ &\leq 2(49C_0\nu) + \overline{\overline{\mu}} \leq 98C_0\nu + \overline{\overline{\mu}} = \overline{\Delta_A}(\nu), \\ \|\Delta_\Sigma\| &\leq \|\Delta_\Sigma\|_{HS} \leq 4C_0\nu\left(1 + \left(\frac{2}{3}\right)^2\right)^{1/2} + \frac{1}{4}\left(1 + \frac{1}{12}\right)16C_0\nu \leq 10C_0\nu = \overline{\Delta_\Sigma}(\nu), \\ \|\Delta_U\| &\leq \|\Delta_U\|_{HS} \leq 4C_0\nu\left(\frac{1}{2} + \frac{1}{8}\right)^{1/2} + \frac{1}{4}\left(1 + \frac{1}{12}\right)47C_0\nu \leq 17C_0\nu = \overline{\Delta_U}(\nu),\end{aligned}$$

and since these bounds are uniform, this completes the proof. QED.

We note the following bounds derived in the above proof for future reference.

6.3.1. COROLLARY. Under the conditions of Proposition 6.3.1, we have

$$\|\overline{\delta a}\|_{L^\infty(A_\sigma)} \leq 12C_0\nu, \quad \|\overline{\delta^{-1} a}\|_{L^\infty(A_\sigma)} \leq 16C_0\nu, \quad \frac{1}{\overline{a}} \leq \frac{4}{3}, \quad \|\overline{\delta h^{-1}}\|_{HS} \leq 47C_0\nu, \quad \|\widetilde{\delta h^{-1}}\|_{HS} \leq 47C_0\nu.$$

Proposition 6.3.1, together with Lemma 4.2.2, also allow us to bound $\overline{I}_X[f]$.

6.3.2. COROLLARY. Under the conditions of Proposition 6.3.1, we have

$$\begin{aligned}\frac{1}{6}\|f\|_{H^1_\circ(\Sigma_0)} &\leq \overline{I}_{\Sigma_0}[f] \leq \|f\|_{H^1_\circ(\Sigma_0)},\\ \frac{1}{6}\|f\|_{H^1_\circ(U_0)} &\leq \overline{I}_{U_0}[f] \leq \|f\|_{H^1_\circ(U_0)},\end{aligned}$$

where the norms were defined in (4.4.1 – 4.4.2) above.

Proof. Given Lemma 4.2.2, this follows from Proposition 6.3.1 as well as equation (6.3.1). We show it for $\overline{I}_{\Sigma_0}[f]$; the proof for $\overline{I}_{U_0}[f]$ is exactly analogous. We have

$$\overline{I}_{\Sigma_0}[f] = \int_{\Sigma_0} \frac{1}{\sqrt{2}}\left[\frac{1}{2}(\partial_{\overline{x}} f)^2 + (\partial_{\overline{v}} f)^2\right] + k^{-1}\Delta_\Sigma^{cd}\partial_c f \partial_d f \, d\overline{v}\, d\overline{x}.$$

Now (letting $c$ and $d$ represent $\overline{x}$ and $\overline{v}$)

$$\left|\Delta_\Sigma^{cd}\partial_c f\partial_d f\right| \leq \|\Delta_\Sigma\|_{HS}\|\partial_c f\partial_d f\|_{HS} \leq \overline{\Delta_\Sigma}(\nu)|\partial_c f|^2 = \overline{\Delta_\Sigma}(\nu)\left[(\partial_{\overline{x}} f)^2 + (\partial_{\overline{v}} f)^2\right],$$

whence

$$\begin{aligned}\overline{I}_{\Sigma_0}[f] &\leq \int_{\Sigma_0} \frac{1}{\sqrt{2}}\left[\frac{1}{2}(\partial_{\overline{x}} f)^2 + (\partial_{\overline{v}} f)^2\right] + \frac{1}{20}\left[(\partial_{\overline{x}} f)^2 + (\partial_{\overline{v}} f)^2\right] d\overline{v}\, d\overline{x}\\ &\leq \int_{\Sigma_0} (\partial_{\overline{x}} f)^2 + (\partial_{\overline{v}} f)^2 \, d\overline{v}\, d\overline{x} = \|f\|_{H^1_\circ(\Sigma_0)},\end{aligned}$$

and similarly

$$\begin{aligned}\overline{I}_{\Sigma_0}[f] &\geq \int_{\Sigma_0} \frac{1}{2\sqrt{2}}\left[(\partial_{\overline{x}} f)^2 + (\partial_{\overline{v}} f)^2\right] - \frac{1}{20}\left[(\partial_{\overline{x}} f)^2 + (\partial_{\overline{v}} f)^2\right] d\overline{v}\, d\overline{x}\\ &\geq \frac{1}{6}\int_{\Sigma_0} (\partial_{\overline{x}} f)^2 + (\partial_{\overline{v}} f)^2 \, d\overline{v}\, d\overline{x},\end{aligned}$$

as claimed. QED.

We now have the following proposition.

6.3.2. PROPOSITION. Suppose that $\overline{\Omega}$ satisfies the bootstrap condition (6.2.28) and the bounds on initial data (6.2.30).

(a) For any $f \in C^\infty(\Gamma)$ and any $\sigma \in [0, \varsigma]$,

$$\frac{9}{10}\left[\|f\|^2_{H^1_\circ(A_\sigma)} + \|\partial_\tau f\|^2_{L^2(A_\sigma)}\right] \leq 2\epsilon[f](\sigma) \leq \frac{11}{10}\left[\|f\|^2_{H^1_\circ(A_\sigma)} + \|\partial_\tau f\|^2_{L^2(A_\sigma)}\right],$$

and thus

$$\frac{9}{10}\left[\|f\|^2_{H^n_\circ(A_\sigma)}+\|\partial_\tau f\|^2_{H^{n-1}_\circ(A_\sigma)}\right]\le 2\overline{E}_n[f](\sigma)\le\frac{11}{10}\left[\|f\|^2_{H^n_\circ(A_\sigma)}+\|\partial_\tau f\|^2_{H^{n-1}_\circ(A_\sigma)}\right].$$

(b) If $\overline{\gamma}$ satisfies the bootstrap condition (6.2.28) and the bounds on initial data (6.2.30), then for $\sigma\in[0,\varsigma]$

$$\left\|\partial_i^\ell\overline{\gamma}\right\|_{L^\infty(A_\sigma)}\le 4C_0\nu,$$

where as in Proposition 6.3.1 $\partial_i$ denotes any of the derivatives $\partial_\tau$, $\partial_\xi$, $\partial_\zeta$, and $\ell\in\{0,1\}$.

Proof. (a) By Proposition 6.3.1(c) and the bounds (6.3.1) on $k$, we have that $k^{-1}\|\Delta_A(\sigma)\|\le 1/20$ for $\sigma\in[0,\varsigma]$; thus for such $\sigma$ and any $f\in C^\infty(\Gamma)$ we have by Lemma 4.2.2,

$$\begin{aligned}\frac{1}{k}\left|\Delta_A^c d(\sigma)\partial_c f\partial_d f\right| &\le \frac{1}{k}\|\Delta_A(\sigma)\|\left[(\partial_\tau f)^2+(\partial_\xi f)^2+(\partial_\zeta f)^2\right]\\ &\le\frac{1}{20}\left[(\partial_\tau f)^2+(\partial_\xi f)^2+(\partial_\zeta f)^2\right];\end{aligned}$$

thus

$$\begin{aligned}\epsilon[f](\sigma)&=\int_{A_\sigma}\frac{1}{2}\left[(\partial_\tau f)^2+(\partial_\xi f)^2+(\partial_\zeta f)^2\right]+k^{-1}\Delta_A^{cd}\partial_c f\partial_d f\,d\xi\,d\zeta\\ &\ge\int_{A_\sigma}\frac{1}{2}\cdot\frac{9}{10}\left[(\partial_\tau f)^2+(\partial_\xi f)^2+(\partial_\zeta f)^2\right]\,d\xi\,d\zeta,\end{aligned}$$

and similarly

$$\epsilon[f](\sigma)\le\int_{A_\sigma}\frac{1}{2}\cdot\frac{11}{10}\left[(\partial_\tau f)^2+(\partial_\xi f)^2+(\partial_\zeta f)^2\right]\,d\xi\,d\zeta.$$

This gives the first set of inequalities. Substituting into the definition of $\overline{E}_n[f](\sigma)$ then gives the second set.

(b) As in Proposition 6.3.1(b), note that $\underline{I}[f](\sigma)$ is effectively the sum over $i$ and $\ell$ of $\|\partial_i^\ell f\|^2_{H^1(A_\sigma\cap\Sigma_0)}$. By Proposition 4.4.2 and part (a), then, we have

$$\begin{aligned}\left\|\partial_i^\ell\overline{\gamma}\right\|_{L^\infty(A_\sigma)}&\le C_0\left[\|\partial_i^\ell\overline{\gamma}\|_{H^{n-2}_\circ(A_\sigma)}+\|\partial_i^\ell\overline{\gamma}\|_{H^1(A_\sigma\cap\Sigma_0)}\right]\\ &\le C_0\left[\frac{11}{10}\overline{E}_n[\overline{\gamma}](\sigma)^{1/2}+\underline{I}[\overline{\gamma}](\sigma)^{1/2}\right]\le 4C_0\nu,\end{aligned}$$

as claimed. QED.

We wish to express Lemmata 4.4.1 – 4.4.3 in our current setting. Now technically $A_\sigma$ is a subset of $\mathbf{R}^3$; to avoid messy but inconsequential notational issues, we shall also use $A_\sigma$ to denote the projection of this set onto the plane $\tau=0$, and when convenient consider functions on $A_\sigma$ as functions on this projection. (In particular, this is the sense in which statement (i) in the first lemma below is to be understood.) Then for $\sigma\in[0,\varsigma]$ we may apply the lemmata with $A_\sigma$ in place of $\Omega_L$; in this case, we have also (since in this context $y=\zeta$)

$$\sum_{\ell=0}^m\|\partial_y^\ell f|_{\tau=\sigma}\|_{H^{m-\ell}(\underline{\partial}\Omega_L)}\le\sqrt{m}\underline{I}^1_m[f](\sigma)^{1/2},$$

so absorbing the factor of $\sqrt{m}$ into the various constants, we have the following.

6.3.1. LEMMA. Let $\sigma \in [0,\varsigma]$, let $m \geq 0$, and let $\phi \in C^\infty(\mathbf{R}^1)$ have support contained in $[-1,3]$ and satisfy $\phi|_{[-1/2,1/2]} = 1$. Then there is an extension map $e : C_c^\infty(A_\sigma) \to H_0^m(\mathbf{R}^1 \times [-1,3])$ such that

(i) $e(f)|_{A_\sigma} = f$,

(ii) $\|e(f)\|_{H^m(\mathbf{R}^1\times[-1,3])} \leq C^e \left[\|f\|_{H^m(A_\sigma)} + \underline{I}^1_m[f](\sigma)^{1/2}\right]$,

where $C^e$ is a constant depending only on $m$ and $\phi$ (in particular, $C^e$ is independent of $\sigma$ and the size of the support of $f$).

6.3.2. LEMMA. Let $m \geq 2$, $\sigma \in [0,\varsigma]$. Let $f_1, \cdots, f_p \in H^m(A_\sigma)$ satisfy $\partial_\zeta^\ell f_i \in H^{m-\ell}(\underline{\partial} A_\sigma)$, $\ell = 0, \cdots, m$, $i = 1, \cdots, p$, and let $I_1, \cdots, I_p$ be multiindices with $|I_1 + \cdots + I_p| \leq m$. Then there is a constant $C$ such that

$$\left\|\partial^{I_1} f_1 \cdots \partial^{I_p} f_p\right\|_{L^2(A_\sigma)} \leq C\,(C^e)^p \prod_{i=1}^p \left(\underline{I}^1[f_i]^{1/2} + \|f_i\|_{H^m(A_\sigma)}\right).$$

6.3.3. LEMMA. Let $m \geq 2$, $\sigma \in [0,\varsigma]$. There is a constant $C \geq 1$ such that if $f_1, f_2, \cdots, f_p \in H^m(A_\sigma)$ satisfy $\partial_y^\ell f_i \in H^{m-\ell}(\underline{\partial}\Omega_L)$, $\ell = 0, \cdots, m$, $i = 1, \cdots, p$,

$$\|f_1 \cdots f_p\|_{H^m(A_\sigma)} \leq C^p \prod_{i=1}^p \left(\underline{I}^1[f_i]^{1/2} + \|f_i\|_{H^m(A_\sigma)}\right).$$

Lemma 6.3.1 gives the following corollary:

6.3.3. COROLLARY. If $f \in \overline{\Omega}$ (see (6.2.20)) or $f \in \{\partial_{\overline{s}}^{\ell_1}\partial_i^{\ell_2}\overline{\gamma}\,|\,\ell_1,\ell_2 \in \{0,1\}\}$, then for $m \leq n-1$

$$\|e(f)\|_{H^m(\{\sigma\}\times\mathbf{R}^2)} \leq C^e \left[\|f\|_{H^m(A_\sigma)} + \nu\right].$$

Proof. This follows from the Lemma 6.3.1 and the bounds (6.2.30). QED.

Since we can bound $\|f\|_{H^{n-1}(A_\sigma)}$ for $f$ as in the corollary by something like $\nu$, this shows that for us the additive term does not fundamentally change the size of the norm.

We have also (see Lemma 4.2.4):

6.3.4. COROLLARY. Let $m \geq 0$. There is a constant $C_H$, depending on $m$, such that the following holds. Let $O \subset \mathbf{R}^p$ be open, let $F : O \to \mathbf{R}^q$ be $C^\infty$, and suppose that $f_1, \cdots, f_p : A_\sigma \to \mathbf{R}^1$ satisfy the conditions in Lemma 6.3.2. Suppose that $D = \sup_{\substack{|J|\leq m\\ \mathbf{x}\in O}} |\partial^J F(\mathbf{x})| < \infty$. Then

$$\|F(f_1(\mathbf{x}), \cdots, f_p(\mathbf{x}))\|_{H^m(A_\sigma)} \leq CD \prod_{i=1}^p \left(\underline{I}^1[f_i]^{1/2} + \|f_i\|_{H^m(A_\sigma)}\right)^m.$$

Proof. This follows from Lemma 4.2.4 and Lemma 6.3.2. QED.

Obviously this bound is far from optimal, but it will be sufficient for our purposes.

We also have the following corollary which will be needed when treating the wave equation (6.2.11).

6.3.5. COROLLARY. If $\overline{\Omega}$ is such that (6.2.28) and (6.2.29) are satisfied, then for $m \leq n-1$ there is a combinatorial constant $C^\delta_m$ depending only on $m$ such that ($\ell \in \{0,1\}$)

$$\|\partial^\ell_{\overline{s}}\overline{\delta^{-1}a}\|_{H^m(A_\sigma)} \leq C^\delta_m C_M (C^e)^m \left(\frac{4}{3}\right)^{\frac{1}{2}m+2} \left(\underline{I}^1_m[\overline{\delta\ell}](\sigma)^{1/2} + \|\overline{\delta\ell}\|_{H^m(A_\sigma)}\right)^m,$$
$$\underline{I}^1_m[\partial^\ell_{\overline{s}}\overline{\delta^{-1}a}](\sigma) \leq C^\delta_m \left(\frac{4}{3}\right)^m \left(\underline{I}^1_m[\overline{a}](\sigma)\right)^m.$$

In particular, there is a constant $C_{\overline{\delta^{-1}a}} \geq 1$ such that

$$\|\partial^\ell_{\overline{s}}\overline{\delta^{-1}a}\|_{H^{n-1}(A_\sigma)} \leq C_{\overline{\delta^{-1}a}}\nu, \qquad \underline{I}^1_m[\partial^\ell_{\overline{s}}\overline{\delta^{-1}a}](\sigma) \leq C_{\overline{\delta^{-1}a}}\nu^2.$$

Proof. We have, since $\overline{a} = (1 + k^{-1}\overline{\delta\ell})^2$,

$$\overline{\delta^{-1}a} = k\left(\frac{1}{\overline{a}} - 1\right) = \frac{2\overline{\delta\ell} + k^{-1}\overline{\delta\ell}^2}{(1+k^{-1}\overline{\delta\ell})^2} = \frac{2\overline{\delta\ell} + k^{-1}\overline{\delta\ell}^2}{\overline{\ell}^2} = \frac{\overline{\delta\ell}}{\overline{\ell}} + \frac{\overline{\delta\ell}}{\overline{\ell}^2},$$
$$\partial_{\overline{s}}\overline{\delta^{-1}a} = -k\frac{\partial_{\overline{s}}\overline{a}}{\overline{a}^2} = -2\frac{\partial_{\overline{s}}\overline{\delta\ell}}{(1+k^{-1}\overline{\delta\ell})^3} = -2\frac{\partial_{\overline{s}}\overline{\delta\ell}}{\overline{\ell}^3}.$$

Now by the product rule for differentiation, if $I$ is any multiindex, then

$$\partial^I(fh) = \sum_{J\leq I}\binom{I}{J}\partial^{I-J}f\partial^J h,$$

where $(j_1,j_2) \leq (i_1,i_2)$ means $j_1 \leq i_1$, $j_2 \leq i_2$, and

$$\binom{(i_1,i_2)}{(j_1,j_2)} = \frac{i_1!i_2!}{j_1!(i_1-j_1)!j_2!(i_2-j_2)!}.$$

Further, if $g \neq 0$, $J$ is a multiindex and $p \geq 1$, then, by Lemma 4.2.1, letting $\mathcal{K}$ denote the set of all collections of multiindices $\{K_k\}$ whose sum equals $J$, there is a collection of combinatorial constants $\{C^p_{\{K_k\}} \mid \{K_k\} \in \mathcal{K}\}$ such that

$$\partial^J\frac{1}{g^p} = \sum_{\{K_k\}\in\mathcal{K}} C^p_{\{K_k\}}\frac{\prod_{K\in\{K_k\}}\partial^K g}{g^{|\{K_k\}|+p}},$$

where $|\{K_k\}|$ denotes the cardinality of $\{K_k\}$.

If we now take $f = \partial^\ell_{\overline{s}}\overline{\delta\ell}$ and $h = \overline{\ell}^{-p}$, for $p = 1,2,3$, then since $\partial_i\overline{\ell} = k^{-1}\partial_i\overline{\delta\ell}$, we see that $\partial^I(\partial^\ell_{\overline{s}}\overline{\delta\ell}/\overline{\ell}^p)$ is a sum of terms of the form

$$k^{-e}C\overline{\ell}^{-(P+p)}\prod\partial^{I_i}\partial^\ell_{\overline{s}}\overline{\delta\ell}, \tag{6.3.3}$$

where $e \geq 0$, $C$ is combinatorial, $P \leq |I|$, and $\sum I_i = I$. Since the number of such terms is also combinatorial in nature, $k \geq 1$, and the number of factors in each term is no greater than $|I|$, we may apply Lemma 6.3.2to find

$$\left\|\partial^I\frac{\partial^\ell_{\overline{s}}\overline{\delta\ell}}{\overline{\ell}^p}\right\|_{L^2(A_\sigma)} \leq CC_M\left(C^e\right)^{|}I|\left(\frac{4}{3}\right)^{(|I|+p)/2}\prod\left(\underline{I}^1_{|I|}[\partial^\ell_{\overline{s}}\overline{\delta\ell}](\sigma)^{1/2} + \left\|\partial^\ell_{\overline{s}}\overline{\delta\ell}\right\|_{H^{|I|}(A_\sigma)}\right)^{|I|}.$$

Since the number of multiindices with order less than or equal to $m$ is combinatorial, the first inequality follows by summing over all such multiindices $I$. The second inequality follows from (6.3.3) and the fact that for any $f$, $g$,

$$\underline{I}^1_m[fg](\sigma) \le \underline{I}^1_m[f](\sigma)\underline{I}^1_m[g](\sigma). \tag{6.3.4}$$

The final set of inequalities now follows from the bootstrap assumptions (6.2.28) and (6.2.29), since $\nu < 1$. QED.

We now define a class of nonlinearities sufficiently broad to encompass everything we will have to deal with. (It would be quite sufficient to consider only functions polynomial in the variables; but the treatment does not seem to be particularly more complicated for the general case. There is evidently some connection between what we do here and jet bundles, but we do not need that machinery here.) Our terminology is borrowed from Ringström [12]. In this definition, $\nu$ is the quantity appearing in the bootstrap (6.2.28) and the condition (6.2.29); in other words, it is a proxy for certain norms of the quantities involved. The spaces are defined in terms of pairs of quadruples $(\overline{a}, \overline{b}, \overline{c}, \overline{\gamma})$ to facilitate the existence proof in Section 6.6, which is by iteration.*

6.3.1. Definition. Fix some particular choice of initial data satisfying (6.2.27) and (6.2.29). Define sets $\mathbf{X}$ and $X$ as follows:

$$\begin{aligned}\mathbf{X} = \{(\overline{a}_1, \overline{b}_1, \overline{c}_1, \overline{\gamma}_1, \overline{a}_2, \overline{b}_2, \overline{c}_2, \overline{\gamma}_2) \in [C^\infty(\Gamma)]^8 \,|\, & \overline{\gamma}_i \text{ solves (6.2.11) with } \overline{a}, \overline{b}, \overline{c} \text{ replaced with } \overline{a}_i, \overline{b}_i, \overline{c}_i,\ i = 1, 2,\\ & \overline{a}_2,\ \overline{b}_2,\ \overline{c}_2 \text{ solve (6.2.12 – 6.2.14) with } \overline{\gamma} \text{ replaced by } \overline{\gamma}_1,\\ & (\overline{a}_i, \overline{b}_i, \overline{c}_i, \overline{\gamma}_i) \text{ satisfy (6.2.28) and (6.2.29)},\ i = 1, 2,\\ & \overline{a}_i, \overline{b}_i, \overline{c}_i, \overline{\gamma}_i \text{ agree on } \overline{s} = 0 \text{ and } \overline{v} = 0\\ & \qquad \text{with the chosen initial data}\},\end{aligned}$$

$$X = \{(\overline{a}, \overline{b}, \overline{c}, \overline{\gamma}) \in [C^\infty(\Gamma)]^4 \,|\, (\overline{a}', \overline{b}', \overline{c}', \overline{\gamma}', \overline{a}, \overline{b}, \overline{c}, \overline{\gamma}) \in \mathbf{X} \text{ for some } (\overline{a}', \overline{b}', \overline{c}', \overline{\gamma}') \in [C^\infty(\Gamma)]^4\}.$$

Note that the requirement $(\overline{a}', \overline{b}', \overline{c}', \overline{\gamma}', \overline{a}, \overline{b}, \overline{c}, \overline{\gamma}) \in \mathbf{X}$ in the definition of $X$ imposes restrictions on the quadruple $(\overline{a}', \overline{b}', \overline{c}', \overline{\gamma}')$. An *admissible nonlinearity of degree $m$ and exponent $p$*, where $2 \le m \le n-1$ and $p > 0$, is a function $\Phi : X \to C^\infty(\Gamma)$ which has the following property: there exist constants $C^m_1(\Phi)$ and $C^m_2(\Phi)$ such that for all $(\overline{a}, \overline{b}, \overline{c}, \overline{\gamma}) \in X$ we have

$$\|\Phi(\overline{a}, \overline{b}, \overline{c}, \overline{\gamma})\|_{H^m(A_\sigma)} \le C^m_1(\Phi)\nu^p, \qquad \underline{I}^1_m[\Phi(\overline{a}, \overline{b}, \overline{c}, \overline{\gamma})](\sigma) \le [C^m_2(\Phi)]^2\nu^{2p}.$$

We denote the set of admissible nonlinearities of degree $m$ by $\widehat{X}^{m,p}$, or simply $\widehat{X}$ if the degree and exponent are clear from the context. It is a normed vector space under the following norm:

$$\|\Phi\|_{\widehat{X}^{m,p}} = \inf\{C_1 + C_2 \,|\, \|\Phi(\overline{a}, \overline{b}, \overline{c}, \overline{\gamma})\|_{H^m(A_\sigma)} \le C_1\nu^p,\ \underline{I}^1_m[\Phi](\sigma) \le C_2^2\nu^{2p} \text{ for all } (\overline{a}, \overline{b}, \overline{c}, \overline{\gamma}) \in X\}.$$

* One could of course also try to prove existence using a fixed-point theorem. We considered that approach but abandoned it as at least as complicated in the current setting since proving the necessary continuity results involved estimates very similar to those used in Theorem 6.6.1 to show convergence of the sequence obtained by iteration.

We further let

$$X_0 = \{(\overline{a},\overline{b},\overline{c},\overline{\gamma}) \in [C^\infty(\Gamma)]^4 \mid (\overline{a},\overline{b},\overline{c},\overline{\gamma}) \text{ satisfy (6.2.28) and (6.2.29),}$$
$$\overline{a},\overline{b},\overline{c},\overline{\gamma} \text{ agree on } \overline{s}=0 \text{ and } \overline{v}=0 \text{ with the initial data}$$
$$\text{constructed in Chapter 5}\},$$

and define a *restricted admissible nonlinearity of degree m and exponent p* to be a $\Phi \in \widehat{X}^{m,p}$ which extends to $X_0$ and satisfies

$$\|\Phi(\overline{a},\overline{b},\overline{c},\overline{\gamma})\|_{H^m(A_\sigma)} \le C_1^m(\Phi)\nu^p, \qquad \underline{I}_m^1[\Phi(\overline{a},\overline{b},\overline{c},\overline{\gamma})](\sigma) \le [C_2^m(\Phi)]^2\nu^{2p}$$

for all $(\overline{a},\overline{b},\overline{c},\overline{\gamma}) \in X_0$. We denote the set of all restricted admissible nonlinearities by $\widehat{X}_0^{m,p}$ (or, as with $\widehat{X}^{m,p}$, simply $\widehat{X}_0$ if the degree and exponent are clear from the context) and define the norm

$$\|\Phi\|_{\widehat{X}_0^{m,p}} = \inf\{C_1 + C_2 \mid \|\Phi(\overline{a},\overline{b},\overline{c},\overline{\gamma})\|_{H^m(A_\sigma)} \le C_1\nu^p,\ \underline{I}_m^1[\Phi](\sigma) \le C_2^2\nu^{2p} \text{ for all } (\overline{a},\overline{b},\overline{c},\overline{\gamma}) \in X_0\}.$$

Note that, by equation (6.2.23), we have also

$$\underline{I}_m^\circ[\Phi(\overline{a},\overline{b},\overline{c},\overline{\gamma})](\sigma) \le [\|\Phi\|_{\widehat{X}^{m,p}}]^2\nu^{2p}.$$

Note also that, if $(\overline{a}',\overline{b}',\overline{c}',\overline{\gamma}',\overline{a},\overline{b},\overline{c},\overline{\gamma}) \in \mathbf{X}$, then $(\overline{a}',\overline{b}',\overline{c}',\overline{\gamma}'), (\overline{a},\overline{b},\overline{c},\overline{\gamma}) \in X_0$.

Since we assume $\nu < 1$, we have $\nu^p < \nu^q$ when $q < p$, so $\widehat{X}^{m,p} \subset \widehat{X}^{m,q}$ and $\widehat{X}_0^{m,p} \subset \widehat{X}_0^{m,q}$ if $q < p$.

We shall principally work with the case $m = n-1$, but we shall have occasion to use $m = n-2$ as well; note that since $n \ge 4$ we have $n-1, n-2 \ge 2$. (The space $\widehat{X}^{m,p}$ could clearly be defined also for $m < 2$, but since some of the results below do not hold for $m < 2$ and we shall never need to consider $\widehat{X}^{m,p}$ for $m < 2$ we exclude that case altogether.) We shall also usually have $p \ge 1$. It is worth noting that $\Phi(\overline{a},\overline{b},\overline{c},\overline{\gamma})$ denotes the image of the solution $(\overline{a},\overline{b},\overline{c},\overline{\gamma})$ under the map $\Phi$, *not* any kind of functional composition. In particular, derivative maps (e.g., $\Phi(\overline{a},\overline{b},\overline{c},\overline{\gamma}) = \partial_{\overline{s}}^\ell\overline{\gamma}$) will be admissible nonlinearities when we can prove the appropriate bounds.

We have the following proposition.

6.3.3. PROPOSITION. (i) Under pointwise multiplication of functions, $\widehat{X}^{m,p} \times \widehat{X}^{m,q} \to \widehat{X}^{m,p+q}$, and if $\Phi \in \widehat{X}^{m,p}$, $\Psi \in \widehat{X}^{m,q}$, then

$$\|\Phi\Psi\|_{\widehat{X}^{m,p+q}} \le C\|\Phi\|_{\widehat{X}^{m,p}}\|\Psi\|_{\widehat{X}^{m,q}}$$

for some constant $C$. Further, the map $(i \in \{1,2\})$

$$\partial_i : \Phi \mapsto \big((\overline{a},\overline{b},\overline{c},\overline{\gamma}) \mapsto \partial_i\left[\Phi(\overline{a},\overline{b},\overline{c},\overline{\gamma})\right]\big)$$

maps $\widehat{X}^{m,p}$ to $\widehat{X}^{m-1,p}$ (when $m \ge 3$).

(ii) Let $F : \mathbf{R}^{17} \to \mathbf{R}^1$ be $C^\infty$, independent of $k$, and satisfy $F(0) = 0$; then the map*

$$(\overline{a}, \overline{b}, \overline{c}, \overline{\gamma}) \mapsto F(\overline{\delta^{-1}a}, \overline{\Omega}, \partial_{\overline{s}}\overline{\Omega}_0, \partial_{\overline{v}}\overline{b}, \partial_{\overline{x}}\overline{c}, \partial_{\overline{v}}\overline{c}, \overline{\gamma}, \partial_\tau\overline{\gamma}, \partial_\xi\overline{\gamma}, \partial_\zeta\overline{\gamma}),$$

which we denote by $F^\circ$, is in $\widehat{X}^{m,1}$. If $Q = [-16C_0\nu, 16C_0\nu] \times [-4C_0\nu, 4C_0\nu]^{16}$ and $C_F = \sup\{\partial^J F(\mathbf{x}) \,|\, \mathbf{x} \in Q, |J| \leq m\}$, then there is a combinatorial constant $C$ depending only on $m$ such that

$$\|F^\circ\|_{\widehat{X}^{m,1}} \leq C_F C_0 C (C_{\overline{\delta^{-1}a}} C^M)^m.$$

Proof. (i) Let $\Phi$, $\Psi \in \widehat{X}$, let $(\overline{a}, \overline{b}, \overline{c}, \overline{\gamma}) \in X$, and let $f = \Phi(\overline{a}, \overline{b}, \overline{c}, \overline{\gamma})$, $g = \Psi(\overline{a}, \overline{b}, \overline{c}, \overline{\gamma})$; then by Lemma 6.3.3 we have

$$\begin{aligned}\|\Phi(\overline{a}, \overline{b}, \overline{c}, \overline{\gamma})\Psi(\overline{a}, \overline{b}, \overline{c}, \overline{\gamma})\|_{H^m(A_\sigma)} &\leq C' \left[\underline{I}^1_m[f](\sigma)^{1/2} + \|f\|_{H^m(A_\sigma)}\right] \left[\underline{I}^1_m[g](\sigma)^{1/2} + \|g\|_{H^m(A_\sigma)}\right] \\ &\leq C' \|\Phi\|_{\widehat{X}^{m,p}} \|\Psi\|_{\widehat{X}^{m,q}} \nu^{p+q},\end{aligned}$$

while by the product rule there is clearly some combinatorial constant $C''$ such that

$$\underline{I}^1_m[fg] \leq C''^2 \underline{I}^1_m[f] \underline{I}^1_m[g] \leq C''^2 [C^m_2(\Phi) C^m_2(\Psi)]^2 \nu^{2(p+q)} < C''^2 \|\Phi\|^2_{\widehat{X}^{m,p}} \|\Psi\|^2_{\widehat{X}^{m,p}} \nu^{2(p+q)};$$

from these two inequalities the stated result follows:

$$\begin{aligned}\|\Phi\Psi\|_{\widehat{X}^{m,p}} &\leq \inf\{C_1 \,|\, \|\Phi(\overline{a}, \overline{b}, \overline{c}, \overline{\gamma})\Psi(\overline{a}, \overline{b}, \overline{c}, \overline{\gamma})\|_{H^m(A_\sigma)} \leq C_1 \nu^{p+q}\} \\ &\qquad + \inf\{C_2 \,|\, \underline{I}^1_m[\Phi(\overline{a}, \overline{b}, \overline{c}, \overline{\gamma})\Psi(\overline{a}, \overline{b}, \overline{c}, \overline{\gamma})] \leq C_2^2 \nu^{2(p+q)}\} \\ &\leq C' \|\Phi\|_{\widehat{X}^{m,p}} \|\Psi\|_{\widehat{X}^{m,q}} + C'' \|\Phi\|_{\widehat{X}^{m,p}} \|\Psi\|_{\widehat{X}^{m,q}} = (C' + C'') \|\Phi\|_{\widehat{X}^{m,p}} \|\Psi\|_{\widehat{X}^{m,q}}.\end{aligned}$$

The second part follows from the observation that

$$\|\partial_i f\|_{H^{m-1}(A_\sigma)} \leq \|f\|_{H^m(A_\sigma)}, \qquad \underline{I}^1_{m-1}[\partial_i f](\sigma) \leq \underline{I}^1_m[f](\sigma).$$

(ii) By the chain rule and product rule, if $I$ is a multiindex in $\xi$ and $\zeta$ with $|I| \leq m$, then the most stupendous quantity

$$\partial^I \left[F(\overline{\delta^{-1}a}, \Omega, \partial_{\overline{s}}\Omega_0, \partial_{\overline{v}}\overline{b}, \partial_{\overline{x}}\overline{c}, \partial_{\overline{v}}\overline{c}, \overline{\gamma}, \partial_\tau\overline{\gamma}, \partial_\xi\overline{\gamma}, \partial_\zeta\overline{\gamma})\right]$$

can be written as a sum of terms of the form (denoting for convenience the argument of $F$ by $\check{X} = (\overline{\delta^{-1}a}, \Omega, \partial_{\overline{s}}\Omega_0, \partial_{\overline{v}}\overline{b}, \partial_{\overline{x}}\overline{c}, \partial_{\overline{v}}\overline{c}, \overline{\gamma}, \partial_\tau\overline{\gamma}, \partial_\xi\overline{\gamma}, \partial_\zeta\overline{\gamma})$)

$$(\partial^J F)(\check{X}) \prod \partial^{I'} \check{\chi}, \tag{6.3.5}$$

where the sum of all multiindices $I'$ in each product is exactly equal to $I$, and $\check{\chi}$ denotes some element of $\check{X}$. (Here a particular element of $\check{X}$ may appear multiple times.) Now by Proposition 6.3.1(a) and 6.3.2(a), if $\check{\chi}$ is any component of $\check{X}$ except $\overline{\delta^{-1}a}$, then

$$\|\check{\chi}\|_{H^m(A_\sigma)} \leq 2\nu, \qquad \underline{I}^1_m[\check{\chi}](\sigma)^{1/2} \leq \nu,$$

* Recall – see (6.2.20) – that $|\overline{\Omega}_0| = 3$ and $|\overline{\Omega}| = 9$, so that $F$ does indeed have 17 arguments. The specific number is of course not important.

while by Corollary 6.3.5

$$\|\partial_{\overline{s}}^{\ell}\overline{\delta^{-1}a}\|_{H^m(A_\sigma)} \leq C_{\overline{\delta^{-1}a}}\nu, \qquad \underline{I}^1_m[\partial_{\overline{s}}^{\ell}\overline{\delta^{-1}a}](\sigma) \leq C_{\overline{\delta^{-1}a}}\nu^2,$$

so that for all $\check{\chi}$ in $\check{X}$ we have

$$\underline{I}^1_m[\check{\chi}](\sigma)^{1/2} + \|\check{\chi}\|_{H^m(A_\sigma)} \leq 3C_{\overline{\delta^{-1}a}}\nu.$$

By Lemma 6.3.2, then, since there are at most $|I| \leq m$ terms in the product,

$$\Big\|\prod \partial^{I'}\check{\chi}\Big\|_{L^2(A_\sigma)} \leq (C^M)^m(3C_{\overline{\delta^{-1}a}})^m\nu^m \leq (3C_{\overline{\delta^{-1}a}}C^M)^m\nu,$$

so that we have finally that

$$\big\|\partial^I F(\check{X})\big\|_{L^2(A_\sigma)} \leq C_F C'(3C_{\overline{\delta^{-1}a}}C^M)^m\nu,$$

where $C'$ is a combinatorial constant. Similarly, since $F(0) = 0$, we may write, doing a Lipschitz estimate,

$$F(\check{X}) \leq \sum_{\check{\chi}\in\check{X}} C_F \cdot |\check{\chi}|, \tag{6.3.6}$$

so by Proposition 6.3.1(a), Proposition 6.3.2(a), Proposition 4.4.1, and the bootstrap conditions (6.2.30), we have for some numerical constant $C''$

$$\|F(\check{X})\|_{L^2(A_\sigma)} \leq 17C_F C''\nu.$$

Thus, letting $C = \max\{17C'', 3^m C'\}$, we see that

$$\|F(\check{X})\|_{H^m(A_\sigma)} \leq C_F C_0 C(C_{\overline{\delta^{-1}a}}C^M)^m\nu.$$

The bound on $\underline{I}^1_m[F(\check{X})]$ follows from (6.3.5) and (6.3.6) by using (6.2.29) and multiplicativity of $\underline{I}^1$ (see (6.3.4)). QED.

We note that our proof of the multiplicative property in (i) did not require $\Phi$ and $\Psi$ to be evaluated at the same points in $\widehat{X}$, though as we shall only have limited use for this result we shall not take the time to formalise it. We note further that we made no use of the equations satisfied by $(\overline{a}, \overline{b}, \overline{c}, \overline{\gamma})$, only that the bounds (6.2.28 – 6.2.30) were satisfied. Thus we have the following corollary:

6.3.6. Corollary. Proposition 6.3.3 holds for restricted admissible nonlinearities if $X$ and $\widehat{X}$ are everywhere replaced by $X_0$ and $\widehat{X}_0$.

Recall the second part of (6.2.26):

$$\begin{aligned} E_{n,\ell}[h](\sigma) &= \sum_{|I|\leq n-1}\sum_{\overline{\omega}\in\overline{\Omega}} \big|\partial^I\partial_{\overline{s}}^{\ell}\overline{\omega}\big|^2_{L^2(A_\sigma)}, \\ E_n[h](\sigma) &= \sum_{\ell=0}^{1} E_{n,\ell}[h](\sigma). \end{aligned} \tag{6.3.7}$$

As noted after (6.2.26) above, this notation needs clarification when we are dealing with multiple metrics simultaneously, as we do when working with admissible nonlinearities. We make the following convention. Elements of $X$ are always denoted by quadruples consisting of the kernel letters $\overline{a}$, $\overline{b}$, $\overline{c}$, and $\overline{\gamma}$, modified by primes, subscripts, etc., as needed. We shall denote the corresponding $2+1$ metric by performing on the kernel letter $h$ the same modifications; thus $h_1$ is the metric corresponding to $\overline{a}_1$, $\overline{b}_1$, $\overline{c}_1$, and so on. The kernel letter $\overline{\delta\ell}$, modified also in the same way, shall refer to the quantity $k(\overline{a}-1)^{1/2}$ with $\overline{a}$ replaced by the appropriately modified symbol. Then $E_n[h](\sigma)$, with $h$ appropriately modified, will refer to the quantity in (6.3.7) with $\overline{\Omega}$ formed from the appropriately modified $\overline{\delta\ell}$, $\overline{b}$, and $\overline{c}$. This set will be denoted by the kernel letter $\overline{\Omega}$ modified in the same way. For example, if we have $(\overline{a}_1, \overline{b}_1, \overline{c}_1, \overline{\gamma}_1) \in X$, then we would define

$$\overline{\Omega}_1 = \{\overline{\delta\ell}_1, \overline{b}_1, \overline{c}_1, \partial_{\overline{x}}\overline{\delta\ell}_1, \partial_{\overline{v}}\overline{\delta\ell}_1, \partial_{\overline{x}}\overline{b}_1\}$$

$$E_{n,\ell}[h](\sigma) = \sum_{|I|\le n-1} \sum_{\overline{\omega}\in\overline{\Omega}_1} \left|\partial^I \partial_{\overline{s}}^{\ell} \overline{\omega}\right|^2_{L^2(A_\sigma)}$$

$$E_n[h](\sigma) = \sum_{\ell=0}^{1} E_{n,\ell}[h](\sigma).$$

We now produce a litany of admissible nonlinearities* which will be used in the next section to derive energy inequalities. For simplicity, we denote maps such as $(\overline{a}, \overline{b}, \overline{c}, \overline{\gamma}) \mapsto \partial_{\overline{s}}^2 \overline{a}$ by simply $\partial_{\overline{s}}^2 \overline{a}$ (exactly as, for example, one denotes the function $x \mapsto x^2$ on $\mathbf{R}^1$ by $x^2$).

6.3.4. Lemma. The following are admissible nonlinearities of degree $n-1$ and exponent 1:

$$\overline{\delta\ell}, \quad \overline{\delta^{-1}a}, \quad \overline{b}, \quad \overline{c}, \quad \partial_{\overline{s}}\overline{\delta\ell}, \quad \partial_{\overline{s}}\overline{b}, \quad \partial_{\overline{s}}\overline{c}, \quad \partial_{\overline{x}}\overline{\delta\ell}, \quad \partial_{\overline{v}}\overline{\delta\ell}, \quad \partial_{\overline{x}}\overline{b};$$

the following are admissible nonlinearities of degree $n-1$ and exponent 2:

$$\partial_{\overline{s}}^2\overline{\delta\ell}, \quad \partial_{\overline{s}}^2\overline{b}, \quad \partial_{\overline{s}}^2\overline{c}, \quad \partial_{\overline{s}}^2\partial_{\overline{x}}\overline{\delta\ell}, \quad \partial_{\overline{s}}^2\partial_{\overline{v}}\overline{\delta\ell}, \quad \partial_{\overline{s}}^2\partial_{\overline{x}}\overline{b};$$

all tilde-barred correspondents of the above are admissible nonlinearities of the same degree and exponent, respectively;

the following are admissible nonlinearities of degree $n-1$ and exponent 1:

$$\overline{\gamma}, \quad \partial_\tau\overline{\gamma}, \quad \partial_\xi\overline{\gamma}, \quad \partial_\zeta\overline{\gamma}, \quad \partial_{\overline{s}}\overline{\gamma}, \quad \partial_{\overline{s}}\partial_\tau\overline{\gamma}, \quad \partial_{\overline{s}}\partial_\xi\overline{\gamma}, \quad \partial_{\overline{s}}\partial_\zeta\overline{\gamma}, \quad \partial_\xi^2\overline{\gamma};$$

and finally

$$\partial_{\overline{s}}^3\overline{\gamma}$$

* We advise the reader not to be put off by the fact that all of these quantities appear to be rather *linear.* The space $X$, inasmuch as it involves solutions to nonlinear equations, is itself not linear, so it does not really make sense to speak of a map from $X$ being linear. More fundamentally, though, the quantities in the lemma below are really the building blocks from which the elements of $\widehat{X}^{m,p}$ we shall have the most use for later are built by multiplication.

is an admissible nonlinearity of degree $n-2$ and exponent 1.

Proof. We note first of all that all of these quantities do indeed give maps $X \to C^\infty(\Gamma)$, and thus the only question is whether the relevant bounds hold. Next, multiplying by $\chi(\overline{s}/T')$ will change the norms we use by at most $\underline{T}'^{-N}$ for some positive integer $N$, and such a factor will not impact the result.

That $\overline{\delta^{-1}a}$ is an admissible nonlinearity follows from Corollary 6.3.5. All other quantities in the first two lines except $\overline{\delta^{-1}a}$ are trivially admissible nonlinearities by definition of $E_n[h](\sigma)$ and (6.2.28). Similarly, and for the same reason, the quantities $\overline{\gamma}$, $\partial_\tau\overline{\gamma}$, $\partial_\xi\overline{\gamma}$, and $\partial_\zeta\overline{\gamma}$ (and hence also $\partial_{\overline{s}}\overline{\gamma}$ and $\partial_{\overline{v}}\overline{\gamma}$), as well as their $\overline{s}$ derivatives, are admissible nonlinearities. In fact, all of these are actually restricted admissible nonlinearities.

To deal with the other terms, let $(\overline{a}, \overline{b}, \overline{c}, \overline{\gamma}) \in X$, and let $(\overline{a}', \overline{b}', \overline{c}', \overline{\gamma}')$ be such that $(\overline{a}', \overline{b}', \overline{c}', \overline{\gamma}', \overline{a}, \overline{b}, \overline{c}, \overline{\gamma}) \in$ **X**. Recall that this implies that $(\overline{a}', \overline{b}', \overline{c}', \overline{\gamma}') \in X_0$, and in particular the quantities $\overline{a}', \overline{b}', \overline{c}', \overline{\gamma}'$ must satisfy the bounds (6.2.28 – 6.2.30).

To deal with $\partial_{\overline{s}}^2\overline{\delta\ell}$, $\partial_{\overline{s}}^2\overline{b}$, and $\partial_{\overline{s}}^2\overline{c}$, we make use of the Riccati equations:

$$\partial_{\overline{s}}^2\overline{\delta\ell} = -2\overline{\ell}k^{1-2\iota}(\partial_{\overline{s}}\overline{\gamma}')^2, \qquad \partial_{\overline{s}}^2\overline{b} = \frac{1}{\overline{\ell}}k^{-1}(\partial_{\overline{s}}\overline{\delta\ell})(\partial_{\overline{s}}\overline{b}) - 4k^{1-2\iota}\partial_{\overline{s}}\overline{\gamma}'(\partial_{\overline{x}}\overline{\gamma}' + k^{-1}\overline{b}\partial_{\overline{s}}\overline{\gamma}'),$$
$$\partial_{\overline{s}}^2\overline{c} = k^{-1}\frac{(\partial_{\overline{s}}\overline{b})^2}{2\overline{a}} - 2k^{1-2\iota}\partial_{\overline{s}}\overline{\gamma}'\left(2\partial_{\overline{v}}\overline{\gamma}' + 2k^{-1}\frac{\overline{b}}{\overline{a}}\partial_{\overline{x}}\overline{\gamma}' + k^{-1}\left(\overline{c} + k^{-1}\frac{\overline{b}^2}{\overline{a}}\right)\partial_{\overline{s}}\overline{\gamma}'\right) - k^{1-2\iota}\frac{2}{\overline{a}}(\partial_{\overline{x}}\overline{\gamma}')^2.$$

Note that $1/\overline{a} = 1 + k^{-1}\overline{\delta^{-1}a}$. Now the function $\overline{\gamma}'$ need not be unique given $(\overline{a}, \overline{b}, \overline{c}, \overline{\gamma})$, but clearly the right-hand sides of each of the above equations must be. Now $(\overline{a}, \overline{b}, \overline{c}, \overline{\gamma}') \in X_0$, and since $k \geq 1$, every quantity on the right-hand side of these equations, considered as a function of $(\overline{a}, \overline{b}, \overline{c}, \overline{\gamma}')$, is in $\widehat{X}_0^{n-1,1}$. Since multiplication maps $\widehat{X}_0^{n-1,1} \times \widehat{X}_0^{n-1,1} \to \widehat{X}_0^{n-1,2}$, we see that the quantities $\partial_{\overline{s}}^2\overline{\delta\ell}$, $\partial_{\overline{s}}^2\overline{b}$ and $\partial_{\overline{s}}^2\overline{c}$, considered as functions of $(\overline{a}, \overline{b}, \overline{c}, \overline{\gamma}')$, are also in $\widehat{X}_0^{n-1,2}$; since these functions do not depend at all on $\overline{\gamma}'$, the quantities $\partial_{\overline{s}}^2\overline{\delta\ell}$, $\partial_{\overline{s}}^2\overline{b}$, and $\partial_{\overline{s}}^2\overline{c}$ must be in $\widehat{X}_0^{n-1,2}$ when considered as functions of $(\overline{a}, \overline{b}, \overline{c}, \overline{\gamma})$, and hence must be in $\widehat{X}^{n-1,2}$, as claimed.

Differentiating the equation for $\overline{\delta\ell}$, we have by the same logic (*not* by applying the second part of Proposition 6.3.3(i)!) that $\partial_{\overline{s}}^2\partial_{\overline{x}}\overline{\delta\ell}$ and $\partial_{\overline{s}}^2\partial_{\overline{v}}\overline{\delta\ell}$ are admissible nonlinearities of degree $n-1$ and exponent 2.

Differentiating the equation for $\overline{b}$ with respect to $\overline{x}$, we see that $\partial_{\overline{s}}^2\partial_{\overline{x}}\overline{b}$ will also be an admissible nonlinearity of degree $n-1$ and exponent 2 if $\partial_{\overline{x}}^2\overline{\gamma} = \partial_\xi^2\overline{\gamma} \in \widehat{X}^{n-1,1}$. To show this, consider the wave equation:

$$\begin{aligned}
0 &= \left(-2\partial_{\overline{s}}\partial_{\overline{v}}\overline{\gamma} + \partial_{\overline{x}}^2\overline{\gamma}\right) \\
&+ \frac{1}{k}\left(\widetilde{\overline{\delta^{-1}a}}\partial_{\overline{x}}^2\overline{\gamma} - \widetilde{\overline{c}}\partial_{\overline{s}}^2\overline{\gamma} + 2\frac{\widetilde{\overline{b}}}{\widetilde{\overline{a}}}\partial_{\overline{s}}\partial_{\overline{x}}\overline{\gamma} - \frac{1}{2}\left(2\partial_{\overline{s}}\widetilde{\overline{c}} - \frac{2}{\widetilde{\overline{a}}}\partial_{\overline{x}}\widetilde{\overline{b}} + \frac{\partial_{\overline{v}}\widetilde{\overline{\delta a}}}{\widetilde{\overline{a}}}\right)\partial_{\overline{s}}\overline{\gamma} + \frac{1}{\widetilde{\overline{a}}}\partial_{\overline{s}}\widetilde{\overline{b}}\partial_{\overline{x}}\overline{\gamma} - \frac{1}{2}\frac{\partial_{\overline{x}}\widetilde{\overline{\delta a}}}{\widetilde{\overline{a}}^2}\partial_{\overline{x}}\overline{\gamma} - \frac{\partial_{\overline{s}}\widetilde{\overline{\delta a}}}{\widetilde{\overline{a}}}\partial_{\overline{v}}\overline{\gamma}\right) \\
&\qquad + k^{-2}\left(\frac{\widetilde{\overline{b}}^2}{\widetilde{\overline{a}}}\partial_{\overline{s}}^2\overline{\gamma} - \frac{1}{2}\left(\widetilde{\overline{c}}\frac{\partial_{\overline{s}}\widetilde{\overline{\delta a}}}{\widetilde{\overline{a}}} - 4\frac{\widetilde{\overline{b}}\partial_{\overline{s}}\widetilde{\overline{b}}}{\widetilde{\overline{a}}} + \frac{\widetilde{\overline{b}}\partial_{\overline{x}}\widetilde{\overline{\delta a}}}{\widetilde{\overline{a}}^2}\right)\partial_{\overline{s}}\overline{\gamma} - \frac{1}{2}\frac{\widetilde{\overline{b}}}{\widetilde{\overline{a}}^2}\partial_{\overline{s}}\widetilde{\overline{\delta a}}\partial_{\overline{x}}\overline{\gamma}\right) - k^{-3}\left(\frac{1}{2}\frac{\widetilde{\overline{b}}^2}{\widetilde{\overline{a}}^2}\partial_{\overline{s}}\widetilde{\overline{\delta a}}\partial_{\overline{s}}\overline{\gamma}\right).
\end{aligned}$$

This can be solved for $\partial_{\bar{x}}^2\bar{\gamma}$ to obtain (recall that $1+k^{-1}\widetilde{\overline{\delta^{-1}a}} = \tilde{\bar{a}}^{-1}$)

$$\begin{aligned}\partial_{\bar{x}}^2\bar{\gamma} = \tilde{\bar{a}}\Bigg[ & 2\partial_{\bar{s}}\partial_{\bar{v}}\bar{\gamma} \\ & - \frac{1}{k}\left( -\tilde{\bar{c}}\partial_{\bar{s}}^2\bar{\gamma} + 2\frac{\tilde{\bar{b}}}{\tilde{\bar{a}}}\partial_{\bar{s}}\partial_{\bar{x}}\bar{\gamma} - \frac{1}{2}\left( 2\partial_{\bar{s}}\tilde{\bar{c}} - \frac{2}{\tilde{\bar{a}}}\partial_{\bar{x}}\tilde{\bar{b}} + \frac{\partial_{\bar{v}}\widetilde{\overline{\delta a}}}{\tilde{\bar{a}}} \right)\partial_{\bar{s}}\bar{\gamma} + \frac{1}{\tilde{\bar{a}}}\partial_{\bar{s}}\tilde{\bar{b}}\partial_{\bar{x}}\bar{\gamma} - \frac{1}{2}\frac{\partial_{\bar{x}}\widetilde{\overline{\delta a}}}{\tilde{\bar{a}}^2}\partial_{\bar{x}}\bar{\gamma} - \frac{\partial_{\bar{s}}\widetilde{\overline{\delta a}}}{\tilde{\bar{a}}}\partial_{\bar{v}}\bar{\gamma} \right) \\ & - k^{-2}\left( \frac{\tilde{\bar{b}}^2}{\tilde{\bar{a}}}\partial_{\bar{s}}^2\bar{\gamma} - \frac{1}{2}\left( \tilde{\bar{c}}\frac{\partial_{\bar{s}}\widetilde{\overline{\delta a}}}{\tilde{\bar{a}}} - 4\frac{\tilde{\bar{b}}\partial_{\bar{s}}\tilde{\bar{b}}}{\tilde{\bar{a}}} + \frac{\tilde{\bar{b}}\partial_{\bar{x}}\widetilde{\overline{\delta a}}}{\tilde{\bar{a}}^2} \right)\partial_{\bar{s}}\bar{\gamma} - \frac{1}{2}\frac{\tilde{\bar{b}}}{\tilde{\bar{a}}^2}\partial_{\bar{s}}\widetilde{\overline{\delta a}}\partial_{\bar{x}}\bar{\gamma} \right) + k^{-3}\left( \frac{1}{2}\frac{\tilde{\bar{b}}^2}{\tilde{\bar{a}}^2}\partial_{\bar{s}}\widetilde{\overline{\delta a}}\partial_{\bar{s}}\bar{\gamma} \right)\Bigg].\end{aligned}$$

Now since $\partial_{\bar{s}} = \frac{1}{\sqrt{2}}(\partial_\tau + \partial_\zeta)$, we see that $\partial_{\bar{s}}^2\bar{\gamma} = \frac{1}{\sqrt{2}}(\partial_{\bar{s}}\partial_\tau\bar{\gamma} + \partial_{\bar{s}}\partial_\zeta\bar{\gamma}) \in \widehat{X}^{n-1,1}$. By the above equation, then, $\partial_{\bar{x}}^2\bar{\gamma} \in \widehat{X}^{n-1,1}$ as well. (In fact, $\partial_{\bar{x}}^2\bar{\gamma}$ has exponent 2 because of the overall factor of $\tilde{\bar{a}}$.)

To deal with $\partial_{\bar{s}}^3\bar{\gamma}$, we write $\partial_{\bar{v}} = -\sqrt{2}\partial_\zeta + \partial_{\bar{s}}$ in the wave equation and gather all terms involving $\partial_{\bar{s}}^2\bar{\gamma}$ together to obtain

$$\begin{aligned}\left[ 2 + \frac{1}{k}\tilde{\bar{c}} - \frac{1}{k^2}\frac{\tilde{\bar{b}}^2}{\tilde{\bar{a}}} \right]\partial_{\bar{s}}^2\bar{\gamma} = {} & 2\sqrt{2}\partial_{\bar{s}}\partial_\zeta\bar{\gamma} + \partial_{\bar{x}}^2\bar{\gamma} \\ & + \frac{1}{k}\left( -\widetilde{\overline{\delta^{-1}a}}\partial_{\bar{x}}^2\bar{\gamma} + 2\frac{\tilde{\bar{b}}}{\tilde{\bar{a}}}\partial_{\bar{s}}\partial_{\bar{x}}\bar{\gamma} - \frac{1}{2}\left( 2\partial_{\bar{s}}\tilde{\bar{c}} - \frac{2}{\tilde{\bar{a}}}\partial_{\bar{x}}\tilde{\bar{b}} + \frac{\partial_{\bar{v}}\widetilde{\overline{\delta a}}}{\tilde{\bar{a}}} \right)\partial_{\bar{s}}\bar{\gamma} + \frac{1}{\tilde{\bar{a}}}\partial_{\bar{s}}\tilde{\bar{b}}\partial_{\bar{x}}\bar{\gamma} - \frac{1}{2}\frac{\partial_{\bar{x}}\widetilde{\overline{\delta a}}}{\tilde{\bar{a}}^2}\partial_{\bar{x}}\bar{\gamma} - \frac{\partial_{\bar{s}}\widetilde{\overline{\delta a}}}{\tilde{\bar{a}}}\partial_{\bar{v}}\bar{\gamma} \right) \\ & + k^{-2}\left( -\frac{1}{2}\left( \tilde{\bar{c}}\frac{\partial_{\bar{s}}\widetilde{\overline{\delta a}}}{\tilde{\bar{a}}} - 4\frac{\tilde{\bar{b}}\partial_{\bar{s}}\tilde{\bar{b}}}{\tilde{\bar{a}}} + \frac{\tilde{\bar{b}}\partial_{\bar{x}}\widetilde{\overline{\delta a}}}{\tilde{\bar{a}}^2} \right)\partial_{\bar{s}}\bar{\gamma} - \frac{1}{2}\frac{\tilde{\bar{b}}}{\tilde{\bar{a}}^2}\partial_{\bar{s}}\widetilde{\overline{\delta a}}\partial_{\bar{x}}\bar{\gamma} \right) + k^{-3}\left( \frac{1}{2}\frac{\tilde{\bar{b}}^2}{\tilde{\bar{a}}^2}\partial_{\bar{s}}\widetilde{\overline{\delta a}}\partial_{\bar{s}}\bar{\gamma} \right). \qquad (6.3.8)\end{aligned}$$

By Proposition 6.3.1 we have

$$2 + \frac{1}{k}\tilde{\bar{c}} - \frac{1}{k^2}\frac{\tilde{\bar{b}}^2}{\tilde{\bar{a}}} \geq 2 - \frac{1}{3} - \frac{4}{3}\left(\frac{1}{3}\right)^2 \geq \frac{3}{2} > 0,$$

so dividing through and using this bound we have that $\partial_{\bar{s}}^2\bar{\gamma}$ is also an admissible nonlinearity. Differentiating by $\zeta$ we see that $\partial_\zeta\partial_{\bar{s}}^2\bar{\gamma}$ is also an admissible nonlinearity of degree $n-2$ and exponent 1; thus differentiating by $\bar{s}$ shows that $\partial_{\bar{s}}^3\bar{\gamma}$ is an admissible nonlinearity of degree $n-2$ and exponent 1, as claimed. QED.

We shall denote the maximum of the $\widehat{X}$ norms of the above quantities by $C_{BL}$; thus if $\check{\chi}$ is any of the quantities in the foregoing except $\partial_{\bar{s}}^3\bar{\gamma}$, we have

$$\|\check{\chi}\|_{H^{n-1}(A_\sigma)} \leq C_{BL}\nu^p, \qquad \underline{I}_{n-1}^1[\check{\chi}](\sigma) \leq C_{BL}^2\nu^{2p},$$

where $p$ is the appropriate exponent.

We recognise in the above proof the use of the special algebraic structure of the equations which was described in Section 0.4 above (see item 2).

### 6.4. Energy inequalities

We now derive integral inequalities satisfied by the energies defined in Section 6.2. We begin by showing how to use the divergence theorem to derive integral inequalities which will be used in the next section to give bounds on $\overline{E}_n[\overline{\gamma}]$. We work with the metric $\tilde{h}$ (see (6.2.10)); in particular, we let $\Box = \Box_{\tilde{h}}$ denote the wave operator for $\tilde{h}$.

On $A_\sigma$ we have the induced metric

$$\widehat{\tilde{h}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{k} \begin{pmatrix} \widetilde{\widetilde{\delta a}} & -\frac{1}{\sqrt{2}}\widetilde{\widetilde{b}} \\ -\frac{1}{\sqrt{2}}\widetilde{\widetilde{b}} & \frac{\widetilde{\widetilde{c}}}{2} \end{pmatrix},$$

which has determinant

$$|\widehat{\tilde{h}}| = 1 + \frac{1}{k}(\widetilde{\widetilde{\delta a}} + \frac{\widetilde{\widetilde{c}}}{2}) + \frac{1}{2k^2}(\widetilde{\widetilde{c}}\widetilde{\widetilde{\delta a}} - \widetilde{\widetilde{b}}^2).$$

Similarly, it can be shown that the determinant of the full metric is $|\tilde{h}| = -\widetilde{\widetilde{a}}$. On lines $\overline{s} = \text{const}$, $\overline{v} = \text{const}$ we have the induced metric $\widetilde{\widetilde{a}} = 1 + k^{-1}\widetilde{\widetilde{\delta a}}$ with volume form $\sqrt{\widetilde{\widetilde{a}}} = 1 + k^{-1}\widetilde{\widetilde{\delta \ell}}$; thus the surface elements on $\Sigma_0$ and $\Sigma_1$ will be $(1+k^{-1}\widetilde{\widetilde{\delta\ell}})d\overline{x}\, d\overline{v}$, and the normal vectors will be $\partial_{\overline{v}}$ and $-\partial_{\overline{v}}$, respectively, while the surface element on $U_0$ will be $\sqrt{\widetilde{\widetilde{a}}} d\overline{x}\, d\overline{s}$ and the normal vector will be $\partial_{\overline{s}}$. We let $-t^a$ denote the (past-directed) unit normal vector to surfaces $\tau = \text{constant}$ and note that in terms of $\partial_\tau$ we have

$$t = \frac{1}{\sqrt{1 - \frac{\widetilde{\widetilde{c}}}{2k}}}\partial_\tau.$$

Now suppose that $f$ is some function defined on the bulk. Then we define its stress-energy tensor

$$\begin{aligned} Q[f]_{ab} &= \nabla_a f \nabla_b f - \frac{1}{2}\tilde{h}_{ab}\widetilde{h^{-1}}^{cd}\nabla_c f \nabla_d f \\ &= \nabla_a f \nabla_b f - \frac{1}{2}\eta_{ab}\eta^{cd}\nabla_c f \nabla_d f - \frac{1}{2k}\widetilde{\delta h}_{ab}\left(\eta^{cd} + \frac{1}{k}\widetilde{\delta h^{-1}}^{cd}\right)\nabla_c f \nabla_d f \\ &= Q^0[f]_{ab} + \frac{1}{k}\delta Q[f]_{ab}, \end{aligned}$$

where

$$\begin{aligned} Q^0[f]_{ab} &= \nabla_a f \nabla_b f - \frac{1}{2}\eta_{ab}\eta^{cd}\nabla_c f \nabla_d f, \\ \delta Q[f]_{ab} &= -\frac{1}{2}\widetilde{\delta h}_{ab}\left(\eta^{cd} + \frac{1}{k}\widetilde{\delta h^{-1}}^{cd}\right)\nabla_c f \nabla_d f. \end{aligned}$$

The divergence theorem on $\Gamma$ applied to $Q[f]_{ab}\partial_\tau^b$ gives (here, quantities like $\partial_\tau^b$, etc., denote tangent vectors and not differential operators)

$$\begin{aligned} &\int_{A_\sigma} Q[f]_{ab}\partial_\tau^b t^a \sqrt{|\widehat{\tilde{h}}|}\, d\xi\, d\zeta + \int_{\Sigma_1} Q[f]_{ab}\partial_\tau^b \partial_{\overline{v}}^a \sqrt{\widetilde{\widetilde{a}}}\, d\overline{x}\, d\overline{v} - \int_{\Sigma_0} Q[f]_{ab}\partial_\tau^b \partial_{\overline{v}}^a \sqrt{\widetilde{\widetilde{a}}}\, d\overline{x}\, d\overline{v} - \int_{U_0} Q[f]_{ab}\partial_\tau^b \partial_{\overline{s}}^a \sqrt{\widetilde{\widetilde{a}}}\, d\overline{x}\, d\overline{s} \\ &\qquad = -\int_{\Gamma \cap \{\tau \le \sigma\}} \left(Q[f]^a_b \partial_\tau^b\right)_{;a} \sqrt{-|\tilde{h}|}\, d\xi\, d\zeta\, d\tau. \end{aligned} \tag{6.4.1}$$

A standard computation gives

$$\left(Q[f]^a_b \partial_\tau^b\right)_{;a} = \Box_{\tilde{h}} f \partial_\tau f + Q[f]^{ab}\mathcal{L}_{\partial_\tau}\tilde{h}_{ab} = \Box_{\tilde{h}} f \partial_\tau f + Q[f]_{ab}\partial_\tau \widetilde{h^{-1}}^{ab}.$$

Now, noting that $\partial_\tau \widetilde{h^{-1}}^{ab} = k^{-1}\partial_\tau \widetilde{\delta h^{-1}}^{ab}$, we have

$$Q[f]_{ab}\partial_\tau \widetilde{h^{-1}}^{ab} = \frac{1}{k}\left(Q[f]^0_{ab} + \frac{1}{k}\delta Q[f]_{ab}\right)\partial_\tau \widetilde{\delta h^{-1}}^{ab}.$$

To compute the integrals on the boundary surfaces in (6.4.1), we note the following results:

On $A_\sigma$:

$$\begin{aligned}
Q[f]_{ab}\partial_\tau^b t^a\sqrt{|\hat{\widetilde{h}}|} &= \frac{\sqrt{|\hat{\widetilde{h}}|}}{(1-\frac{\widetilde{\widetilde{c}}}{2k})^{1/2}}\left[(\partial_\tau f)^2 + \frac{1}{2}\left(1-\frac{\widetilde{\widetilde{c}}}{2k}\right)\widetilde{h^{-1}}^{cd}\partial_c f\partial_d f\right] \\
&= \frac{\sqrt{|\hat{\widetilde{h}}|}}{(1-\frac{\widetilde{\widetilde{c}}}{2k})^{1/2}}\left\{(\partial_\tau f)^2 + \frac{1}{2}\left(1-\frac{\widetilde{\widetilde{c}}}{2k}\right)\left[-(\partial_\tau f)^2 + (\partial_\xi f)^2 + (\partial_\zeta f)^2 + \frac{1}{k}\widetilde{\delta h^{-1}}^{cd}\partial_c f\partial_d f\right]\right\} \\
&= \frac{\sqrt{|\hat{\widetilde{h}}|}}{(1-\frac{\widetilde{\widetilde{c}}}{2k})^{1/2}}\left\{\frac{1}{2}\left[(\partial_\tau f)^2 + (\partial_\xi f)^2 + (\partial_\zeta f)^2\right]\right. \\
&\qquad \left. - \frac{\widetilde{\widetilde{c}}}{4k}\left[(\partial_\tau f)^2 + (\partial_\xi f)^2 + (\partial_\zeta f)^2\right] + \frac{1}{2k}\left(1-\frac{\widetilde{\widetilde{c}}}{2k}\right)\widetilde{\delta h^{-1}}^{cd}\partial_c f\partial_d f\right\} \\
&= \frac{1}{2}\left[(\partial_\tau f)^2 + (\partial_\xi f)^2 + (\partial_\zeta f)^2\right] + k^{-1}\Delta_A^{cd}\partial_c f\partial_d f,
\end{aligned}$$

On $\Sigma_0$ and $\Sigma_1$:

$$\begin{aligned}
Q[f]_{ab}\partial_\tau^b\partial_{\overline{v}}^a\sqrt{\widetilde{\widetilde{a}}} &= \frac{1}{\sqrt{2}}\left[(\partial_{\overline{v}} f)^2 + \frac{1}{2\widetilde{\widetilde{a}}}(\partial_{\overline{x}} f)^2\right](1 + k^{-1}\widetilde{\widetilde{\delta\ell}}) \\
&= \frac{1}{\sqrt{2}}\left[\frac{1}{2}(\partial_{\overline{x}} f)^2 + (\partial_{\overline{v}} f)^2\right] + k^{-1}\Delta_\Sigma^{cd}\partial_c f\partial_d f,
\end{aligned}$$

On $U_0$:

$$\begin{aligned}
Q[f]_{ab}\partial_\tau^b\partial_{\overline{s}}^a\sqrt{\widetilde{\widetilde{a}}} &= \left[\partial_\tau f\partial_\zeta f + \frac{1}{2\sqrt{2}}\left(\eta^{cd}\partial_c f\partial_d f + \frac{1}{k}\widetilde{\delta h^{-1}}^{cd}\partial_c f\partial_d f\right)\right]\sqrt{\widetilde{\widetilde{a}}} \\
&= \left[\frac{1}{\sqrt{2}}\left((\partial_{\overline{s}} f)^2 + \frac{1}{2}(\partial_{\overline{x}} f)^2\right) + \frac{1}{k}\frac{1}{2\sqrt{2}}\widetilde{\delta h^{-1}}^{cd}\partial_c f\partial_d f\right]\sqrt{\widetilde{\widetilde{a}}} \\
&= \frac{1}{\sqrt{2}}\left[(\partial_{\overline{s}} f)^2 + \frac{1}{2}(\partial_{\overline{x}} f)^2\right] + k^{-1}\Delta_U^{cd}\partial_c f\partial_d f.
\end{aligned}$$

We note that $\widetilde{\delta h^{-1}}^{cd}\partial_c f\partial_d f$ only involves derivatives $(\partial_{\overline{s}} f)^2$, $\partial_{\overline{s}}\partial_{\overline{x}} f$, and $(\partial_{\overline{x}} f)^2$. In deriving the formula on $\Sigma_0$ and $\Sigma_1$ we use $\widetilde{\widetilde{b}} = \widetilde{\widetilde{c}} = 0$ on those surfaces (these hold on $\Sigma_0$ by the initial conditions, and on $\Sigma_1$ by the cutoff).

Substituting all of this in to equation (6.4.1), and noting that $\widetilde{\widetilde{a}} = 1$ on $\Sigma_1$ by the cutoff, we have thus

$$\begin{aligned}
&\int_{A_\sigma}\frac{1}{2}\left[(\partial_\tau f)^2 + (\partial_\xi f)^2 + (\partial_\zeta f)^2\right] + k^{-1}\Delta_A^{cd}\partial_c f\partial_d f\,d\xi\,d\zeta \\
&\qquad + \int_{\Sigma_1}\frac{1}{\sqrt{2}}\left[(\partial_{\overline{x}} f)^2 + (\partial_{\overline{v}} f)^2\right] + k^{-1}\Delta_\Sigma^{cd}\partial_c f\partial_d f\,d\overline{v}\,d\overline{x} \\
&\qquad - \int_{\Sigma_0}\frac{1}{\sqrt{2}}\left[(\partial_{\overline{x}} f)^2 + (\partial_{\overline{v}} f)^2\right] + k^{-1}\Delta_\Sigma^{cd}\partial_c f\partial_d f\,d\overline{v}\,d\overline{x} \\
&\qquad - \int_{U_0}\frac{1}{\sqrt{2}}\left[(\partial_{\overline{s}} f)^2 + \frac{1}{2}(\partial_{\overline{x}} f)^2\right] + k^{-1}\Delta_U^{cd}\partial_c f\partial_d f\,d\overline{s}\,d\overline{x} \\
&= -\int_{\Gamma\cap\{\tau\le\sigma\}}\left\{\Box f\partial_\tau f + \frac{1}{k}\left(Q[f]^0_{ab} + \frac{1}{k}\delta Q[f]_{ab}\right)\partial_\tau\widetilde{\delta h^{-1}}^{ab}\right\}\sqrt{\widetilde{\widetilde{a}}}\,d\xi\,d\zeta\,d\tau. \qquad (6.4.2)
\end{aligned}$$

Note that the integral over $A_\sigma$ is just $\epsilon[f](\sigma)$ as defined in Section 6.2 above. Note also that the second integral on the left-hand side above is a positive contribution (since $\Delta_\Sigma^{cd} = 0$ on $\Sigma_1$) and can be dropped.

Pulling everything together, we have the inequality

$$\int_{A_\sigma} \frac{1}{2}\left[(\partial_\tau f)^2 + (\partial_\xi f)^2 + (\partial_\zeta f)^2\right] + k^{-1}\Delta_A^{cd}\partial_c f \partial_d f \, d\xi \, d\zeta$$
$$\leq \int_{\Sigma_0} \frac{1}{\sqrt{2}}\left[(\partial_{\overline{x}} f)^2 + (\partial_{\overline{v}} f)^2\right] + k^{-1}\Delta_\Sigma^{cd}\partial_c f \partial_d f \, d\overline{v}\, d\overline{x} + \int_{U_0} \frac{1}{\sqrt{2}}\left[(\partial_{\overline{s}} f)^2 + \frac{1}{2}(\partial_{\overline{x}} f)^2\right] + k^{-1}\Delta_U^{cd}\partial_c f \partial_d f \, d\overline{s}\, d\overline{x}$$
$$- \int_{\Gamma\cap\{\tau\leq\sigma\}} \left\{\Box f \partial_\tau f + \frac{1}{k}\left(Q[f]^0_{ab} + \frac{1}{k}\delta Q[f]_{ab}\right)\partial_\tau \widetilde{\delta h^{-1}}^{ab}\right\}\sqrt{\widetilde{\overline{a}}}\, d\xi\, d\zeta\, d\tau, \tag{6.4.3}$$

from which we obtain finally, noting that on the bulk $\sqrt{\widetilde{\overline{a}}} \leq \sqrt{\overline{a}} = |1 + k^{-1}\overline{\delta\ell}| \leq 4/3$,

$$\epsilon[f](\sigma) \leq \frac{4}{3}\int_{\Gamma\cap\{\tau\leq\sigma\}} \left\{|\Box f \partial_\tau f| + \left|\frac{1}{k}\left(Q[f]^0_{ab} + \frac{1}{k}\delta Q[f]_{ab}\right)\partial_\tau \widetilde{\delta h^{-1}}^{ab}\right|\right\} d\xi\, d\zeta\, d\tau + \overline{I}_{\Sigma_0}[f] + \overline{I}_{U_0}[f]. \tag{6.4.4}$$

The second term in the bulk integral can be bounded as follows:

6.4.1. Lemma. If $\overline{\Omega}$ satisfies (6.2.28) and (6.2.30), then on $\Gamma$

$$\left|\left(Q[f]^0_{ab} + \frac{1}{k}\delta Q[f]_{ab}\right)\partial_\tau \widetilde{\delta h^{-1}}^{ab}\right| \leq 2113 C_0\nu \cdot \frac{C_\chi^2}{2T'\sqrt{2}}\left[(\partial_\tau f)^2 + (\partial_\xi f)^2 + (\partial_\zeta f)^2\right].$$

Proof. Recall from Corollary 6.3.1 that $\left\|\overline{\delta h^{-1}}^{cd}\right\|_{HS}, \left\|\widetilde{\delta h^{-1}}^{cd}\right\|_{HS} \leq 47C_0\nu$. Further, if $\overline{\omega} \in \overline{\Omega}$, and $\widetilde{\overline{\omega}}$ is the corresponding cutoff quantity, then we have

$$|\widetilde{\overline{\omega}}| \leq |\overline{\omega}|, \qquad |\partial_\tau \widetilde{\overline{\omega}}| \leq \frac{C_\chi}{2T'\sqrt{2}}|\overline{\omega}| + |\partial_\tau \overline{\omega}|;$$

this will allow us below to determine bounds by working with the original (un-cutoff) metric components. Since the transformation from the $\{\partial_\tau, \partial_\xi, \partial_\zeta\}$ basis to the $\{\partial_{\overline{s}}, \partial_{\overline{x}}, \partial_{\overline{v}}\}$ basis is constant, we may work in this latter basis. Now in this basis we have

$$\overline{\delta h^{-1}}^{ij} = \begin{pmatrix} k^{-1}\frac{\overline{b}^2}{\overline{a}} - \overline{c} & \frac{\overline{b}}{\overline{a}} & 0 \\ \frac{\overline{b}}{\overline{a}} & \overline{\delta^{-1}a} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so

$$\partial_\tau \overline{\delta h^{-1}}^{ij} = \begin{pmatrix} k^{-1}\left[2\frac{\overline{b}\partial_\tau \overline{b}}{\overline{a}} - \frac{\overline{b}^2\partial_\tau \overline{a}}{\overline{a}^2}\right] - \partial_\tau \overline{c} & \frac{\partial_\tau \overline{b}}{\overline{a}} - \frac{\overline{b}}{\overline{a}^2}\partial_\tau \overline{a} & 0 \\ \frac{\partial_\tau \overline{b}}{\overline{a}} - \frac{\overline{b}}{\overline{a}^2}\partial_\tau \overline{a} & \partial_\tau \overline{\delta^{-1}a} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now we note the following bounds:

$$\partial_\tau \overline{a} = \frac{2}{k}(1 + k^{-1}\overline{\delta\ell})\partial_\tau \overline{\delta\ell}, \quad |\partial_\tau \overline{a}| \leq k^{-1}\frac{32}{3}C_0\nu \leq 1$$
$$\overline{\delta^{-1}a} = k\left(\overline{a}^{-1} - 1\right), \quad \partial_\tau \overline{\delta^{-1}a} = -k\frac{\partial_\tau \overline{a}}{\overline{a}^2}, \quad |\partial_\tau \overline{\delta^{-1}a}| \leq \frac{16}{9}\frac{32}{3}C_0\nu \leq 20C_0\nu,$$
$$\frac{|\partial_\tau \overline{b}|}{\overline{a}} \leq \frac{16}{3}C_0\nu, \quad \left|\frac{\overline{b}}{\overline{a}^2}\partial_\tau \overline{a}\right| \leq \left|\frac{\overline{b}}{\overline{a}^2}\right| \leq \frac{64}{9}C_0\nu, \quad \left|\frac{\overline{b}\partial_\tau \overline{b}}{\overline{a}}\right| \leq \frac{64}{3}C_0\nu, \quad \left|\frac{\overline{b}^2\partial_\tau \overline{a}}{\overline{a}^2}\right| \leq \frac{256}{9}C_0\nu,$$

from which we may bound the Hilbert-Schmidt norm of $\partial_\tau \overline{\delta h^{-1}}$:

$$\begin{aligned}\|\partial_\tau \overline{\delta h^{-1}}\|_{HS} &\leq \left[\left(\frac{128}{3}C_0\nu + \frac{256}{9}C_0\nu\right)k^{-1} + 4C_0\nu + 2\left(\frac{16}{3}C_0\nu + \frac{64}{9}C_0\nu\right) + 20C_0\nu\right] \\ &\leq \left[\frac{224}{9}C_0\nu + 24C_0\nu + 73C_0\nu k^{-1}\right] \leq 122C_0\nu, \end{aligned} \tag{6.4.5}$$

where we have used $k \geq 1$. Thus finally (recalling that we have $C_\chi \geq 1$)

$$\|\partial_\tau \widetilde{\delta h^{-1}}\|_{HS} \leq \|\chi\partial_\tau \overline{\delta h^{-1}}\|_{HS} + C_\chi\|\overline{\delta h^{-1}}\|_{HS} \leq 122C_0\nu + 47C_0\nu C_\chi \leq 169C_\chi C_0\nu. \tag{6.4.6}$$

Now since

$$\overline{\delta h}_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \overline{a} & \overline{b} \\ 0 & \overline{b} & \overline{c} \end{pmatrix},$$

we have clearly $\|\overline{\delta h}\|_{HS} \leq 16C_0\nu$. Now we see by Lemma 4.2.2 that

$$\|\overline{\delta h}_{ab}\overline{\delta h^{-1}}^{ab}\| \leq 16C_0\nu \cdot 169C_\chi C_0\nu. \tag{6.4.7}$$

Further,

$$\left\|\eta + k^{-1}\overline{\delta h^{-1}}\right\|_{HS} \leq 3 + k^{-1}\cdot 47C_0\nu \leq 14. \tag{6.4.8}$$

Using Lemma 4.2.2 again, and combining equations (6.4.5 – 6.4.8), we see that

$$\left|\delta Q[f]_{ab}\partial_\tau \overline{\delta h^{-1}}^{ab}\right| \leq \frac{1}{2}\cdot 16C_0\nu \cdot 169C_\chi C_0\nu \cdot 14 \cdot \left[(\partial_\tau f)^2 + (\partial_\xi f)^2 + (\partial_\zeta f)^2\right]\cdot \frac{C_\chi}{2T'\sqrt{2}},$$

and that

$$\left\|Q[f]^0\right\|_{HS} \leq \left[(\partial_\tau f)^2 + (\partial_\xi f)^2 + (\partial_\zeta f)^2\right]\cdot\left(1+\frac{3}{2}\right)\cdot\frac{C_\chi}{2T'\sqrt{2}} = \frac{5}{2}\left[(\partial_\tau f)^2 + (\partial_\xi f)^2 + (\partial_\zeta f)^2\right]\cdot\frac{C_\chi}{2T'\sqrt{2}},$$

so

$$\left|Q[f]^0_{ab}\partial_\tau\overline{\delta h^{-1}}^{ab}\right| \leq 169C_\chi C_0\nu\cdot\frac{5}{2}\cdot\left[(\partial_\tau f)^2 + (\partial_\xi f)^2 + (\partial_\zeta f)^2\right]\cdot\frac{C_\chi}{2T'\sqrt{2}},$$

and pulling everything together, we have finally

$$\begin{aligned}\left|\left(Q[f]^0_{ab} + k^{-1}\delta Q[f]_{ab}\right)\partial_\tau\overline{\delta h^{-1}}^{ab}\right| &\leq 169C_\chi C_0\nu\left(\frac{5}{2}+10\right)\left[(\partial_\tau f)^2 + (\partial_\xi f)^2 + (\partial_\zeta f)^2\right]\cdot\frac{C_\chi}{2T'\sqrt{2}} \\ &\leq 2113C_0\nu\cdot\frac{C_\chi^2}{2T'\sqrt{2}}\left[(\partial_\tau f)^2 + (\partial_\xi f)^2 + (\partial_\zeta f)^2\right],\end{aligned}$$

as claimed. QED.

The particular number here is obviously not particularly important; the point is that it is possible to find a fixed number.

Combining this with Proposition 6.3.2 and Corollary 6.3.2, we have finally the following result.

6.4.2. LEMMA. Let $f$ be any $C^\infty$ function on $\Gamma \cap \{\tau \le \sigma\}$, $\sigma \in [0, \varsigma]$. Suppose that $\overline{\Omega}$ satisfies the conditions in (6.2.28) and (6.2.30), and that $k$ satisfies (6.3.1). Then we have

$$\epsilon[f](\sigma) \le \int_0^\sigma \left\{ \frac{4}{3} \int_{A_\tau} |\Box f \partial_\tau f| \, d\xi \, d\zeta + k^{-1} \cdot 6344 C_0 \nu \frac{C_\chi^2}{2T'\sqrt{2}} \epsilon[f](\tau) \right\} d\tau + \|f\|^2_{H^1_\circ(\Sigma_0)} + \|f\|^2_{H^1_\circ(U_0)}.$$

Proof. This follows from equation (6.4.4) and Lemma 6.4.1 by noting that Proposition 6.3.2 shows that

$$\int_{A_\sigma} (\partial_\tau f)^2 + (\partial_\xi f)^2 + (\partial_\zeta f)^2 \, d\xi \, d\zeta \le 3\epsilon[f](\sigma).$$

QED.

We note in passing that for $f = \partial^I \overline{\gamma}$ for some multiindex $I$ with $|I| \le n$, by Corollary 6.3.2 the initial data norms $\overline{I}_{\Sigma_0}[f]$ and $\overline{I}_{U_0}[f]$ will have bounds independent of $k$ since the supports of $\partial^I \overline{\gamma}|_{\Sigma_0}$ and $\partial^I \overline{\gamma}|_{U_0}$ are compact and independent of $k$ and the function $\overline{\gamma}$ itself is independent of $k$ on $\Sigma_0$ and $U_0$.

For the metric components $\overline{\delta\ell}, \overline{b}, \overline{c}$, the role of the above result is played by the following much simpler one. Here and below we define $\underline{\upsilon} = \underline{\upsilon}(\sigma) = \sigma - T'\sqrt{2}$.

6.4.3. LEMMA. Let $f$ be any nonnegative $C^\infty$ function on $\Gamma$. Then for any $\sigma \in [0, \varsigma]$ we have

$$\int_{A_\sigma} f \, d\xi \, d\zeta \le \sqrt{2} \int_{\Sigma'_\sigma} f|_{\overline{s}=0} \, d\overline{x} \, d\overline{v} + \sqrt{2} \int_{\underline{\upsilon}}^\sigma d\upsilon \int_{A_\upsilon} |\partial_{\overline{s}} f| \, d\xi \, d\zeta.$$

Proof. Consider the region $D_\sigma$ in the $(\tau, \zeta)$ plane which is an equilateral right triangle with hypotenuse $A_\sigma \cap \{\xi = 0\}$, recall the sets (equation (6.2.4), see Figure 6.2.1)

$$\begin{aligned} \Gamma_\sigma &= \Gamma \cap \{(\tau, \xi, \zeta) \in \Gamma \,|\, (\tau, \zeta) \in D_\sigma\}, \\ \Sigma'_\sigma &= \{(\upsilon, \xi, -\upsilon) \in \Sigma_0 \,|\, \upsilon \in [\underline{\upsilon}, \sigma]\}, \\ B_\upsilon &= \Gamma_\sigma \cap A_\upsilon, \end{aligned}$$

where $\upsilon \in [\underline{\upsilon}, \sigma]$, and set

$$F(\sigma) = \int_{A_\sigma} f \, d\xi \, d\zeta.$$

$\Sigma'_\sigma$ is one of the boundaries of $\Gamma_\sigma$. The other two are the sets $\tau = \sigma$ and $\overline{v} = \underline{\upsilon}\sqrt{2}$; noting that $\tau = \sigma$ is equivalent to $\overline{s} = \sigma\sqrt{2} - \overline{v}$, we may write

$$\begin{aligned} \int_{\Gamma_\sigma} \partial_{\overline{s}} f \, d\tau \, d\xi \, d\zeta &= \int_{-\infty}^{+\infty} d\overline{x} \int_{\underline{\upsilon}\sqrt{2}}^{\sigma\sqrt{2}} d\overline{v} \int_0^{\sigma\sqrt{2} - \overline{v}} d\overline{s} \partial_{\overline{s}} f \\ &= \int_{-\infty}^{+\infty} d\overline{x} \int_{\underline{\upsilon}\sqrt{2}}^{\sigma\sqrt{2}} d\overline{v} \left[ f|_{\overline{s}=\sigma\sqrt{2}-\overline{v}} - f|_{\overline{s}=0} \right] = \frac{1}{\sqrt{2}} F(\sigma) - \int_{\Sigma'_\sigma} f|_{\overline{s}=0} \, d\overline{x} \, d\overline{v}; \end{aligned}$$

since for any nonnegative function $g$ we have moreover

$$\int_{\Gamma_\sigma} g \, d\tau \, d\xi \, d\zeta = \int_{\underline{\upsilon}}^\sigma d\tau \int_{B_\upsilon} g \, d\xi \, d\zeta \le \int_{\underline{\upsilon}}^\sigma d\upsilon \int_{A_\upsilon} g \, d\xi \, d\zeta,$$

this gives the estimate

$$F(\sigma) \le \sqrt{2}\int_{\Sigma'_\sigma} f|_{\overline{s}=0}\, d\overline{x}\, d\overline{v} + \sqrt{2}\int_{\underline{\upsilon}}^{\sigma} d\upsilon \int_{A_\upsilon} |\partial_{\overline{s}} f|\, d\xi\, d\zeta,$$

as desired. QED.

Note that when $f \in \overline{\Omega}$, the integral along $\Sigma'_\sigma$ is entirely determined by the initial data.

Finally, we have the following lemma.

6.4.4. LEMMA.. Let $f$ be any $C^\infty$ function on $\Gamma$, and let $\sigma \in [0, \varsigma]$. Then

$$|f|^2_{L^2(A_\sigma)} \le 2T' \int_{\underline{\upsilon}}^{\sigma} \int_{A_{\sigma'}} |\partial_{\overline{s}} f|^2\, d\xi\, d\zeta\, d\sigma' + \frac{3}{\sqrt{2}} \int_{\Sigma'_\sigma} \left|f|_{\overline{s}=0}\right|^2\, d\overline{x}\, d\overline{v}.$$

Proof. This follows by a straightforward calculation:

$$\begin{aligned}
|f|^2_{L^2(A_\sigma)} &= \int_{A_\sigma} |f(\sigma,\xi,\zeta)|^2\, d\xi\, d\zeta = \int_{A_\sigma} \left| \int_0^{\frac{1}{\sqrt{2}}(\sigma+\zeta)} \partial_{\overline{s}} f\, d\overline{s} + f|_{\overline{s}=0} \right|^2 d\xi\, d\zeta \\
&\le \int_{A_\sigma} \frac{3}{2\sqrt{2}}(\sigma+\zeta) \int_0^{\frac{1}{\sqrt{2}}(\sigma+\zeta)} |\partial_{\overline{s}} f|^2\, d\overline{s}\, d\xi\, d\zeta + \frac{3}{\sqrt{2}} \int_{\Sigma'_\sigma} \left|f|_{\overline{s}=0}\right|^2\, d\overline{x}\, d\overline{v} \\
&\le 2T' \int_{A_\sigma} \int_0^{\frac{1}{\sqrt{2}}(\sigma+\zeta)} |\partial_{\overline{s}} f|^2\, d\overline{s}\, d\xi\, d\zeta + \frac{3}{\sqrt{2}} \int_{\Sigma'_\sigma} \left|f|_{\overline{s}=0}\right|^2\, d\overline{x}\, d\overline{v} \\
&\le 2T' \int_{\underline{\upsilon}}^{\sigma} \int_{A_{\sigma'}} |\partial_{\overline{s}} f|^2\, d\xi\, d\zeta\, d\sigma' + \frac{3}{\sqrt{2}} \int_{\Sigma'_\sigma} \left|f|_{\overline{s}=0}\right|^2\, d\overline{x}\, d\overline{v},
\end{aligned}$$

where we note that by definition $A_{\sigma'} = 0$ for $\sigma' < 0$. QED.

## 6.5. Continuation of the bootstrap

We now show how our results from Sections 6.3 and 6.4 can be applied to derive bounds on $\overline{E}_n[\overline{\gamma}]$ and $E_n[h]$. We have the following lemmata. Recall that the space $X$ was defined in Definition 6.3.1 above.

6.5.1. LEMMA. Let $(\overline{a}, \overline{b}, \overline{c}, \overline{\gamma}) \in X$, and suppose that the bootstrap assumption (6.2.28) and the bounds (6.2.29) hold on an interval $[0, \varsigma]$. Then there is a constant $C_1$ and a positive integer $N$ such that for all $\sigma \in [0, \varsigma]$ we have

$$\overline{E}_n[\overline{\gamma}](\sigma) \le C_1 \underline{T}'^{-N} \nu^3 k^{-1} \sigma + 6\left[\overline{\iota}_{\Sigma_0,n}[\overline{\gamma}] + \overline{\iota}_{U_0,n}[\overline{\gamma}]\right].$$

Proof. First, recall the following inequality from Lemma 6.4.2:

$$\epsilon[f](\sigma) \le \int_0^\sigma \left\{ \frac{4}{3} \int_{A_\tau} |\Box f \partial_\tau f|\, d\xi\, d\zeta + k^{-1} \cdot 6344 C_0 \nu \frac{C_\chi^2}{2T'\sqrt{2}} \epsilon[f](\tau) \right\} d\tau + \|f\|_{H^1_\circ(\Sigma_0)} + \|f\|_{H^1_\circ(U_0)}. \quad (6.5.1)$$

Now let $f = \partial^I \partial^\ell_{\overline{s}} \overline{\gamma}$, where $|I| \le n-1$. By Lemma 6.3.4, $\partial^\ell_{\overline{s}} \overline{\gamma}$, $\partial^\ell_{\overline{s}} \partial_\tau \overline{\gamma} \in \widehat{X}^{n-1}$, so by Proposition 6.3.2(a) we have for $\sigma \in [0, \varsigma]$

$$\|\partial_\tau f\|_{L^2(A_\sigma)} = \|\partial^I \partial^\ell_{\overline{s}} \partial_\tau \overline{\gamma}\|_{L^2(A_\sigma)} \le \|\partial^\ell_{\overline{s}} \partial_\tau \overline{\gamma}\|_{H^{n-1}(A_\sigma)} \le 3\overline{E}_n[\overline{\gamma}](\sigma).$$

Thus, using the bootstrap (6.2.28), we obtain from (6.5.1) in this case

$$
\begin{aligned}
\epsilon[\partial^I \partial_{\overline{s}}^{\ell}\overline{\gamma}](\sigma) &\leq \int_0^\sigma \frac{4}{3}\,\|\Box f\|_{L^2(A_\upsilon)}\,\|\partial_\tau f\|_{L^2(A_\upsilon)} + 6344 C_0 k^{-1}\nu \frac{C_\chi^2}{2T'\sqrt{2}}\epsilon[\partial^I \partial_{\overline{s}}^{\ell}\overline{\gamma}](\upsilon)\,d\upsilon \\
&\qquad + \|\partial^I \partial_{\overline{s}}^{\ell}\overline{\gamma}\|_{H^1_\circ(\Sigma_0)} + \|\partial^I \partial_{\overline{s}}^{\ell}\overline{\gamma}\|_{H^1_\circ(U_0)} \\
&\leq \int_0^\sigma 4\nu \left\|\Box \partial^I \partial_{\overline{s}}^{\ell}\overline{\gamma}\right\|_{L^2(A_\upsilon)} + 6344 C_0 k^{-1}\nu \frac{C_\chi^2}{2T'\sqrt{2}}\epsilon[\partial^I \partial_{\overline{s}}^{\ell}\overline{\gamma}](\upsilon)\,d\upsilon \\
&\qquad + \|\partial^I \partial_{\overline{s}}^{\ell}\overline{\gamma}\|_{H^1_\circ(\Sigma_0)} + \|\partial^I \partial_{\overline{s}}^{\ell}\overline{\gamma}\|_{H^1_\circ(U_0)} \qquad (6.5.2).
\end{aligned}
$$

The last two terms can be bounded in terms of the initial data, and the $\epsilon[\partial^I \partial_{\overline{s}}^{\ell}\overline{\gamma}](\upsilon)$ in the integral can be bounded by the bootstrap; thus we only need to bound $\left\|\Box \partial^I \partial_{\overline{s}}^{\ell}\overline{\gamma}\right\|_{L^2(A_\upsilon)}$. Since $\Box\overline{\gamma} = 0$, we may write $\Box \partial^I \partial_{\overline{s}}^{\ell}\overline{\gamma} = [\Box, \partial^I \partial_{\overline{s}}^{\ell}]\overline{\gamma}$. The wave operator $\Box$ is as given in (6.2.11):

$$
\begin{aligned}
\Box &= \left(-2\partial_{\overline{s}}\partial_{\overline{v}} + \partial_{\overline{x}}^2\right) \\
&+ \frac{1}{k}\left(-\widetilde{\overline{\delta^{-1}a}}\partial_{\overline{x}}^2 - \tilde{\overline{c}}\partial_{\overline{s}}^2 + 2\frac{\tilde{\overline{b}}}{\tilde{\overline{a}}}\partial_{\overline{s}}\partial_{\overline{x}} - \frac{1}{2}\left(2\partial_{\overline{s}}\tilde{\overline{c}} - \frac{2}{\tilde{\overline{a}}}\partial_{\overline{x}}\tilde{\overline{b}} + \frac{\partial_{\overline{v}}\widetilde{\overline{\delta a}}}{\tilde{\overline{a}}}\right)\partial_{\overline{s}} + \frac{1}{\tilde{\overline{a}}}\partial_{\overline{s}}\tilde{\overline{b}}\partial_{\overline{x}} - \frac{1}{2}\frac{\partial_{\overline{x}}\widetilde{\overline{\delta a}}}{\tilde{\overline{a}}^2}\partial_{\overline{x}} - \frac{\partial_{\overline{s}}\widetilde{\overline{\delta a}}}{\tilde{\overline{a}}}\partial_{\overline{v}}\right) \\
&+ k^{-2}\left(\frac{\tilde{\overline{b}}^2}{\tilde{\overline{a}}}\partial_{\overline{s}}^2 - \frac{1}{2}\left(\tilde{\overline{c}}\frac{\partial_{\overline{s}}\widetilde{\overline{\delta a}}}{\tilde{\overline{a}}} - 4\frac{\tilde{\overline{b}}\partial_{\overline{s}}\tilde{\overline{b}}}{\tilde{\overline{a}}} + \frac{\tilde{\overline{b}}\partial_{\overline{x}}\widetilde{\overline{\delta a}}}{\tilde{\overline{a}}^2}\right)\partial_{\overline{s}} - \frac{1}{2}\frac{\tilde{\overline{b}}}{\tilde{\overline{a}}^2}\partial_{\overline{s}}\widetilde{\overline{\delta a}}\partial_{\overline{x}}\right) - k^{-3}\left(\frac{1}{2}\frac{\tilde{\overline{b}}^2}{\tilde{\overline{a}}^2}\partial_{\overline{s}}\widetilde{\overline{\delta a}}\partial_{\overline{s}}\right).
\end{aligned}
$$

Note that the leading-order terms $-2\partial_{\overline{s}}\partial_{\overline{v}} + \partial_{\overline{x}}^2$ have constant coefficients, and will therefore drop out of the commutator $[\Box, \partial^I \partial_{\overline{s}}^{\ell}]$. The remaining terms do not contain the derivative $\partial_{\overline{s}}\partial_v b$, but only the derivatives in the set

$$D = \{\partial_{\overline{s}}^2, \partial_{\overline{s}}\partial_{\overline{x}}, \partial_{\overline{x}}^2, \partial_{\overline{s}}, \partial_{\overline{x}}, \partial_{\overline{v}}\}.$$

Recalling that

$$\frac{1}{\tilde{\overline{a}}} = 1 + k^{-1}\widetilde{\overline{\delta^{-1}a}},$$

we see that for every $\partial \in D$ there is a polynomial $P_\partial$ in the quantities in $\tilde{\overline{\Omega}} \cup \partial_{\overline{s}}\tilde{\overline{\Omega}}_0 \cup \{\widetilde{\overline{\delta^{-1}a}}\}$, with no constant term and coefficients which are constant multiples of powers of $1/k$, and such that

$$\Box - \left[-2\partial_{\overline{s}}\partial_{\overline{v}} + \partial_{\overline{x}}^2\right] = \frac{1}{k}\sum_{\partial\in D} P_\partial \partial.$$

By Proposition 6.3.3, $P_\partial$ will be an admissible nonlinearity of order $n-1$, as will its first $\overline{s}$ derivative. Now by the foregoing

$$[\Box, \partial^I \partial_{\overline{s}}^{\ell}] = \frac{1}{k}\sum_{\partial\in D}[P_\partial \partial, \partial^I \partial_{\overline{s}}^{\ell}]. \qquad (6.5.3)$$

Further,

$$[P_\partial \partial, \partial^I \partial_{\overline{s}}^{\ell}]\overline{\gamma} = \sum_{\substack{|I_1|+\ell_1>0 \\ I_1+I_2=I,\ \ell_1+\ell_2=\ell}} C_{I_1,I_2,\ell_1,\ell_2}\partial^{I_1}\partial_{\overline{s}}^{\ell_1}P_\partial \partial^{I_2}\partial_{\overline{s}}^{\ell_2}\partial\overline{\gamma},$$

where the quantities $C_{I_1,I_2,\ell_1,\ell_2}$ are combinatorial constants. We may split the terms in the sum up into two sets according as $\ell_1 = 0$ or $\ell_1 = 1$. If $\ell_1 = 0$, then $|I_1| > 0$, so we may write $\partial^{I_1} = \partial^{I_1'}\partial_i$ for some index $i \in \{1, 2\}$. Furthermore, $|I_2| \leq n-2$. If $\ell_2 = 1$ (as it will if $\ell = 1$), then we see that

$$\partial_{\overline{s}}^{\ell_2}\partial\overline{\gamma} \in \{\partial_{\overline{s}}^3\overline{\gamma}, \partial_{\overline{s}}^2\partial_{\overline{x}}\overline{\gamma}, \partial_{\overline{s}}\partial_{\overline{x}}^2\overline{\gamma}, \partial_{\overline{s}}^2\overline{\gamma}, \partial_{\overline{s}}\partial_{\overline{x}}\overline{\gamma}, \partial_{\overline{s}}\partial_{\overline{v}}\overline{\gamma}\} \subset \widehat{X}^{n-2};$$

since $P_\partial \in \widehat{X}^{n-1}$, we must have $\partial_i P_\partial \in \widehat{X}^{n-2}$, and since $|I_1 + I_2| \leq n-2$, we have by Lemma 6.3.2 and the definition of $\widehat{X}^{n-2}$ that

$$\|\partial^{I_1}\partial_{\overline{s}}^{\ell_1} P_\partial \partial^{I_2}\partial_{\overline{s}}^{\ell_2}\partial\overline{\gamma}\|_{L^2(A_v)} \leq C_1\nu^2$$

for some constant $C_1$ which is independent of $k$.

Now suppose that $\ell_1 = 1$. In this case we have $\ell_2 = 0$, so that

$$\partial\overline{\gamma} \in \{\partial_{\overline{s}}^2\overline{\gamma}, \partial_{\overline{s}}\partial_{\overline{x}}\overline{\gamma}, \partial_{\overline{x}}^2\overline{\gamma}, \partial_{\overline{s}}\overline{\gamma}, \partial_{\overline{x}}\overline{\gamma}, \partial_{\overline{v}}\overline{\gamma}\} \subset \widehat{X}^{n-1}$$

by Lemma 6.3.4. Since also $P_\partial \in \widehat{X}^{n-1}$, we have again by Lemma 6.3.2 that

$$\|\partial^{I_1}\partial_{\overline{s}}^{\ell_1} P_\partial \partial^{I_2}\partial_{\overline{s}}^{\ell_2}\partial\overline{\gamma}\|_{L^2(A_v)} \leq C_2\nu^2$$

for some constant $C_2$ which is independent of $k$. Pulling everything together, then, we have that there is a constant $C'$ such that for all $I$ with $|I| \leq n-1$, all $\ell \in \{0,1\}$, and all $\partial \in D$, we have

$$\left\|[P_\partial\partial, \partial^I\partial_{\overline{s}}^{\ell}]\overline{\gamma}\right\|_{L^2(A_v)} \leq C'\nu^2,$$

whence by (6.5.3) we have that there is a constant $C''$ such that for all such $I$, $\ell$, and $\partial$,

$$\left\|[\Box, \partial^I\partial_{\overline{s}}^{\ell}]\overline{\gamma}\right\|_{L^2(A_v)} \leq k^{-1}C''\nu^2,$$

whence finally there is a constant $C_{\overline{\gamma}}$ such that

$$\sum_{|I|\leq n-1}\sum_{\ell=0}^{1}\left\|\Box\partial^I\partial_{\overline{s}}^{\ell}\overline{\gamma}\right\|_{L^2(A_v)} \leq k^{-1}C_{\overline{\gamma}}\nu^2.$$

Noting that by the bootstrap (6.2.28)

$$\sum_{|I|\leq n-1}\sum_{\ell=0}^{1}\epsilon[\partial^I\partial_{\overline{s}}^{\ell}\overline{\gamma}](\sigma) = \overline{E}_n[\overline{\gamma}](\sigma) \leq \nu^2,$$

equation (6.5.2) and Corollary 6.3.2 then gives

$$\begin{aligned}\overline{E}_n[\overline{\gamma}](\sigma) &= \sum_{|I|\leq n-1}\sum_{\ell=0}^{1}\epsilon[\partial^I\partial_{\overline{s}}^{\ell}\overline{\gamma}](\sigma)\\ &\leq 4\nu\int_0^\sigma\sum_{|I|\leq n-1}\sum_{\ell=0}^{1}\left\|\Box\partial^I\partial_{\overline{s}}^{\ell}\overline{\gamma}\right\|_{L^2(A_v)}\,dv + 6344C_0k^{-1}\sigma\nu^2\frac{C_\chi^2}{2T'\sqrt{2}} + 6\left[\overline{\iota}_{\Sigma_0,n}[\overline{\gamma}] + \overline{\iota}_{U_0,n}[\overline{\gamma}]\right];\end{aligned}$$

thus

$$\overline{E}_n[\overline{\gamma}](\sigma) \leq 4\nu^3k^{-1}\sigma C_{\overline{\gamma}} + 6344C_0k^{-1}\sigma\nu^3\frac{C_\chi^2}{2T'\sqrt{2}} + 6\left[\overline{\iota}_{\Sigma_0,n}[\overline{\gamma}] + \overline{\iota}_{U_0,n}[\overline{\gamma}]\right].$$

Now the constant $C_{\overline{\gamma}}$ may depend on $T'$, but it can at any event be bounded by a multiple of $\underline{T}'^{-N}$ for some $N > 0$. There is thus a constant $C_1$ and a positive integer $N$ such that

$$\overline{E}_n[\overline{\gamma}](\sigma) \leq C_1\underline{T}'^{-N}\nu^3k^{-1}\sigma + 6\left[\overline{\iota}_{\Sigma_0,n}[\overline{\gamma}] + \overline{\iota}_{U_0,n}[\overline{\gamma}]\right],$$

as desired. QED.

We have a similar result for the energies $E_{n,0}[h](\sigma)$ and $E_{n,1}[h](\sigma)$. Again, recall that the set $\mathbf{X}$ was defined in Definition 6.3.1 above.

6.5.2. LEMMA. Let $(\overline{a}', \overline{b}', \overline{c}', \overline{\gamma}', \overline{a}, \overline{b}, \overline{c}, \overline{\gamma}) \in \mathbf{X}$. Then for $\sigma \in [0, \varsigma]$ we have

$$E_{n,1}[h](\sigma) \leq \iota_{n,1}[h](\sigma) + 2\sqrt{2}\int_{\underline{\upsilon}(\sigma)}^{\sigma} C_{BL}\nu^3\,d\upsilon = 4C_{BL}\nu^3 T' + \iota_{n,1}[h](\sigma),$$
$$E_{n,0}[h](\sigma) \leq 8\sqrt{2}T'^2 C_{BL}\nu^2 + \frac{3}{2}\iota_{n,0}[h](\sigma) + \iota_{n,1}[h](\sigma).$$

Proof. These are both very straightforward. The inequality in Lemma 6.4.3 allows us to write (recalling that $\underline{\upsilon}(\sigma) = \sigma - T'\sqrt{2}$)

$$\begin{aligned}
E_{n,1}[h](\sigma) &= \sum_{|I|\leq n-1}\sum_{\overline{\omega}\in\overline{\Omega}} \left\|\partial^I \partial_{\overline{s}}\overline{\omega}\right\|^2_{L^2(A_\sigma)} = \sum_{|I|\leq n-1}\sum_{\overline{\omega}\in\overline{\Omega}} \int_{A_\sigma} \left|\partial^I \partial_{\overline{s}}\overline{\omega}\right|^2\,d\xi\,d\zeta \\
&\leq \sqrt{2}\sum_{|I|\leq n-1}\sum_{\overline{\omega}\in\overline{\Omega}} \left[\int_{\Sigma'_\sigma} \left|\partial^I \partial_{\overline{s}}\overline{\omega}\right|^2\,d\overline{x}\,d\overline{\upsilon} + \int_{\underline{\upsilon}(\sigma)}^{\sigma}\int_{A_\upsilon} 2\left|\partial^I \partial_{\overline{s}}\overline{\omega}\right|\left|\partial^I \partial^2_{\overline{s}}\overline{\omega}\right|\,d\xi\,d\zeta\,d\upsilon\right] \\
&\leq \iota_{n,1}[h](\sigma) + 2\sqrt{2}\sum_{|I|\leq n-1}\sum_{\overline{\omega}\in\overline{\Omega}} \int_{\underline{\upsilon}(\sigma)}^{\sigma} \left\|\partial^I \partial_{\overline{s}}\overline{\omega}\right\|_{L^2(A_\upsilon)}\left\|\partial^I \partial^2_{\overline{s}}\overline{\omega}\right\|_{L^2(A_\upsilon)}\,d\upsilon \\
&\leq \iota_{n,1}[h](\sigma) + 2\sqrt{2}\int_{\underline{\upsilon}(\sigma)}^{\sigma} E_{n,1}[h](\upsilon)^{1/2}\left\|\partial^2_{\overline{s}}\overline{\omega}\right\|_{H^{n-1}(A_\upsilon)}\,d\upsilon \\
&\leq \iota_{n,1}[h](\sigma) + 2\sqrt{2}\int_{\underline{\upsilon}(\sigma)}^{\sigma} C_{BL}\nu^3\,d\upsilon = 4C_{BL}\nu^3 T' + \iota_{n,1}[h](\sigma),
\end{aligned}$$

which is the first inequality; and this together with Lemma 6.4.4 allows us to write

$$\begin{aligned}
E_{n,0}[h](\sigma) = \sum_{|I|\leq n-1}\sum_{\overline{\omega}\in\overline{\Omega}} \left\|\partial^I \overline{\omega}\right\|^2_{L^2(A_\sigma)} &\leq \sum_{|I|\leq n-1}\sum_{\overline{\omega}\in\overline{\Omega}} 2T'\int_{\underline{\upsilon}(\sigma)}^{\sigma}\int_{A_\upsilon} \left|\partial^I \partial_{\overline{s}}\overline{\omega}\right|^2\,d\xi\,d\zeta\,d\upsilon + \frac{3}{\sqrt{2}}\int_{\Sigma'_\sigma} \left|\partial^I \overline{\omega}\right|^2\,d\overline{x}\,d\overline{\upsilon} \\
&\leq 2T'\int_{\underline{\upsilon}(\sigma)}^{\sigma} E_{n,1}[h](\upsilon)\,d\upsilon + \frac{3}{2}\iota_{n,0}[h](\sigma) \leq 8\sqrt{2}T'^2 C_{BL}\nu^3 + \frac{3}{2}\iota_{n,0}[h](\sigma) + \iota_{n,1}[h](\sigma),
\end{aligned}$$

the second inequality. QED.

We now have the following theorem.

6.5.1. THEOREM. Let $T, T' > 0$. Let $p \in (0,1)$, assume that $\nu \in (0,1)$ satisfies

$$\nu \leq \min\left\{\frac{1}{128T'(1+2\sqrt{2}T')C_{BL}}, \frac{1-p}{C_1 T}\underline{T'}^N\right\},$$

where $C_1$ and $N$ are as in in Lemma 6.5.1, and assume that the initial data satisfy

$$2\sup_{\sigma\in[0,\varsigma]} \iota_n[h] \leq \frac{p}{8}\nu^2, \qquad 6\left[\overline{\iota}_{\Sigma_0,n}[\overline{\gamma}] + \overline{\iota}_{U_0,n}[\overline{\gamma}]\right] \leq p\nu^2.$$

If $(\overline{a}, \overline{b}, \overline{c}, \overline{\gamma}) \in X$, then for $\sigma \in [0, kT]$ we have

$$E_n[h](\sigma) \leq \nu^2, \qquad \overline{E}_n[\overline{\gamma}](\sigma) \leq \nu^2. \tag{6.5.4}$$

Proof. Set

$$\overline{\epsilon} = \inf\{\sigma \mid E_n[h](\sigma) \geq \nu^2, \text{ or } \overline{E}_n[\overline{\gamma}](\sigma) \geq \nu^2\}.$$

Then (6.5.4) must hold for $\sigma \in [0, \overline{\epsilon}]$, and moreover we must have $E_n[h](\overline{\epsilon}) = \nu^2$ or $\overline{E}_n[\overline{\gamma}](\overline{\epsilon}) = \nu^2$. In the first case, Lemma 6.5.2 would give

$$\begin{aligned}\nu^2 = E_n[h](\overline{\epsilon}) &\leq 4C_{BL}T'\nu^3(1+2\sqrt{2}T') + 2\iota_n[h](\sigma) \leq \nu^2\left[4C_{BL}T'(1+2T'\sqrt{2})\nu + \frac{p}{8}\right]\\ &\leq \nu^2\left[\frac{1}{8}+\frac{1}{8}\right] = \frac{1}{4}\nu^2,\end{aligned}$$

a contradiction. Thus we must have $\overline{E}_n[\overline{\gamma}](\overline{\epsilon}) = \nu^2$; then Lemma 6.5.1 gives

$$\nu^2 \leq C_1\underline{T}'^{-N}\nu^3k^{-1}\overline{\epsilon} + p\nu^2 \leq \nu^2\left[p + k^{-1}\overline{\epsilon}\frac{1-p}{T}\right],$$

whence we obtain $\overline{\epsilon} \geq kT$, so that equation (6.5.4) must indeed hold on $[0, kT]$, as desired. QED.

### 6.6. Existence

In this penultimate section we shall show to apply the foregoing results to prove existence of solutions to the system (6.2.11 – 6.2.14) by showing convergence of an iterative approximation in $H^{n-1}$. Suppose given some suitable (i.e., satisfying the conditions in Section 5.1) set of initial data, which we shall denote $\overline{\delta\ell}_0$, $\partial_{\overline{s}}\overline{\delta\ell}_0$, $\overline{b}_0 = 0$, $\partial_{\overline{s}}\overline{b}_0$, $\overline{c}_0 = 0$, $\partial_{\overline{s}}\overline{c}_0 = 0$, and $\overline{\gamma}_0$ (by abuse of notation, we let $\overline{\gamma}_0$ now indicate also the initial data on $U_0$), and which we assume to satisfy the bounds (6.2.27), (6.2.29). The initial data shall be fixed throughout the rest of the argument. The only real difficulty is in starting the iteration. We begin with the metric components. We define $\overline{\delta\ell}_1$, $\overline{b}_1$, $\overline{c}_1$ as follows. First we define $\overline{\delta\ell}'_1$, $\overline{b}'_1$, $\overline{c}'_1$ on $\Sigma_0$ by setting

$$\partial^\ell_{\overline{s}}\overline{\delta\ell}'_1 = \partial^\ell_{\overline{s}}\overline{\delta\ell}_0, \qquad \partial^\ell_{\overline{s}}\overline{b}'_1 = \partial^\ell_{\overline{s}}\overline{b}_0, \qquad \partial^\ell_{\overline{s}}\overline{c}'_1 = \partial^\ell_{\overline{s}}\overline{c}_0, \qquad \ell \in \{0,1\};$$

we then extend to $\Gamma_0$ by requiring $\partial_{\overline{s}}\overline{\delta\ell}'_1$, $\partial_{\overline{s}}\overline{b}'_1$, $\partial_{\overline{s}}\overline{c}'_1$ to be constant in $\overline{s}$ on $\Gamma_0$, and finally define

$$\overline{\delta\ell}_1 = \chi(\overline{s})\overline{\delta\ell}'_1, \qquad \overline{b}_1 = \chi(\overline{s})\overline{b}'_1, \qquad \overline{c}_1 = \chi(\overline{s})\overline{c}'_1.$$

By the definition of $C_\chi$ (see (6.2.7), (6.2.8), (6.2.9)), as well as the conditions (6.2.29), $(\overline{\delta\ell}_1, \overline{b}_1, \overline{c}_1)$ will satisfy the bootstrap condition (see (6.2.28))

$$E_n[h](\sigma) \leq \nu^2$$

for $\sigma \in [0, \varsigma]$. Note that actually this holds with the $L^2$ norms in $E_n$ replaced by $L^\infty$ norms.

Now let $\overline{\gamma}_1$ be the solution to (6.2.11) with $\overline{\delta\ell} = \overline{\delta\ell}_1$, $\overline{b} = \overline{b}_1$, and $\overline{c} = \overline{c}_1$, with $\overline{\gamma}_1 = \overline{\gamma}_0$ on $\Sigma_0 \cup U_0$. Then we have the following result.

6.6.1. Proposition. There is a constant $C'_1$ and a positive integer $N'_1$, independent of $T'$, $T$, $\nu$, and $k$, such that if $\nu \leq \frac{T'^N \log 2}{C'_1 T}$, then for $\sigma \in [0, kT]$ the function $\overline{\gamma}_1$ satisfies the bootstrap condition

$$\overline{E}_n[\overline{\gamma}_1](\sigma) \leq \nu^2.$$

Proof. This is very similar to, but much simpler than, the proof of Lemma 6.5.1. First, summing the inequality in Lemma 6.4.2 over all expressions of the form $\partial^I \partial_{\overline{s}}^\ell \overline{\gamma}_1$, $|I| \leq n-1$, $\ell = 0, 1$, and applying Corollary 6.3.2, we may write

$$\overline{E}_n[\overline{\gamma}](\sigma) \leq \sum_{\substack{|I|\leq n-1 \\ \ell=0,1}} \int_0^\sigma \left\{ \frac{4}{3} \left\| \Box \partial^I \partial_{\overline{s}}^\ell \overline{\gamma}_1 \right\|_{L^2(A_\upsilon)} \left\| \partial_\tau \partial^I \partial_{\overline{s}}^\ell \overline{\gamma}_1 \right\|_{L^2(A_\upsilon)} + k^{-1} \cdot 6344 C_0 \nu \frac{C_\chi^2}{2T'\sqrt{2}} \epsilon[\partial^I \partial_{\overline{s}}^\ell \overline{\gamma}_1](\upsilon) \right\} d\upsilon$$

$$+ 6 \left[ \overline{\iota}_{\Sigma_0,n}[\overline{\gamma}] + \overline{\iota}_{U_0,n}[\overline{\gamma}] \right],$$

where here $\Box = \Box_{h_1}$. Now as in the proof of Lemma 6.5.1 we may write

$$\Box \partial^I \partial_{\overline{s}}^\ell \overline{\gamma}_1 = [\Box, \partial^I \partial_{\overline{s}}^\ell] \overline{\gamma}_1 = \frac{1}{k} \sum_\partial [P_\partial \partial, \partial^I \partial_{\overline{s}}^\ell] \overline{\gamma}_1, \tag{6.6.1}$$

where $P_\partial$, etc., are as there. Now in the present case, all derivatives $\partial^I \partial_{\overline{s}}^\ell P_\partial$ are bounded *in* $L^\infty$ by $\nu^2$, and the only question is how to bound the derivatives of $\overline{\gamma}_1$. This may be dealt with exactly as there: briefly, any derivative of $\overline{\gamma}_1$ appearing in the sum in (6.6.1) will be of the form

$$\partial^J \partial_{\overline{s}}^\ell \overline{\gamma}_1,$$

where $|J| \leq n$, $\ell \leq 3$, and $|J| + \ell \leq n+1$. If $|J| = n$, then $\ell \leq 1$, and such a derivative can be bounded by $\overline{E}_n[\overline{\gamma}]$. If $\ell = 2$, then $|J| \leq n-1$, so writing $\partial_{\overline{s}} = \frac{1}{\sqrt{2}}(\partial_\tau + \partial_\zeta)$, we see again that such a derivative can be bounded by $\overline{E}_n[\overline{\gamma}]$, as before. Finally, if $\ell = 3$, then solving the wave equation $\Box \overline{\gamma}_1 = 0$ for $\partial_{\overline{s}}^2 \overline{\gamma}_1$ as in (6.3.8), differentiating with respect to $\overline{s}$, and using the $L^\infty$ bounds on the metric components, we may reduce to the case $\ell = 2$. Thus there must be a constant $C$ and a positive integer $N'$ such that

$$\overline{E}_n[\overline{\gamma}](\sigma) \leq \int_0^\sigma \frac{1}{k} \nu C(n+2) C_\chi \underline{T}'^{-N'} \overline{E}_n[\overline{\gamma}](\upsilon) + \frac{1}{k} \nu \cdot 6344 C_0 \frac{C_\chi^2}{2T'\sqrt{2}} \overline{E}_n[\overline{\gamma}](\upsilon)\, d\upsilon + 6 \left[ \overline{\iota}_{\Sigma_0,n}[\overline{\gamma}] + \overline{\iota}_{U_0,n}[\overline{\gamma}] \right],$$

or combining terms, that there must be a constant $C_1'$ such that

$$\overline{E}_n[\overline{\gamma}](\sigma) \leq C\nu \underline{T}'^{-N'} \frac{1}{k} \int_0^\sigma \overline{E}_n[\overline{\gamma}](\upsilon)\, d\upsilon + 6 \left[ \overline{\iota}_{\Sigma_0,n}[\overline{\gamma}] + \overline{\iota}_{U_0,n}[\overline{\gamma}] \right],$$

from which a routine application of Grönwall's Lemma (see, e.g., [2], Theorem 1.1) and the bound (6.2.27) gives the desired result. QED.

We now proceed by induction. Assume that $\nu$ satisfies the conditions in Proposition 6.6.1 (for say $p = 1/2$) and Theorem 6.5.1. Suppose that for some $m \geq 1$ we have constructed $\overline{\delta\ell}_m, \overline{b}_m, \overline{c}_m, \overline{\gamma}_m$, such that $\overline{\gamma}_m$ satisfies (6.2.11) with $\overline{\delta\ell} = \overline{\delta\ell}_m$, $\overline{b} = \overline{b}_m$, $\overline{c} = \overline{c}_m$, and such that $\overline{\delta\ell}_m$, $\overline{b}_m$, $\overline{c}_m$, $\overline{\gamma}_m$ satisfy the bootstrap assumptions (6.2.28) on the interval $[0, kT]$, where $T$ is as in Proposition 6.6.1. Then we construct $\overline{\delta\ell}_{m+1}, \overline{b}_{m+1}, \overline{c}_{m+1}$ by solving the Riccati equations (6.2.12 – 6.2.14) with the given initial data and $\overline{\gamma} = \overline{\gamma}_m$, and $\overline{\gamma}_{m+1}$ by solving the wave equation (6.2.11) with the given initial data and $\overline{\delta\ell} = \overline{\delta\ell}_{m+1}$, $\overline{b} = \overline{b}_{m+1}$, $\overline{c} = \overline{c}_{m+1}$. Then by Theorem 6.5.1, $\overline{\delta\ell}_{m+1}, \overline{b}_{m+1}, \overline{c}_{m+1}, \overline{\gamma}_{m+1}$ will also satisfy the bootstrap assumptions (6.2.28) on the interval $[0, kT]$.

We claim that, after potentially shrinking $T$ and $T'$ slightly, this sequence converges in $H^{n-1}$. This follows from a very standard argument using Lipschitz estimates and the Grönwall inequality. Specifically, we have the following theorem.

6.6.1. THEOREM. Let $T > 0$, $T' \geq 1$. Then there exist constants $C_1, C_2 > 0$ and a positive integer $N$, independent of $T$ and $T'$, such that the following holds. Let $k \geq C_1$,

$$\nu \leq C_2 \min\{T'^{-N}, \frac{T'^N}{T}\}, \tag{6.6.2}$$

and suppose that the initial data satisfy (6.2.27), (6.2.29). Then there is a unique solution to (6.2.11 – 6.2.14) on the set

$$\Gamma = \{(\overline{s}, \overline{x}, \overline{v}) \in \mathbf{R}^3 \,|\, \overline{s} \in [0, T'],\, \overline{v} \in [0, kT],\, \tau \leq kT/\sqrt{2}\},$$

and for $\sigma \in [0, kT]$ the bounds

$$E_n[h](\sigma) \leq \nu^2$$
$$\overline{E}_n[\overline{\gamma}](\sigma) \leq \nu^2$$

hold.

Proof. By choosing $C_2$ appropriately we may assume that (6.6.2) implies the bounds on $\nu$ in Proposition 6.6.1 and Theorem 6.5.1. Let $m > 1$, and consider two consecutive elements of the above approximation sequence, $(\overline{\delta\ell}_m, \overline{b}_m, \overline{c}_m, \overline{\gamma}_m)$ and $(\overline{\delta\ell}_{m+1}, \overline{b}_{m+1}, \overline{c}_{m+1}, \overline{\gamma}_{m+1})$. Note that since $m > 1$ both of these sequence elements will be in $X$. Let

$$D = \{\partial_{\overline{s}}^2, \partial_{\overline{s}}\partial_{\overline{x}}, \partial_{\overline{x}}^2, \partial_{\overline{s}}, \partial_{\overline{x}}, \partial_{\overline{v}}\}$$

and $P_\partial^m$, $\partial \in D$ be as in the proof of Lemma 6.5.1, where $P_\partial^m$ is constructed with the metric components $\overline{\delta\ell}_m$, $\overline{b}_m$, and $\overline{c}_m$. Let also $\Box_m = \Box_{\tilde{h}_m}$, and let $\overline{E}_{n,m}[\overline{\gamma}]$ denote the energy $\overline{E}_n$ constructed using $\overline{\delta\ell}_m$, $\overline{b}_m$, and $\overline{c}_m$. Then

$$\Box_{m+1}(\overline{\gamma}_{m+1} - \overline{\gamma}_m) = \frac{1}{k}\sum_{\partial \in D}\left(P_\partial^{m+1} - P_\partial^m\right)\partial\overline{\gamma}_m.$$

We proceed as in the proof of Proposition 6.6.1. Since we now have, for $I$ a multiindex and $\ell \in \{0, 1\}$,

$$\begin{aligned}\Box_{m+1}\partial^I\partial_{\overline{s}}^\ell(\overline{\gamma}_{m+1} - \overline{\gamma}_m) &= \left[\Box_{m+1}, \partial^I\partial_{\overline{s}}^\ell\right](\overline{\gamma}_{m+1} - \overline{\gamma}_m) + \frac{1}{k}\partial^I\partial_{\overline{s}}^\ell\sum_{\partial \in D}\left(P_\partial^{m+1} - P_\partial^m\right)\partial\overline{\gamma}_m \\ &= \frac{1}{k}\left[\sum_{\partial \in D}\left(P_\partial^{m+1} - P_\partial^m\right)\partial, \partial^I\partial_{\overline{s}}^\ell\right](\overline{\gamma}_{m+1} - \overline{\gamma}_m) + \frac{1}{k}\partial^I\partial_{\overline{s}}^\ell\sum_{\partial \in D}\left(P_\partial^{m+1} - P_\partial^m\right)\partial\overline{\gamma}_m,\end{aligned}$$

and since $\overline{\gamma}_{m+1} = \overline{\gamma}_m$ on $\Sigma_0 \cup U_0$, we have

$$\begin{aligned}\overline{E}_{n-1,m+1}[\overline{\gamma}_{m+1} - \overline{\gamma}_m](\sigma) \leq \sum_{\substack{|I| \leq n-2 \\ \ell = 0,1}} \int_0^\sigma 4\frac{1}{k}&\left\|\left[\sum_{\partial \in D}\left(P_\partial^{m+1} - P_\partial^m\right)\partial, \partial^I\partial_{\overline{s}}^\ell\right](\overline{\gamma}_{m+1} - \overline{\gamma}_m)\right\|_{L^2(A_\upsilon)} \\ &\cdot \left\|\partial_\tau\partial^I\partial_{\overline{s}}^\ell(\overline{\gamma}_{m+1} - \overline{\gamma}_m)\right\|_{L^2(A_\upsilon)} \\ &+ 6344 C_0\nu\frac{C_\chi^2}{2T'\sqrt{2}}k^{-1}\epsilon[\partial^I\partial_{\overline{s}}^\ell(\overline{\gamma}_{m+1} - \overline{\gamma}_m)]\, d\upsilon \\ + 4\frac{1}{k}\int_0^\sigma \sum_{\substack{|I| \leq n-2 \\ \ell = 0,1}}\sum_{\partial \in D}&\left\|\partial^I\partial_{\overline{s}}^\ell\left[\left(P_\partial^{m+1} - P_\partial^m\right)\partial\overline{\gamma}_m\right]\right\|_{L^2(A_\sigma)} \\ &\cdot \left\|\partial_\tau\partial^I\partial_{\overline{s}}^\ell(\overline{\gamma}_{m+1} - \overline{\gamma}_m)\right\|_{L^2(A_\upsilon)}\, d\upsilon.\end{aligned}$$

Since in this case we have $|I| \le n-2$, applying Lemma 6.3.2 and the bounds (6.2.28), and taking the supremum over $\sigma$ in $[0,kT]$, we see that there are constants $C_1$ and $C_2$, and a positive integer $N$, such that

$$\begin{aligned}\|\overline{E}_{n-1,m+1}[\overline{\gamma}_{m+1}-\overline{\gamma}_m]\|_{L^\infty([0,kT])} &\le C_1\underline{T}'^{-N}\nu T\|\overline{E}_{n-1,m+1}[\overline{\gamma}_{m+1}-\overline{\gamma}_m](\upsilon)\|_{L^\infty([0,kT])}\\ &\quad + C_2\nu\underline{T}'^{-N}T\|E_{n-1}[h_{m+1}-h_m+\eta]\|^{1/2}_{L^\infty([0,kT])}\|\overline{E}_{n-1,m+1}[\overline{\gamma}_{m+1}-\overline{\gamma}_m]\|^{1/2}_{L^\infty([0,kT])}\end{aligned}$$

(here $E_{n-1}[h_{m+1}-h_m+\eta]$ gives simply the norms of the differences $\overline{\delta\ell}_{m+1}-\overline{\delta\ell}_m$, $\overline{b}_{m+1}-\overline{b}_m$, $\overline{c}_{m+1}-\overline{c}_m$); cancelling a factor of $\|\overline{E}_{n-1,m+1}[\overline{\gamma}_{m+1}-\overline{\gamma}_m]\|^{1/2}_{L^\infty([0,kT])}$ then allows us to conclude that there is a constant $C$ such that for

$$\nu < C\frac{T'^N}{T}$$

we will have

$$\overline{E}_{n-1,m+1}[\overline{\gamma}_{m+1}-\overline{\gamma}_m](\sigma) \le \frac{1}{4}\|E_{n-1}[h_{m+1}-h_m+\eta]\|_{L^\infty([0,kT])}$$

for $\sigma \in [0,kT]$.

We may do something similar to estimate the metric coefficients. Let us define

$$\overline{\Omega}_m = (\overline{\delta\ell}_m, \overline{b}_m, \overline{c}_m, \partial_{\overline{x}}\overline{\delta\ell}_m, \partial_{\overline{v}}\overline{\delta\ell}_m, \partial_{\overline{x}}\overline{b}_m), \qquad \Gamma_m = (\partial_{\overline{s}}\overline{\gamma}_m, \partial_{\overline{x}}\overline{\gamma}_m, \partial_{\overline{v}}\overline{\gamma}_m).$$

Then there are functions $F$ and $G$, $G$ quadratic, which themselves do not depend on $m$, such that the differentiated Riccati equations (6.2.12 – 6.2.14) can be written as (assuming $\iota \ge 1/2$, as usual)

$$\partial^2_{\overline{s}}\overline{\Omega}_m = \frac{1}{k}F(\overline{\Omega}_m, \Gamma_{m-1}) + G(\Gamma_{m-1}).$$

Since each component of $\overline{\Omega}_m$ and $\Gamma_m$ is bounded in $L^\infty$ by $\nu$, we may apply Lemma 4.2.3 and Corollary 6.3.4 to see that there are constants $C_1'$ and $C_2'$ and a positive integer $N'$ such that

$$\|\partial^2_{\overline{s}}\left(\overline{\Omega}_{m+1}-\overline{\Omega}_m\right)\|_{H^{n-2}(A_\sigma)} \le C_1'\nu\frac{1}{k}E_{n-1}[h_{m+1}-h_m+\eta](\sigma)^{1/2} + C_2'\nu\underline{T}'^{-N'}\overline{E}_{n-1,m}[\overline{\gamma}_m-\overline{\gamma}_{m-1}](\sigma)^{1/2},$$

where we use an $L^1$ norm on vectors like $\overline{\Omega}_m$, and we have used the wave equation to solve for $\partial^2_{\overline{x}}\overline{\gamma}_m$ and $\partial^2_{\overline{x}}\overline{\gamma}_{m-1}$ as in the proof of Lemma 6.3.4 (this accounts for the presence of the $\underline{T}'^{-N'}$). Applying Lemma 6.4.4 in the same way we did in the proof of Lemma 6.5.2, we obtain from this

$$\begin{aligned}&E_{n-1,1}[h_{m+1}-h_m+\eta](\sigma)\\ &\quad= \sum_{|I|\le n-2}\sum_{\overline{\omega}\in\overline{\Omega}}\left\|\partial^I\partial_{\overline{s}}(\overline{\omega}_{m+1}-\overline{\omega}_m)\right\|^2_{L^2(A_\sigma)} = \sum_{|I|\le n-2}\sum_{\overline{\omega}\le\overline{\Omega}}\int_{A_\sigma}\left|\partial^I\partial_{\overline{s}}(\overline{\omega}_{m+1}-\overline{\omega}_m)\right|^2\,d\xi\,d\zeta\\ &\quad\le \sqrt{2}\sum_{|I|\le n-2}\sum_{\overline{\omega}\in\overline{\Omega}}\Bigg[\int_{\Sigma'_\sigma}\left|\partial^I\partial_{\overline{s}}(\overline{\omega}_{m+1}-\overline{\omega}_m)\right|^2\,d\overline{x}\,d\overline{v}\\ &\qquad\qquad + \int_{\underline{\upsilon}(\sigma)}^{\sigma}\int_{A_\upsilon}2\left|\partial^I\partial_{\overline{s}}(\overline{\omega}_{m+1}-\overline{\omega}_m)\right|\left|\partial^I\partial^2_{\overline{s}}(\overline{\omega}_{m+1}-\overline{\omega}_m)\right|\,d\xi\,d\zeta\,d\upsilon\Bigg]\\ &\quad\le 2\sqrt{2}\sum_{|I|\le n-2}\sum_{\overline{\omega}\in\overline{\Omega}}\int_{\underline{\upsilon}(\sigma)}^{\sigma}\left\|\partial^I\partial_{\overline{s}}(\overline{\omega}_{m+1}-\overline{\omega}_m)\right\|_{L^2(A_\upsilon)}\left\|\partial^I\partial^2_{\overline{s}}(\overline{\omega}_{m+1}-\overline{\omega}_m)\right\|_{L^2(A_\upsilon)}\,d\upsilon\end{aligned}$$

$$
\begin{aligned}
&\leq 2\sqrt{2}\int_{\underline{\upsilon}(\sigma)}^{\sigma} E_{n-1,1}[h_{m+1}-h_m+\eta](\upsilon)^{1/2}\left\|\partial_{\overline{s}}^2(\overline{\Omega}_{m+1}-\overline{\Omega}_m)\right\|_{H^{n-2}(A_\upsilon)}\,d\upsilon \\
&\leq 2\sqrt{2}\int_{\underline{\upsilon}(\sigma)}^{\sigma} C_1'\nu\frac{1}{k}E_{n-1}[h_{m+1}-h_m+\eta](\upsilon) \\
&\qquad\qquad + 2C_2'\nu\underline{T}'^{-N'}\overline{E}_{n-1,m}[\overline{\gamma}_m-\overline{\gamma}_{m-1}](\upsilon)^{1/2}E_{n-1,1}[h_{m+1}-h_m+\eta](\upsilon)^{1/2}\,d\upsilon,
\end{aligned}
$$

$$
\begin{aligned}
&E_{n-1,0}[h_{m+1}-h_m+\eta](\sigma) \\
&\quad = \sum_{|I|\leq n-2}\sum_{\overline{\omega}\in\overline{\Omega}}\left\|\partial^I(\overline{\omega}_{m+1}-\overline{\omega}_m)\right\|_{L^2(A_\sigma)}^2 \leq \sum_{|I|\leq n-2}\sum_{\overline{\omega}\in\overline{\Omega}} 2T'\int_{\underline{\upsilon}(\sigma)}^{\sigma}\int_{A_\upsilon}\left|\partial^I\partial_{\overline{s}}(\overline{\omega}_{m+1}-\overline{\omega}_m)\right|^2\,d\xi\,d\zeta\,d\upsilon \\
&\quad \leq 2T'\int_{\underline{\upsilon}(\sigma)}^{\sigma}E_{n-1,1}[h_{m+1}-h_m+\eta](\upsilon)\,d\upsilon \leq 2T'\int_{\underline{\upsilon}(\sigma)}^{\sigma}E_{n-1}[h_{m+1}-h_m+\eta](\upsilon)\,d\upsilon,
\end{aligned}
$$

so taking $L^\infty([\underline{\upsilon}(\sigma),\sigma])$ norms, we find that

$$
\begin{aligned}
&\|E_{n-1,0}[h_{m+1}-h_m+\eta]\|_{L^\infty([\underline{\upsilon}(\sigma),\sigma])} \\
&\quad \leq 2\sqrt{2}T'^2\|E_{n-1,1}[h_{m+1}-h_m+\eta]\|_{L^\infty([\underline{\upsilon}(\sigma),\sigma])} \\
&\|E_{n-1,1}[h_{m+1}-h_m+\eta]\|_{L^\infty([\underline{\upsilon}(\sigma),\sigma])} \\
&\quad \leq 4T'C_1'\nu\frac{1}{k}\|E_{n-1}[h_{m+1}-h_m+\eta]\|_{L^\infty([\underline{\upsilon}(\sigma),\sigma])} \\
&\qquad\quad + 2\sqrt{2}C_2'\nu\underline{T}'^{-N'+1}\|\overline{E}_{n-1,m}[\overline{\gamma}_m-\overline{\gamma}_{m-1}]\|_{L^\infty([\underline{\upsilon}(\sigma),\sigma])}^{1/2}\|E_{n-1,1}[h_{m+1}-h_m+\eta]\|_{L^\infty([\underline{\upsilon}(\sigma),\sigma])}^{1/2} \\
&\quad \leq 8\sqrt{2}T'(1+T'^2)C_1'\nu\frac{1}{k}\|E_{n-1,1}[h_{m+1}-h_m+\eta]\|_{L^\infty([\underline{\upsilon}(\sigma),\sigma])} \\
&\qquad\quad + 2\sqrt{2}C_2'\nu\underline{T}'^{-N'+1}\|\overline{E}_{n-1,m}[\overline{\gamma}_m-\overline{\gamma}_{m-1}]\|_{L^\infty([\underline{\upsilon}(\sigma),\sigma])}^{1/2}\|E_{n-1,1}[h_{m+1}-h_m+\eta]\|_{L^\infty([\underline{\upsilon}(\sigma),\sigma])}^{1/2},
\end{aligned}
$$

whence dividing by $\|E_{n-1,1}[h_{m+1}-h_m+\eta]\|_{L^\infty([\underline{\upsilon}(\sigma),\sigma])}^{1/2}$ as before, and using for the first time the assumption $T'\geq 1$, we find that there is a constant $C$ and a positive integer $M$ such that if

$$
\nu \leq \frac{1}{C}T'^{-M}
$$

then we will have

$$
\|E_{n-1}[h_{m+1}-h_m+\eta]\|_{L^\infty([0,kT])} \leq \frac{1}{4}\|\overline{E}_{n-1,m}[\overline{\gamma}_m-\overline{\gamma}_{m-1}]\|_{L^\infty([0,kT])}.
$$

Since the energy norms here are equivalent to Sobolev norms, with constants no greater than 3, this shows that there is a constant $C>0$ and a positive integer $N'$, independent of $T$ and $T'$, such that if $T'\geq 1$ and

$$
\nu \leq C\min\{T'^{-N'},\frac{T'^{N'}}{T}\},
$$

then the sequence $(\overline{\delta\ell}_m,\overline{b}_m,\overline{c}_m,\overline{\gamma}_m)$ must be Cauchy in $H^{n-1}$, and hence must converge, on the set

$$
\Gamma = \{(\overline{s},\overline{x},\overline{v})\in\mathbf{R}^3 \mid \overline{s}\in[0,T'],\ \overline{v}\in[0,kT]\},
$$

as claimed. QED.

### 6.7. Norms of initial data

In this short section we produce a few results concerning norms of initial data corresponding to the solutions constructed in Theorem 6.6.1, in particular for comparing our results to those in the literature, see Proposition 0.4.1. Let $(\overline{a}, \overline{b}, \overline{c}, \overline{\gamma})$ be a solution constructed as per Theorem 6.6.1. Then we recall (see (0.2.1), (0.2.8)) that the metric given in the $sxvy$ coordinate system by

$$g_{ij} = \begin{pmatrix} 0 & 0 & -e^{-2k^{-1/2}\overline{\gamma}} & 0 \\ 0 & (1+k^{-1}\overline{\delta\ell})^2 e^{-2k^{-1/2}\overline{\gamma}} & k^{-1/2}\overline{b}e^{-2k^{-1/2}\overline{\gamma}} & 0 \\ -e^{-2k^{-1/2}\overline{\gamma}} & k^{-1/2}\overline{b}e^{-2k^{-1/2}\overline{\gamma}} & \overline{c}e^{-2k^{-1/2}\overline{\gamma}} & 0 \\ 0 & 0 & 0 & e^{2k^{-1/2}\overline{\gamma}} \end{pmatrix} \tag{6.7.1}$$

will satisfy the Einstein vacuum equations. Now the function $\overline{\gamma}$ on the hypersurface $\Sigma_0$ is supported either on the square $[0,1]\times[0,1]$ or the square $[0,1]\times[kT\sqrt{2}-1, kT\sqrt{2}]$, which in terms of the coordinates $sxvy$ is the set

$$\{(0,x,v,y)\,|\, x\in[0,k^{-1/2}],\ v\in[0,k^{-1}],\ y\in\mathbf{R}\}$$

or

$$\{(0,x,v,y)\,|\, x\in[0,k^{-1/2}],\ v\in[T\sqrt{2}-k^{-1}, T\sqrt{2}],\ y\in\mathbf{R}\},$$

respectively.

We first compute the second fundamental form of the hypersurface $s=0$ for the metric $g$. In coordinates, this is

$$\chi_{ij} = \nabla_i(\partial_v)_j = g_{jk}\nabla_i(\partial_v)^k = \Gamma^k_{i2}g_{jk} = \frac{1}{2}\left(g_{ij,2} + g_{2j,i} - g_{i2,j}\right).$$

We want the projection of this on the $xy$ plane, and thus take $i,j\in\{1,3\}$. Now $g_{ij,k}=0$ if any of $i$, $j$, or $k$ equals 3, unless $i=j=3$, $k\neq 3$; thus

$$\begin{aligned} \chi_{11} &= \frac{1}{2}\left(g_{11,2}+g_{21,1}-g_{12,1}\right) = \frac{1}{2}\partial_v\left(e^{-2\gamma(s,x,v)}a(s,x,v)\right) = e^{-2\gamma}\left(\frac{1}{2}\partial_v a - a\partial_v\gamma\right) \\ \chi_{13} &= \chi_{31} = 0 \\ \chi_{33} &= \frac{1}{2}\partial_v g_{33} = \partial_v\gamma e^{2\gamma}. \end{aligned}$$

With respect to the orthonormal frame $X = e^{\gamma}a^{-1/2}\partial_x$, $Y = e^{-\gamma}\partial_y$, $\chi$ has the matrix representation

$$\chi = \begin{pmatrix} \frac{1}{a}\left(\frac{1}{2}\partial_v a - a\partial_v\gamma\right) & 0 \\ 0 & \partial_v\gamma \end{pmatrix},$$

which has trace

$$\operatorname{tr}\chi = \frac{\partial_v a}{2a}.$$

Thus, letting 1 denote the $2\times 2$ identity matrix, the traceless part of $\chi$ is

$$\hat{\chi} = \chi - \frac{1}{2}\operatorname{tr}\chi 1 = \begin{pmatrix} -\partial_v\gamma + \frac{\partial_v a}{4a} & 0 \\ 0 & \partial_v\gamma - \frac{\partial_v a}{4a} \end{pmatrix}. \tag{6.7.2}$$

We note that since $\partial_v\gamma = 0$ and $a=1$ outside the support of $\gamma$, we have also $\hat{\chi}=0$ there. In terms of $k$, $(\partial_v a)/(4a)$ is of order 1 on its support, while $\partial_v\gamma$ is of order $k^{1-\iota}$ on its support; for us, $\iota = 1/2$, and thus $\hat{\chi}$ is of order $k^{1/2}$ in $L^\infty$. Off the support of $\gamma|_{\{s=0\}}$, $\hat{\chi}=0$.

We have also the following result on $\gamma$. Recall (see (0.2.22)) that ${}^0\Sigma_0$ is the surface corresponding to $\Sigma_0$ in the unscaled picture.

6.7.1. PROPOSITION. Suppose that $\gamma|_{{}^0\Sigma_0}$ is specified as in (5.4.2). Then there are constants $C_1$, $C_2$, depending on $\ell$, $m$, and $\varpi_0$ but not on $k$, such that

$$\|\partial_v^\ell \partial_x^m \gamma\|_{L^\infty({}^0\Sigma_0)} = C_1 k^{\ell+m/2-\iota}, \qquad \|\partial_v^\ell \partial_x^m \gamma\|_{L^2({}^0\Sigma_0)} = C_2 k^{\ell+m/2-\iota-3/4}.$$

Proof. By (5.4.2) and (3.3.3), it clearly suffices to show these results for $k^{-\iota}\varpi_0(k^{1/2}x, kv)$, which is easy: since (using $\partial_1$ and $\partial_2$ to denote differentiation with respect to the first and second variables, respectively, of $\varpi_0$) $\partial_v^\ell \partial_x^m \varpi_0(k^{1/2}x, kv) = k^{\ell+m/2}(\partial_2 \varpi_0)(k^{1/2}x, kv)$, we have

$$\|\partial_v^\ell \partial_x^m \varpi_0(k^{1/2}x, kv)\|_{L^\infty({}^0\Sigma_0)} = k^{\ell+m/2}\|\partial_2^\ell \partial_1^m \varpi_0\|_{L^\infty(\Sigma_0)} = C_1 k^{\ell+m/2},$$

taking $C_1 = \|\partial_2^\ell \partial_1^m \varpi_0\|_{L^\infty(\Sigma_0)}$. This is of the desired form. Similarly,

$$\begin{aligned}\|\partial_v^\ell \partial_x^m \varpi_0(k^{1/2}x, kv)\|^2_{L^2({}^0\Sigma_0)} &= k^{2\ell+m} \int_{-\infty}^{+\infty} \int_0^{T\sqrt{2}} \left|(\partial_2^\ell \partial_1^m \varpi_0)(k^{1/2}x, kv)\right|^2 \, dv\, dx \\ &= k^{2\ell+m-3/2} \int_{\Sigma_0} \left|(\partial_2^\ell \partial_1^m \varpi_0)(\overline{x}, \overline{v})\right|^2 \, d\overline{x}\, d\overline{v} = C_2^2 k^{2\ell+m-3/2},\end{aligned}$$

taking $C_2 = \left\|\partial_2^\ell \partial_1^m \varpi_0\right\|_{L^2(\Sigma_0)}$. Again, this is of the desired form, establishing the result. QED.

In particular, this shows that the $L^2$ norm of $\partial_v^2 \gamma$ on ${}^0\Sigma_0$ is of size $k^{3/4}$, as previously claimed.

As noted in Section 0.4, because the $L^2$ norm used here is in $2+1$ dimensions, the $L^\infty$ norm is the more appropriate one to use for comparisons with the results in $3+1$ dimensions such as Christodoulou [3], Klainerman and Rodnianski [6], and Luk and Rodnianski [9].

### 7.1. Introduction

In this, final, chapter, we apply the results of Chapters 5 and 6 to give a solution $(a, b, c, \gamma)$ to (1.2.9 – 1.2.11), (1.3.1), depending, in addition to $k$, on a parameter $r$, and satisfying the following property: there is a function $\overline{\gamma}_{GB}$ (the *formal Gaussian beam*) which is supported on a neighbourhood of size 1/4 around a null geodesic $\overline{x} = \overline{x}_0$, $\overline{v} = \overline{v}_0$, and a constant $C_g > 0$, independent of $r$, such that for all $r$ sufficiently large,

$$\frac{\|\overline{\gamma} - \overline{\gamma}_{GB}\|_{H^1(\Gamma)}}{\|\overline{\gamma}_{GB}\|_{H^1(\Gamma)}} \leq C_g r^{-1/2}.$$

We shall do this by applying Gaussian beam techniques. It is not our intention to provide an introduction to or general treatment of Gaussian beams, for which we refer the reader elsewhere, e.g., [13]. We only note one peculiarity in our current situation. A Gaussian beam depends on the wave operator for which it is derived. In our case, we wish the Gaussian beam to be part of the solution to (1.2.9 – 1.2.11), (1.3.1), which determines the metric and hence the wave operator; in other words, the wave operator depends on the Gaussian beam, so we cannot, as in the usual treatment, simply determine the Gaussian beam for a given, fixed metric. We shall see, however, that we *can* specify Gaussian-beam initial data for (1.2.9 – 1.2.11), (1.3.1); and assuming that it can be made to satisfy the conditions in Theorem 6.6.1 – which is the case – the resulting function $\gamma$ will, *a posteriori*, be a Gaussian beam for the resulting metric. Since any solution with initial data satisfying the conditions in Theorem 6.6.1 will satisfy the bounds (6.2.28), we may assume that those bounds (and, hence, all of the bounds derived in Section 6.3) hold for the metric components $\overline{a}$, $\overline{b}$, $\overline{c}$ with respect to which we derive the Gaussian beam.

We anticipate that the results obtained here can be refined in various directions.

### 7.2. Construction of the formal Gaussian beam

Let $r > 0$; in the following, we assume all quantities to be independent of $r$ unless otherwise stated. We shall work with initial data supported near $\overline{v} = kT\sqrt{2}$, i.e., at the upper boundary of $\Sigma_0$ rather than the lower one. Thus, let $\overline{x}_0 \in (1/4, 3/4)$, $\overline{v}_0 \in (kT\sqrt{2} - 3/4, kT\sqrt{2} - 1/4)$. Let $\phi$ be any $C^\infty$ function on $\mathbf{R}^2$ which is supported on the disk $B_{1/4}$ of radius 1/4 centred at the origin, and satisfies $\phi(x, y) = 1$ if $x^2 + y^2 \leq 1/64$. For convenience, define the function $\rho : \Gamma \to \mathbf{R}$ by

$$\rho(\overline{s}, \overline{x}, \overline{v}) = \sqrt{(\overline{x} - \overline{x}_0)^2 + (\overline{v} - \overline{v}_0)^2}.$$

For $\overline{s} \in [0, T']$, define

$$\Sigma_{\overline{s}} = (\{\overline{s}\} \times \mathbf{R}^2) \cap \Gamma,$$

i.e., $\Sigma_{\overline{s}}$ are the planes in $\Gamma$ of constant $\overline{s}$. Since $\rho$ does not depend on $\overline{s}$, we occasionally drop the $\overline{s}$ variable and simply write $\rho(\overline{x}, \overline{v})$ for notational simplicity.

When taking square roots of complex numbers, we use the branch with a cut along the negative imaginary axis and argument in $(-\pi, \pi)$.

We have the following results.

7.2.1. LEMMA. Let $f : B_{1/4} \to \mathbf{C}$ satisfy $|f(\overline{x},\overline{v})| \le C\rho^q$ for some constant $C$ and some $q > 0$. Then for any $r > 0$

$$\left\| e^{-r\rho^2} f(\overline{x},\overline{v}) \right\|_{L^2(B_{1/4})} \le \left[ \frac{\pi\Gamma(q)}{2^{2q+1}} \right]^{1/2} C r^{-\frac{1}{2}(q+1)},$$

where

$$\Gamma(q) = \int_0^\infty u^q e^{-u}\, du$$

is the gamma function.

Proof. This is entirely straightforward:

$$\int_{B_{1/4}} \left| e^{-r\rho^2} f(\overline{x},\overline{v}) \right|^2 d\overline{x}\, d\overline{v} \le 2\pi C^2 \int_0^\infty \rho^{2q} e^{-2r\rho^2} \rho\, d\rho = \frac{2\pi C^2}{4^{q+1}} r^{-(q+1)} \int_0^\infty u^q e^{-u}\, du$$
$$= \left[ \frac{\pi}{2^{2q+1}} \Gamma(q) \right] C^2 r^{-(q+1)},$$

from which the result follows immediately. QED.

7.2.2. LEMMA. Let $\theta_0$, $B_1$, $B_2$, $D$, $r \in \mathbf{R}$, $B_2$, $D$, $r > 0$. Then

$$\left\| \sin\left( r\theta_0\overline{v} + \frac{1}{2} r B_1 \overline{x}^2 \right) e^{-\frac{1}{2}r(B_2\overline{x}^2 + D\overline{v}^2)} \right\|^2_{L^2(\mathbf{R}^2)}$$
$$= \frac{\pi}{2(B_2D)^{1/2} r} \left\{ 1 - 2^{1/2} \left[ \left(1 + i\frac{B_1}{B_2}\right)^{-1/2} + \left(1 - i\frac{B_1}{B_2}\right)^{-1/2} \right] e^{-r\theta_0^2/2D} \right\},$$

$$\left\| \cos\left( r\theta_0\overline{v} + \frac{1}{2} r B_1 \overline{x}^2 \right) e^{-\frac{1}{2}r(B_2\overline{x}^2 + D\overline{v}^2)} \right\|^2_{L^2(\mathbf{R}^2)}$$
$$= \frac{\pi}{2(B_2D)^{1/2} r} \left\{ 1 + 2^{1/2} \left[ \left(1 + i\frac{B_1}{B_2}\right)^{-1/2} + \left(1 - i\frac{B_1}{B_2}\right)^{-1/2} \right] e^{-r\theta_0^2/2D} \right\}.$$

Proof. We recall the Gaussian integral

$$\int_{-\infty}^{+\infty} e^{-au^2}\, du = \left(\frac{\pi}{a}\right)^{1/2}.$$

This is true for all $a > 0$, and we see that by analytic continuation it can be continued to all complex $a$ with $\Re a > 0$. Similarly, for $A_1$, $A_2 \in \mathbf{R}$, $A_2 > 0$,

$$\int_{-\infty}^{+\infty} e^{iA_1u - \frac{1}{2}A_2u^2}\, du = \int_{-\infty}^{+\infty} e^{-\frac{1}{2}A_2\left(u - i\frac{A_1}{A_2}\right)^2 - \frac{A_1^2}{2A_2}}\, du = e^{-A_1^2/(2A_2)} \left(\frac{2\pi}{A_2}\right)^{1/2}.$$

Thus we may write

$$\left\| \sin\left( r\theta_0\overline{v} + \frac{1}{2} r B_1 \overline{x}^2 \right) e^{-\frac{1}{2}r(B_2\overline{x}^2 + D\overline{v}^2)} \right\|_{L^2(\mathbf{R}^2)}$$
$$= \frac{1}{2} \int_{\mathbf{R}^2} \left[ 1 - \cos(2r\theta_0\overline{v} + rB_1\overline{x}^2) \right] e^{-r(B_2\overline{x}^2 + D\overline{v}^2)}\, d\overline{x}\, d\overline{v}$$
$$= \frac{1}{2} \left\{ \frac{\pi}{(B_2D)^{1/2} r} - \int_{\mathbf{R}^2} \left[ \cos 2r\theta_0\overline{v}\, \cos rB_1\overline{x}^2 - \sin 2r\theta_0\overline{v}\, \sin rB_1\overline{x}^2 \right] e^{-r(B_2\overline{x}^2 + D\overline{v}^2)}\, d\overline{x}\, d\overline{v} \right\}$$
$$= \frac{\pi}{2(B_2D)^{1/2} r} - \left(\frac{\pi}{2rD}\right)^{1/2} e^{-r\theta_0^2/(2D)} \left(\frac{\pi}{r}\right)^{1/2} \cdot \frac{1}{2} \left[ (B_2 + iB_1)^{-1/2} + (B_2 - iB_1)^{-1/2} \right]$$
$$= \frac{\pi}{2(B_2D)^{1/2} r} \left\{ 1 - 2^{1/2} \left[ \left(1 + i\frac{B_1}{B_2}\right)^{-1/2} + \left(1 - i\frac{B_1}{B_2}\right)^{-1/2} \right] e^{-r\theta_0^2/2D} \right\},$$

as claimed. The other integral follows in exactly the same way but using the formula $\cos^2 u = \frac{1}{2}(1 + \cos 2u)$.

QED.

7.2.1. COROLLARY. Let $L \in \mathbf{R}$, $L > 0$. Under the conditions of Lemma 7.2.2, we have (letting $B_2 \wedge D$ denote the minimum of $B_2$ and $D$)

$$\left\|\sin\left(r\theta_0\overline{v} + \frac{1}{2}rB_1\overline{x}^2\right)e^{-\frac{1}{2}r(B_2\overline{x}^2 + D\overline{v}^2)}\right\|^2_{L^2(B_L(0))}$$
$$\geq \frac{\pi}{2(B_2D)^{1/2}r}\left\{1 - 2^{1/2}\left[\left(1 + i\frac{B_1}{B_2}\right)^{-1/2} + \left(1 - i\frac{B_1}{B_2}\right)^{-1/2}\right]e^{-r\theta_0^2/2D}\right\} - \frac{\pi}{r(B_2 \wedge D)}e^{-r(B_2\wedge D)L^2},$$

$$\left\|\cos\left(r\theta_0\overline{v} + \frac{1}{2}rB_1\overline{x}^2\right)e^{-\frac{1}{2}r(B_2\overline{x}^2 + D\overline{v}^2)}\right\|^2_{L^2(B_L(0))}$$
$$\geq \frac{\pi}{2(B_2D)^{1/2}r}\left\{1 + 2^{1/2}\left[\left(1 + i\frac{B_1}{B_2}\right)^{-1/2} + \left(1 - i\frac{B_1}{B_2}\right)^{-1/2}\right]e^{-r\theta_0^2/2D}\right\} - \frac{\pi}{r(B_2 \wedge D)}e^{-r(B_2\wedge D)L^2},$$

where as usual $B_L(0)$ denotes the disk of radius $L$ centred at the origin in the $\overline{x}\,\overline{v}$ plane.

Proof. We note that, transforming to polar coordinates and using $P$ as the radial coordinate,

$$\int_{\mathbf{R}^2\setminus B_L(0)}\left|\sin\left(r\theta_0\overline{v} + \frac{1}{2}rB_1\overline{x}^2\right)e^{-\frac{1}{2}r(B_2\overline{x}^2 + D\overline{v}^2)}\right|^2 d\overline{x}\,d\overline{v} \leq \int_{\mathbf{R}^2\setminus B_L(0)} e^{-r(B_2\wedge D)(\overline{x}^2 + \overline{v}^2)}\,d\overline{x}\,d\overline{v}$$
$$= 2\pi\int_L^\infty Pe^{-r(B_2\wedge D)P^2}\,dP = \frac{\pi}{r(B_2\wedge D)}e^{-r(B_2\wedge D)L^2},$$

from which the result readily follows. QED.

7.2.2. COROLLARY. Under the conditions of Corollary 7.2.1, for every $\epsilon > 0$ there is an $R_0 > 0$, depending on $\epsilon$ and the parameters of the problem, such that for $r > R_0$ we have

$$\left\|\sin\left(r\theta_0\overline{v} + \frac{1}{2}rB_1\overline{x}^2\right)e^{-\frac{1}{2}r(B_2\overline{x}^2 + D\overline{v}^2)}\right\|^2_{L^2(B_L(0))} \geq \frac{\pi}{2(B_2D)^{1/2}r}(1 - \epsilon)$$
$$\left\|\cos\left(r\theta_0\overline{v} + \frac{1}{2}rB_1\overline{x}^2\right)e^{-\frac{1}{2}r(B_2\overline{x}^2 + D\overline{v}^2)}\right\|^2_{L^2(B_L(0))} \geq \frac{\pi}{2(B_2D)^{1/2}r}(1 + \epsilon).$$

Proof. This is clear because of the exponential decay of the error terms in Corollary 7.2.1. QED.

7.2.1. PROPOSITION. Under the conditions of Corollary 7.2.1, we have

$$\left\|\sin\left(r\theta_0\overline{v} + \frac{1}{2}rB_1\overline{x}^2\right)e^{-\frac{1}{2}r(B_2\overline{x}^2 + D\overline{v}^2)}\right\|^2_{L^2(B_L(0))} \leq \frac{2\pi}{(B_2D)^{1/2}r}$$
$$\left\|\cos\left(r\theta_0\overline{v} + \frac{1}{2}rB_1\overline{x}^2\right)e^{-\frac{1}{2}r(B_2\overline{x}^2 + D\overline{v}^2)}\right\|^2_{L^2(B_L(0)} \leq \frac{2\pi}{(B_2D)^{1/2}r}.$$

Proof. This follows from the standard Gaussian integral. QED.

From the foregoing we obtain the following result. We take the branch of the square root function corresponding to the argument interval $(-\pi, \pi)$, i.e., with a branch cut along the negative real axis.

7.2.3. LEMMA. Let $\overline{h}$ be any Lorentzian metric on $\Gamma$ of the form (6.2.2). Let $A_1$, $B$, $D$, $\theta_0 \in \mathbf{R}$ satisfy* $1/2 < D < 2$, $1 < B < 2$, $0 < A_1 < 1$, $\theta_0 > 0$. Let

$$\theta_1(\overline{s}) = \frac{iB - \frac{1}{\theta_0}\int_0^{\overline{s}} \frac{1}{\overline{a}(\sigma,\overline{x}_0,\overline{v}_0)}\,d\sigma}{B^2 + \left[\frac{1}{\theta_0}\int_0^{\overline{s}} \frac{1}{\overline{a}(\sigma,\overline{x}_0,\overline{v}_0)}\,d\sigma\right]^2}, \tag{7.2.1}$$

$$A(\overline{s}) = A_1\theta_1^{1/2}\frac{\overline{a}^{1/4}(\overline{s},\overline{x}_0,\overline{v}_0)}{\overline{a}^{1/4}(0,\overline{x}_0,\overline{v}_0)}. \tag{7.2.2}$$

Then there is a constant $C_{GB,1} > 1$ depending on $B$ and $T'$ and constants $C_{GB,2}$, $C_{GB,3} > 0$, depending on $A_1$, $B$, $D$, $\theta_0$, and $T'$, $C_{GB,1}$, $C_{GB,2}$, $C_{GB,3}$ independent of $r$, such that if $\theta_0 > C_{GB,1}$ the function

$$\overline{\gamma}_{GB} = r^{-1/2}\Re\left\{\phi(\overline{x}-\overline{x}_0,\overline{v}-\overline{v}_0)A(\overline{s})\exp\left[ir\left[\theta_0\overline{v} + \frac{1}{2}\left(\theta_1(\overline{s})(\overline{x}-\overline{x}_0)^2 + iD(\overline{v}-\overline{v}_0)^2\right)\right]\right]\right\} \tag{7.2.3}$$

satisfies

$$\|\overline{\gamma}_{GB}\|_{H^1(\Sigma_{\overline{s}})} \geq C_{GB,2}, \qquad \left\|\Box_{\overline{h}}\overline{\gamma}_{GB}\right\|_{L^2(\Sigma_{\overline{s}})} < C_{GB,3}r^{-1/2}. \tag{7.2.4}$$

Proof. In the following, we use letters $C$, $C'$, etc., to denote constants whose values may change from line to line. We first note that, since (see Corollary 6.3.1) $\overline{a}^{-1} \leq 4/3$ on $\Gamma$, we have

$$\int_0^{\overline{s}} \frac{1}{\overline{a}(\sigma,\overline{x}_0,\overline{v}_0)}\,d\sigma \leq 4T'/3;$$

thus if

$$\theta_0 > \frac{4T'}{3(B - B^2/2)}$$

(note that $1 < B < 2$ implies $B - B^2/2 > 0$) then

$$\Im\theta_1 > 1/2. \tag{7.2.5}$$

We take

$$C_{GB,1} = \max\left\{\frac{4T'}{3(B-B^2/2)}, 1\right\},$$

so that for $\theta_0 > C_{GB,1}$ the bound (7.2.5) holds. We may also put upper and lower bounds on the modulus of $\theta_1$, as follows. We note first that $\theta_1$ satisfies the differential equation

$$\partial_{\overline{s}}\theta_1 = \frac{1}{\theta_0\overline{a}}\theta_1^2; \tag{7.2.6}$$

thus also

$$\partial_{\overline{s}}^2\theta_1 = \frac{2\theta_1^3}{\overline{a}^2\theta_0^2}. \tag{7.2.7}$$

* These bounds are, for the most part, a matter of convenience in working out the proof below, and simply requiring the relevant values to be nonzero would often be sufficient. For our purposes it would actually suffice to take a particular choice of the parameters, say $D = 1$, $B = 3/2$, $A_1 = 1/2$; but we carry them along to show that there is quite a bit of flexibility in the final answer.

Since $\overline{a}^{-1} \le 4/3$ on $\Gamma$, we have

$$B \le \left| -iB - \frac{1}{\theta_0}\int_0^{\overline{s}} \frac{1}{\overline{a}}\, d\sigma \right| \le \left[ B^2 + \frac{16T'^2}{9\theta_0^2} \right]^{1/2} < B\sqrt{2},$$

so (recalling $B > 1$)

$$\frac{1}{B\sqrt{2}} \le |\theta_1| \le 1. \tag{7.2.8}$$

In particular, $\theta_1$ is never zero. Note that (7.2.8) implies also that (applying (7.2.7))

$$|\partial_{\overline{s}}\theta_1| \le \frac{4}{3B^2\theta_0} \le \frac{4}{3B^2}, \qquad |\partial_{\overline{s}}^2\theta_1| \le \frac{32}{9B}. \tag{7.2.9}$$

We may also derive bounds on $A$. It is straightforward to see that $A$ satisfies the differential equation

$$\partial_{\overline{s}} A = A\left( \frac{\partial_{\overline{s}}\theta_1}{2\theta_1} - \frac{\partial_{\overline{s}}\overline{a}}{4\overline{a}} \right). \tag{7.2.10}$$

Then, since $\overline{a} \le 2$ and $\overline{a} \ge 3/4$ by Corollary 6.3.1 and (6.3.1), we will have for all $\overline{s} \in [0, T']$, writing $A_0 = A_1\overline{a}^{-1/4}(0, \overline{x}_0, \overline{v}_0)$,

$$\frac{3}{4} \le \overline{a}^{1/4} \le 2, \qquad \frac{1}{2}A_1 \le A_0 \le \frac{4}{3}A_1, \qquad \frac{3}{8}A_1 \le A_0\overline{a}^{1/4}(\overline{s}, \overline{x}_0, \overline{v}_0) \le \frac{8}{3}A_1. \tag{7.2.11}$$

Applying (7.2.8), (7.2.9), and (7.2.11), as well as Proposition 6.3.1, Proposition 6.3.2, and (again) (6.3.1), we thus have the bounds

$$|A| \le \frac{8}{3}|A_1|\frac{1}{B^{1/2}} \le \frac{8}{3}|A_1|, \qquad |\partial_{\overline{s}}A| \le \frac{8}{3}|A_1|\frac{1}{B^{1/2}}\left(\frac{2}{3B} + \frac{1}{3}\right) \le \frac{8}{3}|A_1|, \qquad |\partial_{\overline{s}}^2 A| \le \frac{40}{3}|A_1|. \tag{7.2.12}$$

Define on $\Gamma$

$$\Theta(\overline{s}, \overline{x}, \overline{v}) = \theta_0\overline{v} + \frac{1}{2}\left(\theta_1(\overline{s})(\overline{x} - \overline{x}_0)^2 + iD(\overline{v} - \overline{v}_0)^2\right)$$

(note that this is the coefficient of $ir$ in the exponential in (7.2.3)). We are now in a position to derive the second inequality in (7.2.4). By Lemma 7.2.1, it suffices to ensure that the quantity multiplying $e^{ir\Theta}$ in $\Box_{\overline{h}}\overline{\gamma}_{GB}$ vanishes to a sufficiently high order at $(\overline{s}, \overline{x}_0, \overline{v}_0)$. This is the key observation underlying the technique of Gaussian beams.

We first note that all derivatives of $\phi(\overline{x} - \overline{x}_0, \overline{v} - \overline{v}_0)$ vanish to arbitrarily high order at $(\overline{x}_0, \overline{v}_0)$. More precisely, let $n$, $N$ be positive integers; then there is a constant $C$, depending on $n$ and $N$, such that for all multiindices $I$ in $\overline{x}$ and $\overline{v}$ with $|I| \le n$,

$$\left\| \partial^I \phi(\overline{x} - \overline{x}_0, \overline{v} - \overline{v}_0) \right\|_{L^\infty(\mathbf{R}^2)} \le C\rho(\overline{x}, \overline{v})^{-N}. \tag{7.2.13}$$

By the product rule, this means that any terms in $\Box_{\overline{h}}\overline{\gamma}_{GB}$ involving derivatives of $\phi$ will satisfy the bound in (7.2.4) for any $\alpha$, and hence can be disregarded. Thus it suffices to bound

$$\|\Box_{\overline{h}}\left(A(\overline{s})e^{ir\Theta(\overline{x},\overline{v})}\right)\|_{L^2(\{\overline{s}\}\times B_{1/4})}. \tag{7.2.14}$$

A straightforward calculation gives

$$\Box_{\overline{h}}\left(A(\overline{s})e^{ir\Theta(\overline{x}-\overline{x}_0,\overline{v}-\overline{v}_0)}\right)=e^{ir\Theta(\overline{x}-\overline{x}_0,\overline{v}-\overline{v}_0)}\left[\Box_{\overline{h}}A+irA\Box_{\overline{h}}\Theta+2irh^{ij}\partial_iA\partial_j\Theta-r^2h^{ij}\partial_i\Theta\partial_j\Theta\right]. \quad (7.2.15)$$

We take these terms in turn, starting with the last one. We have

$$\begin{aligned} h^{ij}\partial_i\Theta\partial_j\Theta &= -2\partial_{\overline{v}}\Theta\partial_{\overline{s}}\Theta+\frac{1}{\overline{a}}\left(\partial_{\overline{x}}\Theta\right)^2+\frac{1}{k}\left(k^{-1}\frac{\overline{b}^2}{\overline{a}}-\overline{c}\right)\left(\partial_{\overline{s}}\Theta\right)^2+\frac{2}{k}\frac{\overline{b}}{\overline{a}}\partial_{\overline{s}}\Theta\partial_{\overline{x}}\Theta \\ &= -\partial_{\overline{s}}\theta_1(\overline{x}-\overline{x}_0)^2(\theta_0+\theta_2(\overline{v}-\overline{v}_0))+\frac{1}{\overline{a}}\theta_1^2(\overline{x}-\overline{x}_0)^2 \\ &\qquad +\frac{1}{k}\left(k^{-1}\frac{\overline{b}^2}{\overline{a}}-\overline{c}\right)\cdot\frac{1}{4}\left(\partial_{\overline{s}}\theta_1\right)^2(\overline{x}-\overline{x}_0)^4-\frac{1}{k}\frac{\overline{b}}{\overline{a}}\theta_1\partial_{\overline{s}}\theta_1(\overline{x}-\overline{x}_0)^3. \quad (7.2.16)\end{aligned}$$

By (7.2.6), the lowest-order (in $\overline{x}-\overline{x}_0$, $\overline{v}-\overline{v}_0$) term in (7.2.16) must vanish. Substituting (7.2.1) back in to equation (7.2.16), we see that there is a constant $C_1>0$ such that on the support of $\phi(\overline{x}-\overline{x}_0,\overline{v}-\overline{v}_0)$

$$\left|h^{ij}\partial_i\Theta\partial_j\Theta\right|\le\frac{4D}{3B}\rho^3+\frac{1}{k}C_1\rho^3. \quad (7.2.17)$$

By Lemma 7.2.1, then, there is a constant $C$, independent of $\overline{s}$, such that

$$\left\|e^{ir\Theta(\overline{x}-\overline{x}_0,\overline{v}-\overline{v}_0)}h^{ij}\partial_i\Theta\partial_j\Theta\right\|_{L^2(\{\overline{s}\}\times B_{1/4})}\le Cr^{-2}.$$

We may similarly treat the other terms in (7.2.15). We first recall the following expression for the wave operator $\Box_{\overline{h}}$, see (3.3.5):

$$\begin{aligned}&\left(-2\partial_{\overline{s}}\partial_{\overline{v}}+\partial_{\overline{x}}^2\right)+\frac{1}{k}\left(2\frac{\overline{b}}{\overline{a}}\partial_{\overline{s}}\partial_{\overline{x}}-\overline{c}\partial_{\overline{s}}^2-\overline{\delta^{-1}a}\partial_{\overline{x}}^2-\left(\partial_{\overline{s}}\overline{c}-\frac{1}{\overline{a}}\partial_{\overline{x}}\overline{b}+\frac{\partial_{\overline{v}}\overline{\delta\ell}}{\overline{\ell}}\right)\partial_{\overline{s}}+\left(\frac{1}{\overline{a}}\partial_{\overline{s}}\overline{b}-\frac{\overline{\ell}\partial_{\overline{x}}\overline{\delta\ell}}{\overline{a}^2}\right)\partial_{\overline{x}}-\frac{\partial_{\overline{s}}\overline{\delta\ell}}{\overline{\ell}}\partial_{\overline{v}}\right)\\ &\qquad+\frac{1}{k^2}\left(\frac{\overline{b}^2}{\overline{a}}\partial_{\overline{s}}^2-\left(\overline{c}\frac{\partial_{\overline{s}}\overline{\delta\ell}}{\overline{\ell}}-2\frac{\overline{b}}{\overline{a}}\partial_{\overline{s}}\overline{b}\right)\partial_{\overline{s}}-\frac{\overline{b}\overline{\ell}\partial_{\overline{s}}\overline{\delta\ell}}{\overline{a}^2}\partial_{\overline{x}}\right)-\frac{1}{k^3}\frac{\overline{b}^2\overline{\ell}\partial_{\overline{s}}\overline{\delta\ell}}{\overline{a}^2}\partial_{\overline{s}}.\end{aligned}$$

From this, it is evident that there are constants $C_i$ such that (letting $\Gamma_{ij}^k$ denote the Christoffel symbol for $\overline{h}$)

$$\left|\Box_{\overline{h}}A\right|\le\left|\frac{1}{k}\left(k^{-1}\frac{\overline{b}^2}{\overline{a}}-\overline{c}\right)\partial_{\overline{s}}^2A-\overline{h}^{ij}\Gamma_{ij}^0\partial_{\overline{s}}A\right|\le\frac{1}{k}\left(C_2|\partial_{\overline{s}}^2A|+C_3|\partial_{\overline{s}}A|\right) \quad (7.2.18)$$

$$\begin{aligned}&\left|2\overline{h}^{ij}\partial_iA\partial_j\Theta+\Box_{\overline{h}}\Theta\right|\\ &\le\Bigg|-2\partial_{\overline{s}}A(\theta_0+\theta_2\overline{v})+2\frac{1}{k}\partial_{\overline{s}}A\left[2\frac{\overline{b}}{\overline{a}}\theta_1\overline{x}+\frac{1}{2k}\left(k^{-1}\frac{\overline{b}^2}{\overline{a}}-\overline{c}\right)\partial_{\overline{s}}\theta_1\overline{x}^2\right]\\ &\qquad+A\Bigg[\frac{1}{a}\theta_1+\frac{1}{k}\left[\left(k^{-1}\frac{\overline{b}^2}{\overline{a}}-\overline{c}\right)\frac{1}{2}\partial_{\overline{s}}^2\theta_1\overline{x}^2+\frac{\overline{b}}{\overline{a}}\partial_{\overline{s}}\theta_1\theta_1\overline{x}^3\right]\\ &\qquad\qquad-\overline{h}^{ij}\Gamma_{ij}^0\frac{1}{2}\partial_{\overline{s}}\theta_1\overline{x}^2-\overline{h}^{ij}\Gamma_{ij}^1\theta_1\overline{x}-\frac{\partial_{\overline{s}}\overline{a}}{2\overline{a}}(\theta_0+\theta_2\overline{v})\Bigg]\Bigg|\\ &\le\left|-2\partial_{\overline{s}}A\theta_0+A\left(\frac{1}{a}\theta_1-\frac{\partial_{\overline{s}}\overline{a}}{2\overline{a}}\theta_0\right)\right|+\left|2\partial_{\overline{s}}A\theta_2\overline{v}+\frac{\partial_{\overline{s}}\overline{a}}{2\overline{a}}\theta_2\overline{v}\right|\end{aligned}$$

$$+\left|2\frac{1}{k}\partial_{\overline{s}}A\left(2\frac{\overline{b}}{\overline{a}}\theta_1\overline{x}+\frac{1}{2k}\left(k^{-1}\frac{\overline{b}^2}{\overline{a}}-\overline{c}\right)\partial_{\overline{s}}\theta_1\overline{x}^2\right)\right.$$
$$\left.+A\left[\frac{1}{k}\left[\left(k^{-1}\frac{\overline{b}^2}{\overline{a}}-\overline{c}\right)\frac{1}{2}\partial_{\overline{s}}^2\theta_1\overline{x}^2+\frac{\overline{b}}{\overline{a}}\partial_{\overline{s}}\theta_1\theta_1\overline{x}^3\right]-\overline{h}^{ij}\Gamma^0_{ij}\frac{1}{2}\partial_{\overline{s}}\theta_1\overline{x}^2-\overline{h}^{ij}\Gamma^1_{ij}\theta_1\overline{x}-\frac{\partial_{\overline{s}}\overline{a}}{2\overline{a}}\theta_2\overline{v}\right]\right|$$
$$\leq\left|-2\partial_{\overline{s}}A\theta_0+A\left(\frac{1}{a}\theta_1-\frac{\partial_{\overline{s}}\overline{a}}{2\overline{a}}\theta_0\right)\right|+\rho D\left(2|\partial_{\overline{s}}A|+C_4\right)+\frac{1}{k}\left\{|A|C_1\frac{1}{B^3}\rho^2+(|\partial_{\overline{s}}A|+1)\left(C_2\frac{1}{B^2}\rho^2+C_3\frac{1}{B}\rho\right)\right\}. \tag{7.2.19}$$

As before, equation (7.2.10) shows that the term of lowest order in $\overline{x}-\overline{x}_0$ and $\overline{v}-\overline{v}_0$ must vanish. From this and the bounds on $A$ derived in (7.2.11) and (7.2.12) we see that there is a new set of constants $C_i$ such that

$$\left|2\overline{h}^{ij}\partial_iA\partial_j\Theta+\Box_{\overline{h}}\Theta\right|\leq\rho D\left(C_4+C_5|A_1|\right)+\frac{1}{k}\left[C_1|A_1|\rho^2+(1+|A_1|)\left(C_2\rho^2+C_3\rho\right)\right].$$

Recalling that $0<A_1<1$, $D<2$, we have (yet another family of) constants $C_i$ such that

$$\left|\Box_{\overline{h}}A\right|\leq\frac{1}{k}C_1|A_1|,\qquad\left|2\overline{h}^{ij}\partial_iA\partial_j\Theta+\Box_{\overline{h}}\Theta\right|\leq\rho D(C_1+C_2|A_1|)+\frac{1}{k}\rho\left(C_3+C_4|A_1|\right).$$

Thus, by Lemma 7.2.1 again, there is a constant $C$ such that

$$\left\|\Box_{\overline{h}}e^{ir\Theta(\overline{x}-\overline{x}_0,\overline{v}-\overline{v}_0)}A\right\|_{L^2(\{\overline{s}\}\times B_{1/4})}\leq C,\quad\left\|e^{ir\Theta(\overline{x}-\overline{x}_0,\overline{v}-\overline{v}_0)}\left(2\overline{h}^{ij}\partial_iA\partial_j\Theta+\Box_{\overline{h}}\Theta\right)\right\|_{L^2(\{\overline{s}\}\times B_{1/4})}\leq Cr^{-1}. \tag{7.2.20}$$

From equations (7.2.15), (7.2.17), and (7.2.20), we obtain finally that there are constants $C_1$, $C_2$ such that

$$\left\|\Box_{\overline{h}}\left(A(\overline{s})e^{ir\Theta(\overline{x}-\overline{x}_0,\overline{v}-\overline{v}_0)}\right)\right\|_{L^2(\{\overline{s}\}\times B_{1/4})}\leq C_1+\frac{1}{k}C_2;$$

as noted above (see our discussion around (7.2.13) and (7.2.14)), there are (potentially slightly different constants) $C_1$ and $C_2$ such that

$$\left\|\Box_{\overline{h}}\overline{\gamma}_{GB}\right\|_{L^2(\Sigma_{\overline{s}})}\leq\left[C_1+\frac{1}{k}C_2\right]r^{-1/2},$$

which establishes the second inequality in (7.2.4) since $k\geq 1$.

To complete the proof we must compute $\|\overline{\gamma}_{GB}\|_{H^1(\Sigma_{\overline{s}})}$, or at least give a lower bound. We note first that by construction $\Re\theta_1$ and $\theta_0$ will have opposite signs for all $\overline{s}\in(0,T']$ (when $\overline{s}=0$, of course, $\theta_1(\overline{s})=i/B$ so $\Re\theta_1(0)=0$); since $\theta_0>0$, we have $\Re\theta_1<0$ and (since $\Im\theta_1>0$) the argument of $\theta_1$ will lie in the interval $(\pi/2,\pi)$. By our choice of branch, then, $\theta_1^{1/2}$ will have argument in $(\pi/4,\pi/2)$, so

$$0<\Re\theta_1^{1/2}<\Im\theta_1^{1/2}; \tag{7.2.21}$$

since $\overline{a}^{1/4}>0$ and $A_1>0$, this gives

$$\Re A<\Im A \tag{7.2.22}$$

for all $\overline{s} \in (0, T']$, with equality holding at $\overline{s} = 0$. (This relied on the requirements $\theta_0 > 0$, $A_1 > 0$.) Now

$$\begin{aligned}
\Re\left[A(\overline{s})e^{ir\Theta(\overline{s},\overline{x}-\overline{x}_0,\overline{v}-\overline{v}_0)}\right] &= \Re A \Re e^{ir\Theta} - \Im A \Im e^{ir\Theta} \\
= \Re A\Bigg[\cos\left(r\theta_0(\overline{v}-\overline{v}_0) - \frac{1}{2\theta_0}r(\overline{x}-\overline{x}_0)^2 \frac{\int_0^{\overline{s}} \frac{1}{a(\sigma,\overline{x}_0,\overline{v}_0)}\,d\sigma}{B^2 + \left[\frac{1}{\theta_0}\int_0^{\overline{s}} \frac{1}{\overline{a}(\sigma,\overline{x}_0,\overline{v}_0)}\,d\sigma\right]^2}\right) \\
\cdot \exp\left[-\frac{1}{2}r\left(\frac{B}{B^2 + \left[\frac{1}{\theta_0}\int_0^{\overline{s}} \frac{1}{\overline{a}(\sigma,\overline{x}_0,\overline{v}_0)}\,d\sigma\right]^2}(\overline{x}-\overline{x}_0)^2 + D(\overline{v}-\overline{v}_0)^2\right)\right]\Bigg] \\
- \Im A\Bigg[\sin\left(r\theta_0(\overline{v}-\overline{v}_0) - \frac{1}{2\theta_0}r(\overline{x}-\overline{x}_0)^2 \frac{\int_0^{\overline{s}} \frac{1}{a(\sigma,\overline{x}_0,\overline{v}_0)}\,d\sigma}{B^2 + \left[\frac{1}{\theta_0}\int_0^{\overline{s}} \frac{1}{\overline{a}(\sigma,\overline{x}_0,\overline{v}_0)}\,d\sigma\right]^2}\right) \\
\cdot \exp\left[-\frac{1}{2}r\left(\frac{B}{B^2 + \left[\frac{1}{\theta_0}\int_0^{\overline{s}} \frac{1}{\overline{a}(\sigma,\overline{x}_0,\overline{v}_0)}\,d\sigma\right]^2}(\overline{x}-\overline{x}_0)^2 + D(\overline{v}-\overline{v}_0)^2\right)\right]\Bigg].
\end{aligned}$$

If we differentiate just the exponential terms and take an $L^2$ norm, we effectively multiply both parts by $rD(\overline{v} - \overline{v}_0)$, so by Lemma 7.2.1 the result will have a uniform upper bound in $r$. On the other hand, if we differentiate the trigonometric functions, use the fact that $\Re A < \Im A$, and apply Corollary 7.2.2, we find that for $r$ sufficiently large the difference of the $L^2$ norms is bounded below by $C_1 r^{1/2}$ for some constant $C_1$. Putting all of this together, we find that there is a constant $C' > 0$, depending on the parameters $A_1$, $B$, $D$, and $\theta_0$ (and hence, indirectly, on $T'$), such that for $r$ sufficiently large

$$\left\|\partial_{\overline{v}}\Re\left[A(\overline{s})e^{ir\Theta(\overline{s},\overline{x}-\overline{x}_0,\overline{v}-\overline{v}_0)}\right]\right\|_{L^2(\{\overline{s}\}\times B_{1/4}(\overline{x}_0,\overline{v}_0))} \geq C' r^{1/2},$$

from which the first inequality in (7.2.4) follows. QED.

The condition $\Re A < \Im A$ in (7.2.22) is not needed in treatments such as that in [13] since one is able to use the full *complex* Gaussian beam. We are not able to do that here. There are of course other ways of obtaining the same result, but the most straightforward one – noting that $|z| > C$ implies that at least one of $|\Re z|$ and $|\Im z|$ must be greater than $C/\sqrt{2}$ – does not work well for us since the phase of the complex number $A$ is a function of $s$. Thus we opt for the method above.

### 7.3. Energy-focussed solutions

From this we can finally prove the result announced in Section 1 above. We note first of all that when $\overline{s} = 0$ we have

$$\theta_1(0) = iB^{-1}, \qquad A(0) = A_1 B^{-1/2}\frac{1+i}{\sqrt{2}},$$

both of which are independent of the metric $\overline{h}$; thus (see (7.2.3)) $\overline{\gamma}_{GB}|_{\Sigma_0}$ is independent of $\overline{h}$. Let $(\overline{a}, \overline{b}, \overline{c}, \overline{\gamma})$ be the solution given by Theorem 6.6.1 with initial data that constructed in Chapter 5 from

$$\overline{\gamma}|_{\Sigma_0} = \varepsilon(r)\overline{\gamma}_{GB}|_{\Sigma_0},$$

where $\varepsilon(r) > 0$ is sufficiently small (depending on $r$) that the resulting initial data satisfies the assumptions in Theorem 6.6.1. ($\varepsilon(r)$ can clearly be taken to be nonincreasing with $r$, and we shall do so in the following. A careful examination of the construction in Chapter 5 suggests that $o \leq r^{-\eta}$ for some exponent $\eta$, but determining $\eta$ is not straightforward and is also not important for what we wish to do here.) Note that, since $\overline{\gamma}_{GB}|_{\Sigma_0}$ is supported on $\{\overline{v} \geq kT\sqrt{2}-1\}$, the solution $\overline{\gamma}$ will be Minkowskian for $\overline{v} < kT\sqrt{2}-1$. We have the following theorem.

7.3.1. THEOREM. There is a constant $C_g > 0$, independent of $r$, such that for $r$ sufficiently large

$$\frac{\|\overline{\gamma} - \varepsilon(r)\overline{\gamma}_{GB}\|_{H^1(\Gamma)}}{\|\varepsilon(r)\overline{\gamma}_{GB}\|_{H^1(\Gamma)}} \leq C_g r^{-1/2}.$$

Proof. Let $\overline{\gamma}_{GB,o} = \varepsilon(r)\overline{\gamma}_{GB}$. We note first that

$$\|\overline{\gamma}_{GB,o}\|_{H^1(\Gamma)} \geq \|\partial_{\overline{v}}\overline{\gamma}_{GB,o}\|_{L^2(\Gamma)} \geq C_{GB,2}T'\varepsilon(r) \tag{7.3.1}$$

where the last inequality follows from Lemma 7.2.3. Again by Lemma 7.2.3,

$$\|\Box_{\overline{h}}\overline{\gamma}_{GB,o}\|_{L^2(\Gamma)} \leq C_{GB,3}T'r^{-1/2}\varepsilon(r).$$

Now applying Lemma 6.4.2, together with the bootstrap (6.2.28), we have that there are constants $C_i$ such that for any $\sigma \in [0, kT]$ (recall that the seminorm $\epsilon$ was defined in (6.2.25)),

$$\epsilon[\overline{\gamma} - \overline{\gamma}_{GB,o}](\sigma) \leq \int_0^\sigma \frac{4}{3}\|\Box_{\overline{h}}\overline{\gamma}_{GB,o}\|_{L^2(A_\upsilon)}\|\partial_\tau(\overline{\gamma} - \overline{\gamma}_{GB,o})\|_{L^2(A_\upsilon)} + C\nu k^{-1}\epsilon[\overline{\gamma} - \overline{\gamma}_{GB,o}]\, d\upsilon. \tag{7.3.2}$$

Since the support of $\overline{\gamma}_{GB,o}$ and (by domain of dependence arguments) $\overline{\gamma}$ is contained in the region $\{(\overline{s}, \overline{x}, \overline{v}) \in \Gamma \,|\, \overline{v} \geq kT\sqrt{2}-1\}$, the integrand in (7.3.2) will vanish except for $\upsilon \in [kT - 1/\sqrt{2}, kT]$, so taking a supremum over all $\sigma \in [0, kT]$ and recalling that (see Proposition 6.3.2)

$$\|\partial_\tau(\overline{\gamma} - \overline{\gamma}_{GB,o})\|_{L^2(A_\upsilon)} \leq 2\epsilon[\overline{\gamma} - \overline{\gamma}_{GB,o}]^{1/2}(\upsilon),$$

we find additional constants such that

$$\sup_{\sigma\in[0,kT]} \epsilon[\overline{\gamma} - \overline{\gamma}_{GB,o}](\sigma) \leq C \sup_{\sigma\in[0,kT]} \epsilon[\overline{\gamma} - \overline{\gamma}_{GB,o}]^{1/2}(\sigma)\|\Box_{\overline{h}}\overline{\gamma}_{GB,o}\|_{L^2(\Gamma)} + \frac{1}{k}C'\frac{\nu}{T'} \sup_{\sigma\in[0,kT]} \epsilon[\overline{\gamma} - \overline{\gamma}_{GB,o}](\sigma).$$

Now integrating the second inequality in (7.2.4) in $\overline{s}$ from 0 to $T'$ gives

$$\|\Box_{\overline{h}}\overline{\gamma}_{GB,o}\|_{L^2(\Gamma)} < C_{GB,3}{T'}^{1/2}r^{-1/2},$$

so we obtain for yet another set of constants

$$\begin{aligned}&\sup_{\sigma\in[0,kT]} \epsilon[\overline{\gamma} - \overline{\gamma}_{GB,o}](\sigma)\\ &\qquad \leq C{T'}^{1/2}r^{-1/2} \sup_{\sigma\in[0,kT]} \epsilon[\overline{\gamma} - \overline{\gamma}_{GB,o}]^{1/2}(\sigma)\|\Box_{\overline{h}}\overline{\gamma}_{GB,o}\|_{L^2(\Gamma)} + \frac{1}{k}C'\frac{\nu}{T'} \sup_{\sigma\in[0,kT]} \epsilon[\overline{\gamma} - \overline{\gamma}_{GB,o}](\sigma).\end{aligned}$$

Taking $k$ sufficiently large (or, alternatively, taking $\nu$ sufficiently small, and then increasing $r$ if necessary), we find that there is a constant $C$ such that

$$\sup_{\sigma\in[0,kT]} \epsilon[\overline{\gamma} - \overline{\gamma}_{GB,o}]^{1/2}(\sigma) \leq Cr^{-1/2}\varepsilon(r),$$

whence (again since the supports of $\overline{\gamma}$ and $\overline{\gamma}_{GB,o}$ are contained in $\{(\overline{s},\overline{x},\overline{v}) \in \Gamma \,|\, \overline{v} \geq kT\sqrt{2} - 1\}$)

$$\|\overline{\gamma} - \overline{\gamma}_{GB,o}\|_{H^1(\Gamma)} \leq C' r^{-1/2}\varepsilon(r),$$

and the result follows from (7.3.1). QED.

The same result is true in the unscaled coordinates. We reformulate it slightly, as follows.

7.3.1. COROLLARY. Let $\overline{\gamma}_{GB}$, $\varepsilon(r)$ and $\overline{\gamma}$ be as above, and set

$$\gamma_{GB,o}(s,x,v) = k^{-1/2}\varepsilon(r)\overline{\gamma}_{GB}(s,k^{1/2}x,kv), \qquad \gamma(s,x,v) = k^{-1/2}\overline{\gamma}(s,k^{1/2}x,kv).$$

Then for $r$ sufficiently large,

$$\frac{\|\gamma - \gamma_{GB,o}\|_{H^1({}^0\Gamma)}}{\|\gamma\|_{H^1({}^0\Gamma)}} \leq 2C_g r^{-1/2}, \tag{7.3.3}$$

where $C_g$ is the constant in Theorem 7.3.1.

Proof. With $\gamma$ in the denominator replaced by $\gamma_{GB,o}$, this follows from Theorem 7.3.1 by noting the inequalities

$$\|f(s,k^{1/2}x,kv)\|_{H^1({}^0\Gamma)} \leq k^{1/4}\|f(\overline{s},\overline{x},\overline{v})\|_{H^1(\Gamma)}$$
$$\|k^{1/2}\gamma_{GB,o}\|_{H^1({}^0\Gamma)} \geq k^{1/2}\|\partial_v\gamma_{GB,o}\|_{L^2({}^0\Gamma)} = k^{1/4}\|\partial_{\overline{v}}\overline{\gamma}_{GB,o}\|_{L^2(\Gamma)}.$$

Now if we let $r \geq 4C_g^2$, then (7.3.3) follows from the observation that

$$\|\gamma\|_{H^1({}^0\Gamma)} \geq \|\gamma_{GB,o}\|_{H^1({}^0\Gamma)} - \|\gamma - \gamma_{GB,o}\|_{H^1({}^0\Gamma)} \geq \|\gamma_{GB,o}\|_{H^1({}^0\Gamma)}\left[1 - C_g r^{-1/2}\right] \geq \frac{1}{2}\|\gamma_{GB,o}\|_{H^1({}^0\Gamma)}.$$

QED.

Taking $r = \epsilon^{-1/2}$ gives the result in Theorem 0.3.1. In other words, noting that, because of the cutoff function $\phi$, the support of $\gamma_{GB,o}$ *in all of* $\Gamma$ is contained in the set

$$N = \{(\overline{s},\overline{x},\overline{v}) \in \Gamma \,|\, \overline{x} \in [0,1],\, \overline{v} \in [kT\sqrt{2} - 1, kT\sqrt{2}]\}$$

(this is the neighbourhood around a null geodesic mentioned in Section 7.1), the result in Corollary 7.3.1 can be stated in words as follows. For every $\epsilon > 0$ and every $k$ sufficiently large, there is a solution to the Einstein vacuum equations of the form (0.2.1), such that the fraction $1 - \epsilon$ of the $H^1$ norm of $\gamma(s,x,v)$ is contained in the set

$$\{(s,x,v) \in \mathbf{R}^3 \,|\, s \in [0,T'],\, x \in [0,k^{-1/2}],\, v \in [T\sqrt{2} - k^{-1}, T\sqrt{2}]\}.$$

The time $T$, as well as the time $T'$ implicit in the solution, are independent of $k$, though they may depend on $\epsilon$.

Finally, we note that the proof of Theorem 7.3.1 *almost* works for initial data supported near $\overline{v} = 0$, i.e., with $\overline{v}_0 \in (1/4, 3/4)$ instead of $\overline{v}_0 \in (kT\sqrt{2} - 3/4, kT\sqrt{2} - 1/4)$. There are two added complications that come into play here: the initial data for the solution $\overline{\gamma}$ will not vanish on $\overline{v} = 0$, unlike the initial data for the Gaussian beam; and we have no a priori control on the size of the support of $\overline{\gamma}$, hence of $\overline{\gamma} - \overline{\gamma}_{GB}$. Preliminary investigations of these issues suggest that the first issue is not really a problem as the initial data for $\overline{\gamma}$ along $\overline{v} = 0$ can apparently be chosen to be exponentially small in $r$. There does not appear to be any ready way of overcoming the second issue without going beyond our current framework (for example, defining energies over hypersurfaces other than $A_\sigma$). On the other hand, since as shown above $\epsilon[\overline{\gamma} - \overline{\gamma}_{GB}](\sigma)$ is small relative to $\|\overline{\gamma}_{GB}\|_{H^1(\Gamma)}$ for all $\sigma \in [0, kT]$, one suspects that it represents only some kind of 'tail' which does not really detract from the focussed nature of the solution; for example, that perhaps the result in Lemma 7.2.3 might hold in $W^{1,\infty}$.

Note that were we able to produce suitably bounded initial data supported on a set in the middle of $\Sigma_0$ (see our discussion at the end of Section 0.2 above), the above techniques would presumably allow us to show the existence of a solution with energy concentrated along a geodesic through the point $x = 1/2$, $v = T/2$.

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