APM346, Partial differential equations

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This document includes lecture notes, homework sets and solutions, term test and final review sheets, final exam solutions, and review material. It was typed up as the course progressed and has not been subsequently modified, so should be considered a rough draft. Comments or corrections can be sent to the author at ncarruth@math.toronto.edu.

Consider the ordinary differential equation

$$\frac{d^2u}{dt^2} = u.$$

This has solutions $u = \sinh t$, $\cosh t$. These are easily seen to be linearly independent and thus to form a basis for the solution set of the above equation (exercise). Thus the general solution to this equation can be written

$$u(t) = a\cosh t + b\sinh t.$$

Now suppose that we are given initial conditions $u(0) = u_0$, $u'(0) = u_1$. These give

$$\begin{aligned} u_0 &= u(0) = a\cosh 0 + b\sinh 0 = a \\ u_1 &= u'(0) = a\sinh 0 + b\cosh 0 = b. \end{aligned}$$

Thus this particular solution is

$$u(t) = u(0)\cosh t + u'(0)\sinh t.$$

Now it is a general fact (and one with which we shall become much better acquianted as this class goes on) that for evolution equations, such as the one here, the heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, or the wave equation $\frac{\partial^u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$, there is a collection of data, usually termed *Cauchy data*, pertaining to and completely determined by the solution, which, when given at one particular point in time, determine the solution uniquely for all future points in time. For the equation above, Cauchy data at $t = 0$ would be the pair $(u(0), u'(0))$; for the heat equation, it might be $u(0, x)$, for all x; for the wave equation, it might be the pair $(u(0, x), u_t(0, x))$ (where $u_t = \frac{\partial u}{\partial t}$). It is worth noting, though, that there is nothing special about the choice of $t = 0$: specifying Cauchy data at any point in time will allow us to find the solution at all future points. (The equation above and the wave equation can also be solved backwards in time; this is not always possible for the heat equation. But this does not matter for the present considerations.) Moreover, since the Cauchy data is completely determined by the solution, having given it at one initial point, we are able to determine it at all future points. Thus, in some sense, instead of thinking of the evolution of just the solution, we should really think of the evolution of the Cauchy data.

In the context of our original ordinary differential equation, this suggests that we should consider not just the solution u but actually the pair $(u(t), u'(t))$, and see how this pair evolves with time. Suppose, as above, that we are given that at $t = 0$, $(u(0), u'(0)) = (u_0, u_1)$. Then from the foregoing we have

$$\begin{aligned} u(t) &= u_0\cosh t + u_1\sinh t \\ u'(t) &= u_0\sinh t + u_1\cosh t \end{aligned}$$

or in other words

$$(u(t), u'(t)) = u_0(\cosh t, \sinh t) + u_1(\sinh t, \cosh t).$$

In particular, if $u_0 = 1$, $u_1 = 0$, then we get the solution

$$(u^1(t), {u^1}'(t)) = (\cosh t, \sinh t),$$

while if $u_0 = 0$, $u_1 = 1$, we get the solution

$$(u^2(t), {u^2}'(t)) = (\sinh t, \cosh t).$$

Thus the general solution can be understood in this way: we decompose the initial data as $u_0 \cdot (1, 0) + u_1 \cdot (0, 1)$, evolve each piece separately, and sum the results.

In general, when we treat partial differential equations (even, in a modified way, those which are not evolution equations, such as Laplace's equation $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \cdots = 0$), we shall follow a similar procedure: find a particularly convenient decomposition of the Cauchy (or, in general, boundary) data such that each individual piece propagages in a simple way, and then sum all of these propagated pieces to obtain the final solution.

We shall make this much more clear as we go on.

Summary:

- The temperature in a body satisfies the equation $\frac{\partial u}{\partial t} = D\nabla^2 u$ for some constant D, where (in three dimensions) $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$.
- In one dimension, this becomes $\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$.
- If we require initial data $u(x, 0) = \sin kx$ for some constant k, then a corresponding solution to this equation is $u(x, t) = \sin kx e^{-k^2 Dt}$.
- Thus, if we require initial data $u(x, 0) = \sum \sin k_n x$, where the sum is over a finite collection of k_n, then a corresponding solution is $u(x, t) = \sum \sin k_n x e^{-k_n^2 Dt}$.
- This can be extended to infinite sums (and even integrals), which will allow us to represent (almost) any initial data on a bounded interval.

Notation. We use the notations $\frac{\partial u}{\partial x}$, u_x, and $\partial_x u$ to denote the partial derivative of u with respect to x. They are all equivalent.

Example from ordinary differential equations. Let $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$, and consider the equation (where a dot indicates differentiation with respect to t)

$$\dot{\mathbf{x}} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{x}, \qquad \begin{aligned} \dot{x}_1 &= 2x_1 + x_2 \\ \dot{x}_2 &= x_1 + 2x_2 \end{aligned} \tag{1}$$

with initial data $\mathbf{x}(0) = \begin{pmatrix} x_{1,0} \\ x_{2,0} \end{pmatrix}$. As it stands, this is a coupled system, which is difficult to solve directly. It can be decoupled by diagonalising the coefficient matrix $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, as follows. Let us denote this matrix by A. It has characteristic equation

$$\begin{aligned} 0 &= \det\left(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} \\ &= (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3, \end{aligned}$$

which has roots $\lambda = 3$, $\lambda = 1$. We see that

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

are corresponding eigenvectors. It is useful to normalise these; thus we set

$$\mathbf{e}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{e}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

These vectors are clearly linearly independent and hence span $\mathbf{R}^2$; thus for each t there must exist numbers $y_1(t)$, $y_2(t)$ such that

$$\mathbf{x}(t) = y_1(t)\mathbf{e}_1 + y_2(t)\mathbf{e}_2.$$

If we substitute this back in to equation (1) above, we obtain

$$\begin{aligned} \dot{y}_1(t)\mathbf{e}_1 + \dot{y}_2(t)\mathbf{e}_2 &= A\left(y_1(t)\mathbf{e}_1 + y_2(t)\mathbf{e}_2\right) \\ &= y_1(t)A\mathbf{e}_1 + y_2(t)A\mathbf{e}_2 = y_1(t)\left(3\mathbf{e}_1\right) + y_2(t)\mathbf{e}_2 = 3y_1(t)\mathbf{e}_1 + y_2(t)\mathbf{e}_2 \end{aligned}$$

since $\mathbf{e}_1$ and $\mathbf{e}_2$ are eigenvectors of A with eigenvalues 3 and 1, respectively. Thus we obtain the two equations

$$\begin{aligned} \dot{y}_1(t) &= 3y_1(t) \\ \dot{y}_2(t) &= y_2(t) \end{aligned}$$

which are easily solved to give $y_1(t) = y_1(0)e^{3t}$, $y_2(t) = y_2(0)e^t$.What this means is that if our initial data is equal to $\mathbf{e}_1$, so that $y_1(0) = 1$, $y_2(0) = 0$, then our solution will be

$$y_1(t)\mathbf{e}_1 = e^{3t}\mathbf{e}_1,$$

while if our initial data is instead equal to $\mathbf{e}_2$, so that $y_1(0) = 0$, $y_2(0) = 1$, then our solution will be

$$y_2(t)\mathbf{e}_1 = e^t\mathbf{e}_2.$$

In general, our solution will be a linear combination of these, depending on $y_1(0)$ and $y_2(0)$. To find $y_1(0)$ and $y_2(0)$ in terms of $\mathbf{x}(0)$, we may proceed as follows. We have

$$\begin{aligned}\mathbf{x}(0)\cdot\mathbf{e}_1 &= (y_1(0)\mathbf{e}_1 + y_2(0)\mathbf{e}_2)\cdot\mathbf{e}_1\\ &= y_1(0)\mathbf{e}_1\cdot\mathbf{e}_1 + y_2(0)\mathbf{e}_2\cdot\mathbf{e}_1 = y_1(0),\end{aligned}$$

since $\mathbf{e}_1\cdot\mathbf{e}_1 = 1$ and (crucially) $\mathbf{e}_2\cdot\mathbf{e}_1 = 0$. (We see that had we not normalised, this procedure would still work; we would just have to divide $\mathbf{x}(0)\cdot\mathbf{e}_1$ by $\mathbf{e}_1\cdot\mathbf{e}_1$ to find $y_1(0)$.) In exactly the same way, we see that

$$\begin{aligned}\mathbf{x}(0)\cdot\mathbf{e}_2 &= (y_1(0)\mathbf{e}_1 + y_2(0)\mathbf{e}_2)\cdot\mathbf{e}_2\\ &= y_1(0)\mathbf{e}_1\cdot\mathbf{e}_2 + y_2(0)\mathbf{e}_2\cdot\mathbf{e}_2 = y_2(0).\end{aligned}$$

Thus we may write the final solution for $\mathbf{x}$ as

$$\mathbf{x}(t) = (\mathbf{x}(0)\cdot\mathbf{e}_1)\,\mathbf{e}_1e^{3t} + (\mathbf{x}(0)\cdot\mathbf{e}_2)\,\mathbf{e}_2e^t. \qquad \square$$

It is instructive to compare this to the general result (true for any vector $\mathbf{x}$ in $\mathbf{R}^2$), whose demonstration we leave as an exercise:

$$\mathbf{x} = (\mathbf{x}\cdot\mathbf{e}_1)\,\mathbf{e}_1 + (\mathbf{x}\cdot\mathbf{e}_2)\,\mathbf{e}_2. \tag{2}$$

The general theory of systems of ordinary differential equations will not be needed in the rest of the course. The point of the above is to give a concrete example of the method of breaking multidimensional initial data into components which evolve in a simple fashion, and then writing the solution to the original problem as the sum of these evolved parts. Thus we decomposed $\mathbf{x}(0)$ according to (2), evolved each piece separately (this only required multiplying by e^{3t} and e^t, respectively), and then summed the results.

Derivation of the heat equation. Pages 99–100 of the textbook give a nice derivation of the heat equation which we followed quite closely in class. The derivation in the textbook is actually a bit more general since it allowed for heat sources located within the body. (If our sphere really were a cow, for example, these could represent heat due to metabolisation of food or muscle contractions.) Here we shall only use the so-called homogeneous heat equation, meaning the heat equation without sources, which we write as (we want to use k for something else in a moment)

$$\frac{\partial u}{\partial t} = D\nabla^2 u.$$

Examples of solutions to the one-dimensional heat equation without sources[1]. In this case we seek a function $u(x,t)$ which satisfies the equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}. \tag{3}$$

[1]The point of the first example is to motivate the introduction of separated solutions, while that of the second is to motivate the idea that a general solution can be written as a sum of separated ones.

We need to know something more than just this equation if we wish to determine u throughout all of space and time. By analogy with the system of ordinary differential equations above, we try specifying initial datum[2] $u(x,0) = \sin kx$, for some constant[3]. Now at $t = 0$, the equation (3) will then become

$$\partial_t u(x,0) = D\partial_x^2 (\sin kx) = -k^2 D \sin kx = -k^2 D u(x,0). \tag{4}$$

This by itself does not really tell us much. However, it leads us to guess that we might be able to find a solution to the original heat equation by requiring (4) to hold for all t, not just $t = 0$. This gives the equation

$$\partial_t u(x,t) = -k^2 D u(x,t).$$

Now fix some $x = x_0$, and let u denote $u(x_0, t)$, so that the equation becomes the ordinary differential equation

$$\frac{du}{dt} = -k^2 D u.$$

By the theory of first-order linear equations, the solution to this will be $u = Ce^{-k^2 Dt}$, where C is some constant which can be determined by evaluating both sides at $t = 0$:

$$C = Ce^{-k^2 D \cdot 0} = u(0) = u(x_0, 0) = \sin kx_0.$$

Thus we obtain, for all x_0, $u(x_0, t) = \sin kx_0 e^{-k^2 Dt}$, which gives the function

$$u(x,t) = \sin kx e^{-k^2 Dt}.$$

We must now check whether this is actually a solution to the heat equation. We have

$$\begin{aligned} \frac{\partial u}{\partial t} &= \sin kx \left(-k^2 D e^{-k^2 Dt}\right) \\ D\frac{\partial^2 u}{\partial x^2} &= D\left(-k^2 \sin kx\right) e^{-k^2 Dt}, \end{aligned}$$

which are easily seen to be equal. Thus the function $u(x,t) = \sin kx e^{-k^2 Dt}$ is a solution to the heat equation satisfying $u(x,0) = \sin kx$, as desired. (Compare this to the solution to the system of ordinary differential equations with initial datum $\mathbf{x}(0) = \mathbf{e}_1$.)

Let us now consider the initial datum $u = \sin k_1 x + \sin k_2 x$, $k_1 \neq k_2$. Unfortunately the above approach does not work in this case, since

$$\partial_x^2 u = -k_1 \sin k_1 x - k_2 \sin k_2 x$$

is not simply a multiple of u. In order to treat this case, we note that the heat equation is *linear*, by which we mean that any linear combination of solutions is also a solution. To see this for the case of two solutions, suppose that u_1 and u_2 are solutions of the heat equation (with the same constant D), and that a_1 and a_2 are constants. Then we see that

$$\begin{aligned} \frac{\partial}{\partial t}\left(a_1 u_1 + a_2 u_2\right) &= a_1 \frac{\partial u_1}{\partial t} + a_2 \frac{\partial u_2}{\partial t} \\ &= a_1 \left(D\nabla^2 u_1\right) + a_2 \left(D\nabla^2 u_2\right) \\ &= D\nabla^2 \left(a_1 u_1 + a_2 u_2\right), \end{aligned}$$

[2]The word 'data' is actually a Latin plural (so please never make the too-clever-by-half mistake of writing datæ as though data were a Latin singular; the author encountered this once!). The singular is datum. One could argue whether giving $u(x,0)$ is giving a singular or a plural quantity. We use the singular here because we shall want to talk about multiple distinct *data* below.

[3]Throughout this course, when we say 'constant' we mean a number which does not depend on any of the variables in the question; in other words, a quantity constant in both space and time.

since a_1 and a_2, being constants, can be brought through the differentiation signs. Since our initial datum is a sum of initial data both of which we know how to handle, this suggests that we do something similar to what worked in the case of the system of ordinary differential equations above and work out the solution for each of the initial data separately. More specifically, let u_1 be the solution to the heat equation determined above with $u_1(x, 0) = \sin k_1 x$, and u_2 be that with $u_2(x, 0) = \sin k_2 x$, so that

$$u_1(x,t) = \sin k_1 x e^{-k_1^2 Dt}$$
$$u_2(x,t) = \sin k_2 x e^{-k_2^2 Dt}.$$

Since these are both solutions, their sum $u_1(x,t) + u_2(x,t)$ will also be; moreover, at $t = 0$ it will agree with the initial datum given above. Thus the solution to the heat equation with initial datum $\sin k_1 x + \sin k_2 x$ is

$$u(x,t) = \sin k_1 x e^{-k_1^2 Dt} + \sin k_2 x e^{-k_2^2 Dt}.$$

Note the similarity to the solution to the system of ordinary differential equations above.

Commentary. It should be clear from the foregoing how to handle the case of an initial datum which is a sum of any finite number of sine functions. A review of our method shows that it also works for cosine functions; hence we now know how to find a solution to the heat equation with initial datum any finite sum of sine and cosine functions. By itself this is still not much use. However, it turns out that almost any function on a finite interval (and in particular, any continuous function on a closed interval) can be expressed as a series – an infinite sum – of sine and cosine functions. Thus, if we can find a way of expressing our initial datum as such a sum, we can apply the above method to determine the solution for all future times. The reason why sine and cosine functions (and, as we shall see later, Legendre polynomials and Bessel functions) are particularly useful is that they turn out to be orthogonal with respect to certain inner products (generalisations of the dot product we are familiar with in $\mathbf{R}^2$ and $\mathbf{R}^3$ to spaces of functions). Recalling our method for computing $y_1(0)$, $y_2(0)$ in the first example above, we see that this should allow us to compute the coefficients in the expansion of our initial datum as a series in sine and cosine functions using inner products. Thus we must first discuss what we mean by an inner product, and what kind of inner product we can put on a space of functions.

Inner Products. We are all familiar with the dot product in $\mathbf{R}^3$: if $\mathbf{v} = v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}$, $w = w_1\mathbf{i}+w_2\mathbf{j}+w_3\mathbf{k}$, then

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3.$$

The dot product is useful for finding projections (this is basically how we used it in the first example above). In particular, we have

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{i})\mathbf{i} + (\mathbf{v} \cdot \mathbf{j})\mathbf{j} + (\mathbf{v} \cdot \mathbf{k})\mathbf{k}.$$

It would be helpful to be able to extend this formula to spaces of functions. Now a vector has only a finite number of components while a function has essentially infinitely many components (speaking very loosely); thus it seems reasonable that the sum over components which worked to give us the dot product of two vectors should become an integral when we work with functions. More specifically, consider the function space

$$X = \{f : [a,b] \to \mathbf{R} | f \text{ is integrable and bounded}\};$$

on X we may define an inner product

$$(f,g) = \int_a^b f(x)g(x)dx.$$

It turns out that this inner product has many of the same properties as the dot product, and in particular can be used to separate out the different sine and cosine components of a function under appropriate circumstances. We shall take this up on Tuesday.

APM346, Homework 1. Solutions.

1. Calculate the indicated derivatives.

(a) $\frac{d}{dx}\left(10x^6 - 5x^3 + 4x^2 - 7x + 1\right) = 60x^5 - 15x^2 + 8x - 7.$

(b) $\frac{d}{dx}\left(\ln\left[5x^2 - 3x + 100\right]\right) = \frac{10x-3}{5x^2-3x+100}.$

(c) $\frac{d}{dx}\left(e^{5x^{10}-10x^5+102}\right) = \left(50x^9 - 50x^4\right)e^{5x^{10}-10x^5+102}.$

(d) $\frac{d}{dx}(\sin 2x) = 2\cos 2x.$

(e) $\frac{d}{dx}(\cos kx) = -k\sin kx$, k a constant.

(f) $\frac{\partial}{\partial y}(\cos k_1 x \, \sin k_2 y) = k_2 \cos k_1 x \cos k_2 y$, k_1, k_2 constants.

(g)

$$\begin{aligned}
&\frac{\partial}{\partial z}\left(\sin^{-1}\left(\ln\left(\cos\left(\tan\left(xyz + x^2 + 10xy - 100\right)\right)\right)\right)\right) \\
&\quad = -\frac{1}{\sqrt{1 - \ln^2\left(\cos\left(\tan\left(xyz + x^2 + 10xy - 100\right)\right)\right)}} \frac{\sin\left(\tan\left(xyz + x^2 + 10xy - 100\right)\right)}{\cos\left(\tan\left(xyz + x^2 + 10xy - 100\right)\right)} \\
&\qquad \cdot xy\sec^2\left(xyz + x^2 + 10xy - 100\right) \\
&= -\frac{1}{\sqrt{1 - \ln^2\left(\cos\left(\tan\left(xyz + x^2 + 10xy - 100\right)\right)\right)}} \tan\left(\tan\left(xyz + x^2 + 10xy - 100\right)\right) \\
&\qquad \cdot xy\sec^2\left(xyz + x^2 + 10xy - 100\right).
\end{aligned}$$

2. Evaluate the following expressions.

(a) $\nabla\left(x^2 + y^2\right) = 2x\mathbf{i} + 2y\mathbf{j}.$

(b) $\nabla\left(x^2 + y^2 - 2z^2\right) = 2x\mathbf{i} + 2y\mathbf{j} - 4z\mathbf{k}.$

(c) $\operatorname{div}(x\mathbf{i} + y\mathbf{j} + 10\mathbf{k}) = 1 + 1 + 0 = 2.$

(d) $\operatorname{div}\left(\nabla\left(x^2 + y^2 - 2z^2\right)\right) = \operatorname{div}(2x\mathbf{i} + 2y\mathbf{j} - 4z\mathbf{k}) = 0.$

(e) $\operatorname{div}(\nabla(e^y \sin x)) = \operatorname{div}(e^y \cos x\mathbf{i} + e^y \sin x\mathbf{j}) = -e^y \sin x + e^y \sin x = 0.$

3. Evaluate the following integrals. (You must show your work to get credit.)

(a) We use integration by parts:

$$\begin{aligned}
\int_0^{2\pi} x^2 \sin x \, dx &= -x^2 \cos x\Big|_0^{2\pi} + \int_0^{2\pi} 2x \cos x \, dx = -4\pi^2 + \left(2x\sin x\Big|_0^{2\pi} - \int_0^{2\pi} 2\sin x \, dx\right) \\
&= -4\pi^2 + 2\cos x\Big|_0^{2\pi} = -4\pi^2.
\end{aligned}$$

(b) We use integration by parts again:

$$\begin{aligned}
\int_0^{2\pi} x\sin(kx) \, dx &= -\frac{1}{k}x\cos(kx)\Bigg|_0^{2\pi} + \int_0^{2\pi} \frac{1}{k}\cos(kx) \, dx = -\frac{2\pi}{k}\cos(2\pi k) + \frac{1}{k^2}\sin(kx)\Bigg|_0^{2\pi} \\
&= -\frac{2\pi}{k}\cos(2\pi k) + \frac{1}{k^2}\sin(2\pi k).
\end{aligned}$$

(c) $\int_0^{+\infty} xe^{-x}\,dx = -xe^{-x}\big|_0^{+\infty} + \int_0^{+\infty} e^{-x}\,dx = -e^{-x}\big|_0^{+\infty} = 1$, where we can use L'Hôpital's rule to conclude $\lim\limits_{x\to+\infty} xe^{-x} = 0$.

(d) This problem can be done two ways, one using a double integration by parts, and the other (for those who are comfortable working with complex functions) using complex exponentials. The first is as follows. We work with indefinite integrals:

$$\begin{aligned}\int e^x \cos x\,dx &= e^x \cos x + \int e^x \sin x\,dx \\ &= e^x \cos x + \left(e^x \sin x - \int e^x \cos x\,dx\right)\end{aligned}$$

from which we easily see that $\int e^x \cos x\,dx = \frac{1}{2} e^x (\cos x + \sin x)$. From this it follows that

$$\int_0^{2\pi} e^x \cos x\,dx = \frac{1}{2}\left(e^{2\pi} - 1\right).$$

The other method is as follows:

$$\begin{aligned}\int e^x \cos x\,dx &= \int e^x \frac{e^{ix} + e^{-ix}}{2}\,dx = \frac{1}{2}\int e^{(1+i)x} + e^{(1-i)x}\,dx = \frac{1}{2}\left(\frac{e^{(1+i)x}}{1+i} + \frac{e^{(1-i)x}}{1-i}\right) \\ &= \frac{1}{4}\left((1-i)e^{(1+i)x} + (1+i)e^{(1-i)x}\right) = \frac{1}{4} 2\mathrm{Re}\,(1-i)e^x (\cos x + i\sin x) \\ &= \frac{1}{2} e^x (\cos x + \sin x).\end{aligned}$$

From this the definite integral follows as before.

(e) $\int_0^{2\pi} \sin k_1 x \sin k_2 x\,dx$, $k_1, k_2 \in \mathbf{Z}$, $k_1 \neq k_2$.

This integral can be evaluated by using the trigonometric identity $\sin a \sin b = \frac{1}{2}(\cos(a-b) - \cos(a+b))$. In the present case, this gives

$$\begin{aligned}\int_0^{2\pi} \sin k_1 x \sin k_2 x\,dx &= \frac{1}{2}\int_0^{2\pi} \cos((k_1 - k_2)x) - \cos((k_1 + k_2)x)\ dx \\ &= \frac{1}{2}\left(\frac{\sin\,((k_1 - k_2)x)}{k_1 - k_2} - \frac{\sin\,((k_1 + k_2)x)}{k_1 + k_2}\right)\Bigg|_0^{2\pi} \\ &= 0,\end{aligned}$$

assuming $k_1 \neq -k_2$, $k_1, k_2 \neq 0$, and since $k_1 \neq k_2$. The case $k_1 = -k_2$ is essentially identical to $k_1 = k_2$ (since sin is odd) and is covered (up to a minus sign) by the solution to (f), and when either k_1 or k_2 is zero the integral is zero. (The author apologises for these oversights in setting the original problem; he should have written $k_1, k_2 \in \mathbf{Z}$, $k_1, k_2 > 0$.)

(f) Same as (e), but with $k_1 = k_2$.

Again, we assume $k_1 \neq 0$ (otherwise the integrand is 0). In this case the identity above becomes $\sin^2 k_1 x = \frac{1}{2}(1 - \cos(2k_1 x))$, and the above integral becomes

$$\int_0^{2\pi} \sin^2 k_1 x\,dx = \frac{1}{2}\int_0^{2\pi} 1 - \cos(2k_1 x)\ dx = \pi,$$

since the integral of cos will vanish as in (e).

(g) $\int_0^{2\pi} \sin k_1 x \cos k_2 x\,dx$, k_1, k_2 any two integers.

This is very similar to (e) but uses instead the identity $\sin a \cos b = \frac{1}{2}(\sin(a+b) + \sin(a-b))$. The integral becomes, for $k_1 \neq \pm k_2$,

$$\begin{aligned}\int_0^{2\pi} \sin k_1 x \cos k_2 x &= \frac{1}{2}\int_0^{2\pi} \sin((k_1+k_2)x) + \sin((k_1-k_2)x)\ dx \\ &= -\frac{1}{2}\left(\frac{\cos((k_1+k_2)x)}{k_1+k_2} + \frac{\cos((k_1-k_2)x)}{k_1-k_2}\right)\Bigg|_0^{2\pi} \\ &= 0.\end{aligned}$$

If $k_1 = k_2$, then $\sin((k_1-k_2)x) = 0$ for all x, so its integral still vanishes, while the above shows that the remaining integral vanishes as before. If $k_1 = -k_2$ then the integral of $\sin((k_1+k_2)x)$ vanishes, while the other integral vanishes as above. If $k_1 = k_2 = 0$ then the entire integrand vanishes. Thus the result above holds for all $k_1, k_2 \in \mathbf{Z}$.

4. Evaluate the following integrals.

(a) If $R = [0,\pi] \times [0,\pi]$, then

$$\begin{aligned}\iint_R \sin x \sin y \, dA &= \int_0^{\pi}\int_0^{\pi} \sin x \sin y \, dx\, dy \\ &= \int_0^{\pi} \sin y \left(-\cos x\right)\big|_0^{\pi}\ dy = 2\int_0^{\pi} \sin y \, dy = 4.\end{aligned}$$

(b) $\iint_R e^{-\left(x^2+y^2\right)}\, dA$, R the unit disk in the xy-plane.
In polar coordinates, R is represented by the set $\{(r,\theta)|r \leq 1\}$, and the integral becomes

$$\int_0^{2\pi}\int_0^1 e^{-r^2} r\, dr\, d\theta = \int_0^{2\pi} -\frac{1}{2}e^{-r^2}\Big|_0^1\ d\theta = \pi\left(1 - e^{-1}\right).$$

(c) $\iiint_R \sin\left(x^2+y^2+z^2\right)^{\frac{3}{2}}\ dV$, R the unit ball in xyz-space.
In spherical polar coordinates, R is represented by the set $\{(r,\theta,\phi)|r \leq 1\}$, and the integral becomes

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi}\int_0^1 \sin r^3\, r^2 \sin\theta dr\, d\theta\, d\phi &= 2\pi\int_0^{\pi} \sin\theta\left(-\frac{1}{3}\cos r^3\right)\Big|_0^1\ d\theta \\ &= \frac{2\pi}{3}\left(1-\cos 1\right)\left(-\cos\theta\right)\big|_0^{\pi} = \frac{4\pi}{3}\left(1-\cos 1\right).\end{aligned}$$

5. Consider the two-dimensional vector space of functions on the interval $[0,1]$

$$V = \{a\sin\pi x + b\cos\pi x | a, b \in \mathbf{R}\}.$$

Let $B = \{\sin\pi x, \cos\pi x\} \subset V$.

(a) Prove that B is a basis for V. (Hint: Wronskian!)

The Wronskian of the functions $\sin\pi x$ and $\cos\pi x$ is given by

$$W(x) = \begin{vmatrix} \sin\pi x & \cos\pi x \\ \pi\cos\pi x & -\pi\sin\pi x \end{vmatrix} = -\pi,$$

which is not zero on any interval. Thus the functions $\sin\pi x$ and $\cos\pi x$ are linearly independent on any open interval, and hence on the interval $[0,1]$ itself, by the contrapositive of the proposition in the notes on the Wronskian on the course website. Since they span the space V by definition, they must then be a basis for V.

(b) Find the matrix representation $[T]_B$ of the operator T in the basis B, for (i) $T = \frac{d}{dx}$; (ii) $T = \frac{d^2}{dx^2}$.

(i) We evaluate T on the basis elements:

$$T(\sin\pi x) = \frac{d}{dx}\sin\pi x = \pi\cos\pi x, \qquad T(\cos\pi x) = -\pi\sin\pi x.$$

From this we see that (cf. the notes on linear algebra on the course website)

$$[T]_B = \begin{pmatrix} 0 & -\pi \\ \pi & 0 \end{pmatrix}.$$

(ii) Again, we evaluate T on the basis elements:

$$T(\sin\pi x) = \frac{d}{dx}\pi\cos\pi x = -\pi^2\sin\pi x, \qquad T(\cos\pi x) = -\pi^2\cos\pi x.$$

From this we see that

$$[T]_B = \begin{pmatrix} -\pi^2 & 0 \\ 0 & -\pi^2 \end{pmatrix}.$$

We note that this is the square of the matrix in (i), as it should be.

6. Consider the differential equation $\frac{d^2y}{dx^2} = -4y$.

(a) Find the set of all solutions to this equation.

Writing the equation as $\frac{d^2y}{dx^2} + 4y = 0$, we have the characteristic equation $r^2 + 4 = 0$, which has the imaginary roots $r = \pm 2i$. This means that (as we could have determined by inspection in this case) the equation has solutions $\sin 2x$, $\cos 2x$; since (as we show in (b) in a moment) these are linearly independent, the solution set is $\{a\sin 2x + b\cos 2x | a, b \in \mathbf{R}\}$.

(b) Find a basis for this solution set. (You must prove that your answer is in fact a basis.)

We claim that $\{\sin 2x, \cos 2x\}$ is a basis for the solution set to this equation. We know from the theory of ordinary differential equations that the set of solutions to this equation is two-dimensional, so to show this it suffices to show that $\{\sin 2x, \cos 2x\}$ is linearly independent. This can be effected by computing its Wronskian:

$$W(x) = \begin{vmatrix} \sin 2x & \cos 2x \\ 2\cos 2x & -2\sin 2x \end{vmatrix} = -2,$$

so as in 5(a) above this set is indeed linearly independent and hence (as noted in part (a) of this problem) a basis for the set of solutions to the equation.

(c) (Optional) Can you find the set of all solutions to $\frac{d^2y}{dx^2} + 4y = \sin 4x$?

By the theory of ordinary differential equations, the general solution to this equation will be the sum of a particular solution and the general solution to the corresponding homogeneous equation from (a), which we already know. Now we note that

$$\frac{d^2}{dx^2}\sin 4x = -16\sin 4x,$$

so that if $y = -\frac{1}{12}\sin 4x$,

$$\frac{d^2y}{dx^2} + 4y = -\frac{1}{12}(-16\sin 4x + 4\sin 4x) = \sin 4x,$$

and the set of all solutions to $\frac{d^2y}{dx^2} + 4y = \sin 4x$ is $\{-\frac{1}{12}\sin 4x + a\sin 2x + b\cos 2x | a, b \in \mathbf{R}\}$.

7. Find all (a) local and (b) global maxima of $f(x, y) = e^y \cos x$ on the rectangle $[0, 2\pi] \times [0, 1]$.

To find any local extrema, we compute the gradient and set it to zero:

$$\nabla e^y \cos x = -e^y \sin x\mathbf{i} + e^y \cos x\mathbf{j} = 0.$$

Since $e^y \neq 0$ for any y, this gives the system $\sin x = \cos x = 0$; but since $\sin^2 x + \cos^2 x = 1$, this is impossible. Thus this function has no local extrema in the rectangle (or anywhere in the plane, for that matter).

To find global extrema, we thus need only consider the function on the boundary. Now if $x = 0$ or $x = 2\pi$, we have $f(x, y) = e^y$, which (on $[0, 1]$) has a minimum of 1 at $y = 0$ and a maximum of e at $y = 1$. If $y = 0$ then $f(x, y) = \cos x$, which has a maximum of 1 at $x = 0$ and a minimum of -1 at $x = \pi$, while if $y = 1$ then $f(x, y) = e \cos x$, which has a maximum of e at $x = 0$ and a minimum of $-e$ at $x = \pi$. Putting all of this together, we see that the global maximum of $e^y \cos x$ is e, at the point $(0, 1)$, and the global minimum is $-e$, at $(\pi, 1)$. (Only the global maximum was required for this problem; the author put in the solution for the global minimum by mistake.)

Summary:

- One of the main goals of this course is to understand how to write functions as series in a collection of mutually orthogonal functions which are such that the original partial differential equation (or whatever other problem we are dealing with) becomes simple.
- This bears some analogies to the process of diagonalising matrices and writing arbitrary vectors as linear combinations of the eigenvectors of a matrix.
- This process – whether for matrices or for the differential operators which shall be our main concern here – works best when we have an *inner product*, which gives us a way of generalising the notion of *projection* and hence allows us to compute the coefficients in the series expansions mentioned above.

EXAMPLE. Consider the matrix from last time, $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.[1] We recall that this has eigenvalues 3 and 1 and corresponding eigenvectors $\mathbf{e}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{e}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, which are orthonormal. Now suppose that we wish to solve the equation $A\mathbf{x} = \mathbf{y}$, for some given vector $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. Now since $\mathbf{e}_1$ and $\mathbf{e}_2$ span $\mathbf{R}^2$, there are numbers a_1 and a_2 such that $\mathbf{y} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$; in fact, we may write

$$\mathbf{y} \cdot \mathbf{e}_1 = (a_1\mathbf{e}_1 + a_2\mathbf{e}_2) \cdot \mathbf{e}_1 = a_1\mathbf{e}_1 \cdot \mathbf{e}_1 + a_2\mathbf{e}_2 \cdot \mathbf{e}_1 = a_2,$$

and similarly $\mathbf{y} \cdot \mathbf{e}_2 = a_2$; thus

$$\mathbf{y} = (\mathbf{y} \cdot \mathbf{e}_1)\mathbf{e}_1 + (\mathbf{y} \cdot \mathbf{e}_2)\mathbf{e}_2. \tag{1}$$

Similarly, we may write $\mathbf{x} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2$; then the equation $A\mathbf{x} = \mathbf{y}$ becomes

$$A(b_1\mathbf{e}_1 + b_2\mathbf{e}_2) = b_1A\mathbf{e}_1 + b_2A\mathbf{e}_2 = 3b_1\mathbf{e}_1 + b_2\mathbf{e}_2 = a_1\mathbf{e}_1 + a_2\mathbf{e}_2.$$

Since $\{\mathbf{e}_1, \mathbf{e}_2\}$ forms a basis for $\mathbf{R}^2$, we see that we must have $3b_1 = a_1$, $b_2 = a_2$, i.e.,

$$\mathbf{x} = \frac{1}{3}(\mathbf{y} \cdot \mathbf{e}_1)\mathbf{e}_1 + (\mathbf{y} \cdot \mathbf{e}_2)\mathbf{e}_2.$$

The main point here, though, is not this last formula, but rather that if we take what was originally a difficult problem (solving a system of equations) and rewrite it using the eigenvectors $\mathbf{e}_1$ and $\mathbf{e}_2$, it becomes a very simple problem. For the problems in partial differential equations which we wish to tackle, rewriting them in terms of (what we shall term) *eigenfunctions* is often about the only real way to approach the problem (at least if what we want is a formula for the solution, which is usually the case for us in this class).

COMMENTARY. To extend the above example to the problems in partial differential equations which we wish to treat, we see that we need to extend the notion of dot product to functions, in such a way that our expansion formulas will look like equation (1) above. The following is a particular example of such an extension. (We shall have occasion to use others, but they will be closely related to this one.)

DEFINITION. Suppose that $f, g : [a, b] \to \mathbf{C}$ are integrable[2]. We define their *inner product* to be the complex number[3]

$$(f, g) = \int_a^b f(x)\overline{g(x)}\, dx.$$

[1] In class I used a slightly different matrix, $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Either matrix can be used to make the points here.

[2] 'Integrable' for us means that their Riemann integral exists, which in particular means that they are bounded. It is enough to think of continuous or piecewise continuous functions for the moment.

[3] The integral here was motivated in the notes from last Thursday's lecture. The complex conjugate here can be motivated by observing that if $z = \alpha + i\beta$ is a complex number, then $z\overline{z} = \alpha^2 + \beta^2$, which is the square of the distance from $(0, 0)$ to (α, β) in the plane; in other words, $\sqrt{z\overline{z}}$ represents the length of z when considered as a vector in the plane.

We also define the L^2 *norm*[4] of f to be

$$\|f\| = \left(\int_a^b |f(x)|^2 \, dx \right)^{\frac{1}{2}},$$

where $|f(x)| = \sqrt{f(x)\overline{f(x)}}$ is the modulus of the complex number $f(x)$.

PROPERTIES. The inner product defined above satisfies the following properties (see the review sheet on linear algebra):

1. $(\alpha f + \beta g, h) = \alpha(f, h) + \beta(g, h)$ for all $\alpha, \beta \in \mathbf{C}$ and all integrable f, g, h.
2. $(f, g) = \overline{(g, f)}$ for all integrable f, g.
3. $(f, f) \geq 0$ for all integrable f, and $(f, f) = 0$ if and only if f is zero except on a set of content zero[5].

These can be proved directly from the definition of the inner product; for example, the first property may be proved as follows:

$$\begin{aligned}
(\alpha f + \beta g, h) &= \int_a^b (\alpha f(x) + \beta g(x))\overline{h(x)} \, dx \\
&= \int_a^b \alpha f(x)\overline{h(x)} + \beta g(x)\overline{h(x)} \, dx \\
&= \alpha \int_a^b f(x)\overline{h(x)} \, dx + \beta \int_a^b g(x)\overline{h(x)} \, dx = \alpha(f, h) + \beta(g, h).
\end{aligned}$$

As noted in the review sheet on linear algebra, the first and second properties show that $(f, \alpha g + \beta h) = \overline{\alpha}(f, h) + \overline{\beta}(g, h)$ (this can also be shewn more simply directly). We see that the inner product is *linear* in the first argument and *conjugate linear* in the second.

BESSEL'S INEQUALITY. The inner product also satisfies the following: suppose that $\{e_1, e_2, \ldots\}$ is a collection of pairwise orthonormal integrable functions; in other words, that

$$(e_i, e_j) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}.$$

Let f be any integrable function. Then we have

$$\sum_i |(f, e_i)|^2 \leq \|f\|^2. \tag{2}$$

Intuitively, this may be understood as follows. If our above inner product works as desired, the quantities (f, e_i) will be the coefficients in the expansion of f in terms of the e_i, or what amounts to the same thing, the (scalar) projections of f along the e_i; the above relation says that the sum of the squares of these projections can be no greater than the square of the length of the function f itself. This can be understood by considering the example in $\mathbf{R}^2$ of $\mathbf{e}_1 = \mathbf{i}$, $\mathbf{e}_2 = \mathbf{j}$, $\mathbf{x} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$; in this case the left-hand expression will be $1^2 + 2^2 = 5$ while the right-hand will be $1^2 + 2^2 + (-1)^2 = 6$.

This example suggests something else: note that the inequality is strict because we did not include enough vectors in our set – had we also defined $\mathbf{e}_3 = \mathbf{k}$, then the left-hand side would have become $1^2 + 2^2 + (-1)^2 = 6$, the same as the right-hand side. Thus perhaps equality in (2) holds exactly when our collection $\{e_1, e_2, \ldots\}$ has 'enough functions' in some sense – enough to write f as a series in the e_i. This turns out to be quite close to the truth, as we shall discuss on Thursday.

[4]It is possible to define L^p norms for all $p \geq 1$. These are important in advanced analysis but we shall not need them here.

Looking forward. Our goal, as stated multiple times, is to understand how to expand arbitrary functions in series of suitable orthonormal sets of functions. We have seen that the trigonometric functions on appropriate intervals give orthogonal sets, and it turns out that basically all functions we shall be interested in dealing with can be expanded in series of trigonometric functions; these are called *Fourier series* and are the topic of chapter 1 in the textbook. However, later on in the course we shall be interested in series in more general orthogonal sets, such as those arising from Bessel functions, Legendre polynomials, and spherical harmonics. It turns out that all of these arise as solutions to ordinary differential equations obtained by separating variables for one of our standard partial differential equations (Laplace's equation, the heat equation, and the wave equation) in different coordinate systems. In particular, the trigonometric functions arise from separating variables for Laplace's equation in rectangular coordinates, the Bessel functions arise when doing so in cylindrical coordinates, and the Legendre polynomials and spherical harmonics arise when doing so in spherical polar coordinates. Thus we shall first take some time to write down Laplace's equation in these three coordinate systems and discuss the kinds of ordinary differential equations which arise when looking for their separated solutions. This will lead us to the topic of *Sturm-Liouville problems*, and general expansions in terms of eigenfunctions of so-called *self-adjoint differential operators*. This will then allow us to discuss expansions in the above-mentioned functions.

Summary:

- Any integrable function can be expanded in a series of complete orthogonal functions, and the coefficient of the function e_i is simply the inner product $\frac{(f,e_i)}{(e_i,e_i)}$, where the inner product is given by $(f,g) = \int_a^b f(x)\overline{g(x)}\,dx$.
- On the interval $[0,1]$, two complete orthogonal sets are $\{1, \cos 2k\pi x, \sin 2k\pi x | k \in \mathbf{Z}, k > 0\}$.
- This allows us to determine the *Fourier series* of a function f by computing the inner products $(f,1)$, $(f, \cos 2k\pi x)$, $(f, \sin 2k\pi x)$, as well as the lengths $(1,1)$, etc.

More Later

APM 346, Homework 2. Due Monday, May 20, at 6 AM EDT. To be marked completed/not completed.

1. Use the identity $e^{3i\theta} = \left(e^{i\theta}\right)^3$ ($\theta \in \mathbf{R}$) to find an expression for $\cos 3\theta$ in terms of $\cos\theta$ and $\sin\theta$.
We have

$$\begin{aligned}\cos 3\theta + i\sin 3\theta = e^{3i\theta} &= \left(e^{i\theta}\right)^3 = (\cos\theta + i\sin\theta)^3\\ &= \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3\\ &= \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta).\end{aligned}$$

Since two complex numbers are equal if and only if their real and imaginary parts are equal, we see that

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta.$$

2. Find all numbers $\lambda > 0$ for which there is a nonzero function f on $(0,1)$ satisfying

$$f'' = -\lambda^2 f, \qquad f(0) = 0, \qquad f'(1) = -f(1).$$

Also find the corresponding functions f. (Note: it is enough to find an equation which λ must satisfy. It is in general not possible to solve this equation.)

The general solution to the given differential equation is (using x as the independent variable) $f(x) = a\sin\lambda x + b\cos\lambda x$. The first boundary condition gives

$$f(0) = a\sin 0 + b\cos 0 = b = 0,$$

so that we may write $f(x) = a\sin\lambda x$. The second boundary condition then gives

$$f'(1) = a\lambda\cos\lambda = -f(1) = -a\sin\lambda.$$

Since we want $f \neq 0$ (note that this means that f and 0 are not the same function, i.e., that f is not identically zero; it does *not* mean that there is no x for which $f(x) = 0$!), we cannot have $a = 0$; thus we may cancel the a from this equation to obtain

$$\lambda = -\tan\lambda.$$

Thus, if $\lambda > 0$ is any solution to this equation, then $f(x) = a\sin\lambda x$ will satisfy the given boundary value problem for any a. (In principle, a could even be a complex number.)

3. (You need only do one of problems 3 and 4.) Suppose that $A_n \in \mathbf{R}$, $n = 0, 1, 2, \ldots$, $B_n \in \mathbf{R}$, $n = 1, 2, \ldots$, are such that

$$x = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n\cos 2n\pi x + B_n\sin 2n\pi x\right)$$

for $x \in (0,1)$. Find an expression for the A_n and B_n.

The set

$$\{1\} \cup \{\cos 2n\pi x,\ \sin 2n\pi x | n \in \mathbf{Z}, n > 0\}$$

is an orthogonal set, so we may calculate as follows, letting $(f,g) = \int_0^1 f(x)\overline{g(x)}\,dx$ denote the standard inner product on functions:

$$\begin{aligned}\frac{1}{2}A_0 &= \frac{(x,1)}{(1,1)}\\ &= \frac{\int_0^1 x\,dx}{\int_0^1 dx} = \frac{\frac{1}{2}x^2\big|_0^1}{1} = \frac{1}{2},\end{aligned}$$

so that $A_0 = 1$, while if $n > 0$

$$\begin{aligned}
A_n &= \frac{(x, \cos 2n\pi x)}{(\cos 2n\pi x, \cos 2n\pi x)} \\
&= \frac{\displaystyle\int_0^1 x \cos 2n\pi x \, dx}{\displaystyle\int_0^1 \cos^2 2n\pi x \, dx} = \frac{\displaystyle x \frac{1}{2n\pi} \sin 2n\pi x \Big|_0^1 - \int_0^1 \frac{1}{2n\pi} \sin 2n\pi x \, dx}{\displaystyle\int_0^1 \frac{1}{2} + \frac{1}{2} \cos 4n\pi x \, dx} \\
&= \frac{\displaystyle\frac{1}{4n^2\pi^2} \cos 2n\pi x \Big|_0^1}{\frac{1}{2}} = 0,
\end{aligned}$$

where we have used the fact that the integral of cosine over any integer number of periods is zero, and that $\cos 2n\pi = 1$, $\sin 2n\pi = 0$ for all integers n. Finally, we have

$$\begin{aligned}
B_n &= \frac{(x, \sin 2n\pi x)}{(\sin 2n\pi x, \sin 2n\pi x)} \\
&= \frac{\displaystyle\int_0^1 x \sin 2n\pi x \, dx}{\displaystyle\int_0^1 \sin^2 2n\pi x \, dx} = \frac{\displaystyle -\frac{1}{2\pi n} x \cos 2\pi n x \Big|_0^1 + \int_0^1 \frac{1}{2\pi n} \cos 2\pi n x \, dx}{\displaystyle\int_0^1 \frac{1}{2}(1 - \cos 4\pi n x) \, dx} \\
&= -\frac{1}{\pi n}.
\end{aligned}$$

4. (You need only do one of problems 3 and 4.) Suppose that $A_n \in \mathbf{C}$, $n = 0, 1, 2, \ldots$, are such that

$$x = \sum_{n=0}^{\infty} A_n e^{2in\pi x}$$

for $x \in (0, 1)$. Find an expression for the A_n.

Since $\{e^{2i\pi n x} | n \in \mathbf{Z}, n \geq 0\}$ is an orthonormal set, we may calculate as follows:

$$A_0 = (x, 1) = 1,$$

while for $n \neq 0$,

$$\begin{aligned}
A_n = (x, e^{2i\pi n x}) &= \int_0^1 x e^{-2i\pi n x} \, dx \\
&= -\frac{1}{2i\pi n} x e^{-2i\pi n x} \Big|_0^1 + \int_0^1 \frac{1}{2i\pi n} e^{-2i\pi n x} \, dx \\
&= -\frac{1}{2i\pi n} + \frac{1}{4\pi^2 n^2} e^{-2i\pi n x} \Big|_0^1 = -\frac{1}{2i\pi n},
\end{aligned}$$

where we have used $e^{2i\pi n} = 1$ for all integers n.

(Note. There was in fact a typographical error in the original problem, and the sum should have been extended from $-\infty$ to ∞; in other words, there is in fact no expansion of the form indicated in the problem statement. Technically, though, this does not affect our ability to solve the problem; and anyway the above calculation works for $n < 0$ just as well as for $n > 0$.)

APM 346, Homework 3. Due Monday, May 27, at 6.05 AM EDT. To be marked completed/not completed.

1. Recall the following boundary-value problem on the interval [0, 1] from Homework 2:

$$f'' = -\lambda^2 f, \qquad f(0) = 0, \qquad f'(1) = -f(1).$$

Show that if (λ_1, f_1) and (λ_2, f_2) are two solutions to this boundary-value problem, $\lambda_1, \lambda_2 > 0$, $\lambda_1 \neq \lambda_2$, then f_1 and f_2 are orthogonal with respect to the standard inner product $(f, g) = \int_0^1 f(x)\overline{g(x)}\, dx$. (You may use the solution posted on the course website, or work directly from the equation and boundary conditions above.)

There are two ways of doing this problem. First, we know that we may write (letting $i = 1, 2$)

$$f_i = a_i \sin \lambda_i x, \qquad \lambda_i = -\tan \lambda_i.$$

Thus

$$\begin{aligned}
(f_1, f_2) &= \int_0^1 f(x)\overline{f_2(x)}\, dx = \int_0^1 a_1\overline{a_2} \sin \lambda_1 x \sin \lambda_2 x\, dx = a_1\overline{a_2} \cdot \frac{1}{2} \in_0^1 \cos\left[(\lambda_1 - \lambda_2)x\right] - \cos\left[(\lambda_1 + \lambda_2)x\right]\, dx \\
&= \frac{1}{2} a_1\overline{a_2} \left[\left.\frac{\sin\left[(\lambda_1 - \lambda_2)x\right]}{\lambda_1 - \lambda_2}\right|_0^1 - \left.\frac{\sin\left[(\lambda_1 + \lambda_2)x\right]}{\lambda_1 + \lambda_2}\right|_0^1 \right] = \frac{1}{2} a_1\overline{a_2} \left[\frac{\sin(\lambda_1 - \lambda_2)}{\lambda_1 - \lambda_2} - \frac{\sin(\lambda_1 + \lambda_2)}{\lambda_1 + \lambda_2} \right] \\
&= \frac{1}{2} a_1\overline{a_2} \left[\frac{\sin\lambda_1 \cos\lambda_2 - \cos\lambda_1 \sin\lambda_2}{-\tan\lambda_1 + \tan\lambda_2} + \frac{\sin\lambda_1 \cos\lambda_2 + \cos\lambda_1 \sin\lambda_2}{\tan\lambda_1 + \tan\lambda_2} \right] \\
&= \frac{1}{2} a_1\overline{a_2} \left[\frac{\sin\lambda_1 \cos\lambda_2 - \cos\lambda_1 \sin\lambda_2}{(-\sin\lambda_1 \cos\lambda_2 + \cos\lambda_1 \sin\lambda_2)\frac{1}{\cos\lambda_1 \cos\lambda_2}} + \frac{\sin\lambda_1 \cos\lambda_2 + \cos\lambda_1 \sin\lambda_2}{(\sin\lambda_1 \cos\lambda_2 + \cos\lambda_1 \sin\lambda_2)\frac{1}{\cos\lambda_1 \cos\lambda_2}} \right] \\
&= \frac{1}{2} a_1\overline{a_2} \left[-\cos\lambda_1 \cos\lambda_2 + \cos\lambda_1 \cos\lambda_2 \right] = 0.
\end{aligned}$$

Alternatively, we may work directly from the equation. Since $\lambda_1 \neq \lambda_2$, at least one of $\lambda_1, \lambda_2 \neq 0$; we may assume that $\lambda_1 \neq 0$ without loss of generality (since our inner product satisfies $(f_1, f_2) = \overline{(f_2, f_1)}$). Then (note that we may assume that f_1 and f_2 are real, but this is not really necessary; we do assume however that λ is real, as we assumed in Homework 2)

$$\begin{aligned}
\int_0^1 f_1(x)\overline{f_2(x)}\, dx &= -\frac{1}{\lambda_1} \int_0^1 f_1''(x)\overline{f_2(x)}\, dx = -\frac{1}{\lambda_1} \left[\left. f_1'(x)\overline{f_2(x)} \right|_0^1 - \int_0^1 f_1'(x)\overline{f_2'(x)}\, dx \right] \\
&= -\frac{1}{\lambda_1} \left[\left. f_1'(x)\overline{f_2(x)} \right|_0^1 - \left[\left. f_1(x)\overline{f_2'(x)} \right|_0^1 - \int_0^1 f_1(x)\overline{f_2''(x)}\, dx \right] \right] \\
&= -\frac{1}{\lambda_1} \left[\left. f_1'(x)\overline{f_2(x)} \right|_0^1 - \left. f_1(x)\overline{f_2'(x)} \right|_0^1 - \lambda_2 \int_0^1 f_1(x)\overline{f_2(x)}\, dx \right],
\end{aligned}$$

whence we see that, solving for $\int_0^1 f_1(x)\overline{f_2(x)}\, dx$,

$$\left(1 - \frac{\lambda_2}{\lambda_1}\right) \int_0^1 f_1(x)\overline{f_2(x)}\, dx = -\frac{1}{\lambda_1} \left[-f_1(1)\overline{f_2(1)} - f_1(1) \left[-\overline{f_2(1)} \right] \right] = 0,$$

where we have used the boundary conditions. Since $\lambda_1 \neq \lambda_2$, this shows that $(f_1, f_2) = \int_0^1 f_1(x)\overline{f_2(x)}\, dx = 0$, as desired.

2. Solve the following boundary-value problem on $[0,1]\times[0,1]$:

$$\nabla^2 u = 0, \qquad f(x,0) = \begin{cases} 1, & x\in[0,\frac{1}{2}) \\ 0, & x\in(\frac{1}{2},1] \end{cases}, \qquad f(x,1) = \begin{cases} 0, & x\in[0,\frac{1}{2}) \\ 1, & x\in(\frac{1}{2},1] \end{cases},$$
$$f(0,y) = 0, \qquad f(1,y) = 0.$$

(You may use the expansion of $f(x,0)$ given in the lecture notes.)

[Erratum: please read 'u' for 'f' at each occurence in the foregoing. We apologise and hope this did not cause too much confusion.]

We begin by looking for separated solutions: suppose that $u(x,y) = X(x)Y(y)$; then we have

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0,$$

whence as discussed in lecture we must have $X'' = -\lambda^2 X$, $Y'' = \lambda^2 Y$, for some constant λ (which will be real since the boundary conditions force us to have $\frac{X''}{X} < 0$, and which we may then take to be positive[1]). These equations have solutions $X = a_\lambda \cos\lambda x + b_\lambda \sin\lambda x$, $Y = c_\lambda \cosh\lambda y + d_\lambda \sinh\lambda y$. Thus we posit that the full solution will have the form

$$u = \sum_\lambda \left(a_\lambda \cos\lambda x + b_\lambda \sin\lambda x\right)\left(c_\lambda \cosh\lambda y + d_\lambda \sinh\lambda y\right).$$

We may now apply the boundary conditions to determine λ and the coefficients in the above expansion. First of all, we apply the homogeneous conditions:

$$u(0,y) = \sum_\lambda a_\lambda \left(c_\lambda \cosh\lambda y + d_\lambda \sinh\lambda y\right) = 0$$

whence we take $a_\lambda = 0$;

$$u(1,y) = \sum_\lambda b_\lambda \sin\lambda \left(c_\lambda \cosh\lambda y + d_\lambda \sinh\lambda y\right) = 0,$$

whence we take $\lambda = n\pi$, $n\in\mathbf{Z}$, $n>0$. Absorbing b_λ by writing

$$\alpha_n = b_{n\pi}c_{n\pi}, \quad \beta_n = b_{n\pi}d_{n\pi},$$

we may now write

$$u = \sum_{n=1}^{\infty} \sin n\pi x \left(\alpha_n \cosh n\pi y + \beta_n \sinh n\pi y\right).$$

We may now apply the other boundary conditions:

$$u(x,0) = \sum_{n=1}^{\infty} \sin n\pi x\,(\alpha_n) = \begin{cases} 1, & x\in[0,\frac{1}{2}) \\ 0, & x\in(\frac{1}{2},1] \end{cases}$$

We let $h(x)$ denote the function on the right-hand side above. Since, as discussed in lecture, the set $\{\sin n\pi x | n\in\mathbf{Z}, n>0\}$ is complete on $[0,1]$, and since it is also orthogonal[2], we may calculate α_n as follows (exactly as was done in lecture):

$$\alpha_n = \frac{(u,\sin n\pi x)}{\sin n\pi x, \sin n\pi x)} = \frac{\int_0^1 h(x)\sin n\pi x\ dx}{\int_0^1 \sin^2 n\pi x\, dx} = \frac{\int_0^{\frac{1}{2}} \sin n\pi x\, dx}{\int_0^1 \frac{1}{2}\left(1-\cos 2n\pi x\right)\, dx}$$
$$= \frac{-\frac{1}{n\pi}\cos n\pi x\Big|_0^{\frac{1}{2}}}{\frac{1}{2}} = -\frac{2}{n\pi}\left[\cos\frac{n\pi}{2} - 1\right].$$

[1] It should be noted that in principle $\lambda = 0$ should also be considered. However, it is readily seen that the solution for X in this case is of the form $ax+b$, which cannot satisfy the boundary conditions at $(0,y)$ and $(1,y)$ unless $a=b=0$ and may thus be dropped.

[2] The instructor thinks he may have forgotten to demonstrate this point in class. It may be shewn easily as follows: $\int_0^1 \sin n\pi x \sin m\pi x = \frac{1}{2}\left[\frac{\sin[(n-m)\pi x]}{n-m}\Big|_0^1 - \frac{\sin[(n+m)\pi x]}{n+m}\Big|_0^1\right] = 0.$

Finally, the last boundary condition gives

$$u(x,1)=\sum_{n=1}^{\infty}\sin n\pi x\,(\alpha_n\cosh n\pi+\beta_n\sinh n\pi)=1-h,$$

whence we have

$$\begin{aligned}\alpha_n\cosh n\pi+\beta_n\sinh n\pi&=\frac{(1-h,\sin n\pi x)}{\sin n\pi x,\sin n\pi x)}=\frac{(1,\sin n\pi x)-(h,\sin n\pi x)}{(\sin n\pi x,\sin n\pi x)}\\&=2\int_0^1\sin n\pi x\,dx-\alpha_n=-\frac{2}{n\pi}\cos n\pi x\big|_0^1-\alpha_n\\&=-\frac{2}{n\pi}\left[(-1)^n-1\right]-\frac{2}{n\pi}\left[1-\cos\frac{n\pi}{2}\right]=-\frac{2}{n\pi}\left[(-1)^n-\cos\frac{n\pi}{2}\right],\end{aligned}$$

whence

$$\beta_n=-\alpha_n\coth n\pi-\frac{2}{n\pi\sinh n\pi}\left[(-1)^n-\cos\frac{n\pi}{2}\right]=-\frac{2}{n\pi\sinh n\pi}\left[\cosh n\pi\left(1-\cos\frac{n\pi}{2}\right)+(-1)^n-\cos\frac{n\pi}{2}\right].$$

Thus we have finally the grand expression[3]

$$u(x,y)=\sum_{n=1}^{\infty}\frac{2}{n\pi}\left[\left(1-\cos\frac{n\pi}{2}\right)\cosh n\pi y+\frac{1}{\sinh n\pi}\left(\cosh n\pi\left(\cos\frac{n\pi}{2}-1\right)+\cos\frac{n\pi}{2}-(-1)^n\right)\sinh n\pi y\right]\cdot\sin n\pi x.$$

3. (a) Write x^4 on $(-1,1)$ as a series of Legendre polynomials. (Hint: the series has only finitely many terms. But you need to prove this!)

(b) (Optional) Is the series expansion from (a) valid outside of the interval $(-1,1)$? Is this likely to matter for our applications of Legendre polynomials?

(a) We have the first five Legendre polynomials (see p. 254 in the textbook)

$$P_0(x)=1,\quad P_1(x)=x,\quad P_2(x)=\frac{3}{2}x^2-\frac{1}{2},$$
$$P_3(x)=\frac{5}{2}x^3-\frac{3}{2}x,\quad P_4(x)=\frac{35}{8}x^4-\frac{15}{4}x^2+\frac{3}{8}.$$

Thus we may write $x^2=\frac{2}{3}\left(P_2+\frac{1}{2}P_0\right)$, whence

$$\begin{aligned}x^4&=\frac{8}{35}\left(P_4+\frac{15}{4}x^2-\frac{3}{8}P_0\right)=\frac{8}{35}\left(P_4+\frac{5}{2}\left(P_2+\frac{1}{2}P_0\right)-\frac{3}{8}P_0\right)\\&=\frac{8}{35}\left(P_4+\frac{5}{2}P_2+\frac{7}{8}P_0\right)=\frac{8}{35}P_4+\frac{4}{7}P_2+\frac{1}{5}P_0.\end{aligned}$$

[3]This is typical of the kinds of solutions one obtains by separation of variables. We should get some satisfaction out of our ability to construct such an expression! The author once read a biography of one Hugh Nibley ("A Consecrated Life", probably published by Deseret Book in 2002 or 2003, though the remaining bibliographical details escape me at the moment) in which he is reported to have written to his mother during training in meteorology (if my memory serves me correctly) in the US military prior to deployment in World War II, expressing the following sentiment: "We have become quite the little mathematician, and work great big problems sometimes passing within sight, almost, of the correct answer"! One of the author's colleagues at UC Berkeley expressed a similar sentiment regarding their common graduate quantum mechanics class, that she was learning how to actually solve quantum mechancis problems. For those of you who go on to study electrodynamics at the graduate level, the experience gained in producing solutions of this type will be extremely useful.

Alternatively, we may use the fact that the Legendre polynomials are orthogonal on the interval $[-1,1]$ – since we have not yet discussed this we shall omit it for the moment. (The above calculation shows that the expansion can have only finitely many terms.)

(b) [NB This was added when it was anticipated that we would be able to discuss the orthogonality of the Legendre polynomials on $[-1,1]$ before this homework was due. In that case, the point was that the expression in (a) would be derived using our general orthogonal function theory, in the which case it would not be clear a priori that it would hold outside of $[-1,1]$. To prove that it does hold everywhere, though, it would be sufficient to note that polynomials equal on an interval are equal on the entire real line. This is not relevant for our applications of Legendre polynomials, though, since (as we shall see shortly) we are interested in Legendre polynomials of $\cos\theta$, and $\cos\theta \in [-1,1]$ for all θ.]

Summary:

- When solving problems with boundary data specified on circles, cylinders, or spheres, it is useful to work in coordinate systems adapted to the boundary surfaces at hand.
- The gradient in cylindrical coordinates is given by

$$\nabla f = \frac{\partial f}{\partial \rho}\boldsymbol{\rho} + \frac{1}{r}\frac{\partial f}{\partial \phi}\boldsymbol{\phi} + \frac{\partial f}{\partial z}\mathbf{k},$$

 and in spherical coordinates by

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\boldsymbol{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\boldsymbol{\phi}.$$

- The divergence in cylindrical coordinates of a vector field $\mathbf{F} = F_\rho\boldsymbol{\rho} + F_\phi\boldsymbol{\phi} + F_z\mathbf{k}$ is given by

$$\frac{\partial F_\rho}{\partial \rho} + \frac{1}{\rho}F_\rho + \frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z},$$

 and the divergence in spherical coordinates of a vector field $\mathbf{F} = F_r\mathbf{r} + F_\theta\boldsymbol{\theta} + F_\phi\boldsymbol{\phi}$ is given by

$$\frac{\partial F_r}{\partial r} + \frac{2}{r}F_r + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{1}{r}\cot\theta F_\theta + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi}.$$

- In cylindrical coordinates, Laplace's equation becomes

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = 0,$$

 and in spherical coordinates,

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = 0.$$

- When we separate variables in Laplace's equation in spherical coordinates, we get solutions $u = R\Theta\Phi$, where R, Θ, and Φ are of the following form:

$$R = ar^\ell + br^{-(\ell+1)}, \qquad \Theta = P_\ell^m(\cos\theta), \qquad \Phi = c\cos m\theta + d\sin m\theta,$$

 where ℓ and m are nonnegative integers and P_ℓ^m is a *Legendre function*. The simplest case is when $m = 0$, in the which case we write $\Theta = P_\ell(\cos\theta)$, where P_ℓ is the *Legendre polynomial of degree ℓ*.

MOTIVATION. We have by now seen a few examples of the use of the separation-of-variables technique to solve Laplace's equation on a square. Exactly similar methods would work to solve it on a rectangle, and in three (or even higher) dimensions we could solve it on a cube with exactly analogous techniques. Suppose however that our boundary data were given on a circle, or a sphere – this would be a very different matter. Thinking back to our general series solution to Laplace's equation on the unit square,

$$u(x, y) = \sum_{n=1}^{\infty} \sin n\pi x\left(a_n\sinh n\pi y + b_n\cosh n\pi y\right),$$

if we were given boundary data on a circle, we would need to satisfy a requirement of the form

$$u(x, \sqrt{1-x^2}) = f(x) = \sum_{n=1}^{\infty} \sin n\pi x\left(a_n\sinh n\pi\sqrt{1-x^2} + b_n\cosh n\pi\sqrt{1-x^2}\right),$$

and now not only does it look hopeless to try to integrate this series against $\sin m\pi x$, it seems pretty clear that that is not even the right thing to try since now y depends on x rather than being constant, and it is not at all clear that integrating against $\sin m\pi x$ will allow us to deduce the expansion coefficients a_n and b_n. Thus it seems that in cases like this something else is required. It turns out that the correct way forwards is to do a change of variables and work in polar, cylindrical, or spherical coordinates. This is analogous to how we change integrals to integrate over circular or spherical regions in multivariable calculus.

NOTE. The derivations of the expressions for the gradient and divergence below are rather technical. Since in this class we only really need the end results of these derivations, i.e., the expressions for the Laplacian in spherical and cylindrical coordinates, the derivations themselves are of secondary importance and may be skipped without essential loss of continuity. They are given here for the sake of completeness, and also because the author feels that the existence (at least) of the techniques demonstrated is worth knowing.

The main subject-matter of the course continues on p. 6 below.

GRADIENT IN GENERAL COORDINATE SYSTEMS. Let $f : \mathbf{R}^n \to \mathbf{R}$ be a differentiable function (one can think of $n = 2$ or $n = 3$ if one likes). The *gradient* of f is defined to be the vector ∇f in $\mathbf{R}^n$ such that, for any unit vector $\mathbf{n}$, the rate of change of f in the direction $\mathbf{n}$ is equal to $\mathbf{n} \cdot \nabla f$; in other words, such that

$$\lim_{h\to 0} \frac{f(\mathbf{x} + h\mathbf{n}) - f(\mathbf{x})}{h} = \mathbf{n} \cdot \nabla f(\mathbf{x}). \tag{1}$$

In rectangular coordinates in $\mathbf{R}^3$, the gradient has the well-known expression

$$\nabla f(\mathbf{x}) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Now fix some point $\mathbf{x} \in \mathbf{R}^n$ and suppose that $\boldsymbol{\gamma} : (-\epsilon, \epsilon) \to \mathbf{R}^n$ (for some $\epsilon > 0$) is such that $\boldsymbol{\gamma}(0) = \mathbf{x}$, $\boldsymbol{\gamma}'(0) = \mathbf{n}$ (where $\boldsymbol{\gamma}'$ denotes the derivative of $\boldsymbol{\gamma}$ with respect to its parameter). Then by the chain rule we see that

$$\frac{d}{dt} f(\boldsymbol{\gamma}(t))\bigg|_{t=0} = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\bigg|_{\mathbf{x}} \frac{d\gamma^i}{dt}\bigg|_{t=0} = \boldsymbol{\gamma}'(0) \cdot \nabla f(\mathbf{x}) = \mathbf{n} \cdot \nabla f(\mathbf{x});$$

in other words, to determine $\mathbf{n} \cdot \nabla f(\mathbf{x})$, we do not need to use the straight-line path in the definition in (1) above; differentiating along any other curve which passes through the point in the correct direction with unit speed (i.e., satisfying $\boldsymbol{\gamma}'(0) = \mathbf{n}$; unit speed means that $|\boldsymbol{\gamma}'(0)| = |\mathbf{n}| = 1$) will also do.

In particular, let us consider how to express the gradient in curvilinear coordinates. Suppose that $y^1, \ldots, y^n$ is a set of coordinates on some (open) subset of $\mathbf{R}^n$ – this means that we have two sets of functions (letting $x^1, \ldots, x^n$ denote the standard coordinates on $\mathbf{R}^n$)

$$\begin{aligned} y^1 &= y^1(x^1, \ldots, x^n), & x^1 &= x^1(y^1, \ldots, y^n), \\ y^2 &= y^2(x^1, \ldots, x^n), & x^2 &= x^2(y^1, \ldots, y^n), \\ &\vdots & &\vdots \\ y^n &= y^n(x^1, \ldots, x^n), & x^n &= x^n(y^1, \ldots, y^n); \end{aligned}$$

if we think of spherical coordinates on $\mathbf{R}^3$, for example (and readers who feel uncomfortable with the level of generality are highly advised to think only of spherical or cylindrical coordinates in the following), we have

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2}, & x &= r\sin\theta\cos\phi, \\ \theta &= \arctan\frac{\sqrt{x^2 + y^2}}{z}, & y &= r\sin\theta\sin\phi, \\ \phi &= \pm\arctan\frac{y}{x}, & z &= r\cos\theta, \end{aligned}$$

where the $\pm$ in the equation for ϕ is the normal ambiguity in determining ϕ from the ratio $\frac{y}{x}$.

Let us now fix some point $\mathbf{x}_0 \in \mathbf{R}^n$ which has coordinates $(y_0^1, y_0^2, \ldots, y_0^n)$. Now for each $j = 1, 2, \ldots, n$ we have the curve

$$\boldsymbol{\gamma}_j(t) = \left(x^1(y_0^1, \ldots, y_0^j + t, \ldots, y_0^n), x^2(y_0^1, \ldots, y_0^j + t, \ldots, y_0^n), \ldots, x^n(y_0^1, \ldots, y_0^j + t, \ldots, y_0^n)\right),$$

which is just the curve obtained by holding all but the jth coordinate constant and letting the jth coordinate change at unit speed. The unit tangent vector to this curve at $t = 0$, $\frac{\boldsymbol{\gamma}_j'(0)}{|\boldsymbol{\gamma}_j'(0)|}$, is called the *unit coordinate vector in the jth direction at the point* $\mathbf{x}$; we denote it by $\mathbf{y}_j$. It is not hard to see that the vector $\mathbf{y}_j$ is the unit normal to the surface $y^j = y_0^j$ passing through the point $\mathbf{x}$. *Calculating the gradient* in the y coordinate system means representing ∇f in the basis $\{\mathbf{y}_j\}$ at each point. For simplicity in these calculations, we shall when convenient reparametrise the above curves by arclength and let $\boldsymbol{\gamma}_j(s)$ denote the j curve parametrised by arclength $s(t) = \int_0^t |\boldsymbol{\gamma}_j'(t')|dt'$; then we have simply $\mathbf{y}_j = \frac{d\boldsymbol{\gamma}_j}{ds}$.

For example, in spherical coordinates we have the three curves and unit vectors

$$\begin{aligned}
\boldsymbol{\gamma}_1(t) &= ((r_0 + t)\sin\theta_0\cos\phi_0, (r_0 + t)\sin\theta_0\sin\phi_0, (r_0 + t)\cos\theta_0) & \mathbf{r} &= \sin\theta_0\cos\phi_0\mathbf{i} + \sin\theta_0\sin\phi_0\mathbf{j} + \cos\theta_0\mathbf{k} \\
\boldsymbol{\gamma}_2(t) &= (r_0\sin(\theta_0 + t)\cos\phi_0, r_0\sin(\theta_0 + t)\sin\phi_0, r_0\cos(\theta_0 + t)) & \boldsymbol{\theta} &= \cos\theta_0\cos\phi_0\mathbf{i} + \cos\theta_0\sin\phi_0\mathbf{j} - \sin\theta_0\mathbf{k} \\
\boldsymbol{\gamma}_3(t) &= (r_0\sin\theta_0\cos(\phi_0 + t), r_0\sin\theta_0\sin(\phi_0 + t), r_0\cos\theta_0) & \boldsymbol{\phi} &= -\sin\phi_0\mathbf{i} + \cos\phi_0\mathbf{j}
\end{aligned}$$

and the reparametrisation by arclength can be obtained by noting that $\gamma_1(t) = r_0\mathbf{r} + t\mathbf{r}$, and hence is already parametrised by arclength; that $\gamma_2(t)$ represents a circle of radius r_0, so an arclength parameter is $s = r_0 t$; and that $\gamma_3(t)$ represents a circle of radius $r_0\sin\theta_0$, so that an arclength parameter is $s = r_0\sin\theta_0 t$, so that finally we have the parametrisations by arclength –

$$\begin{aligned}
\boldsymbol{\gamma}_1(s) &= ((r_0 + s)\sin\theta_0\cos\phi_0, (r_0 + s)\sin\theta_0\sin\phi_0, (r_0 + s)\cos\theta_0) \\
\boldsymbol{\gamma}_2(s) &= \left(r_0\sin\left(\theta_0 + \frac{s}{r_0}\right)\cos\phi_0, r_0\sin\left(\theta_0 + \frac{s}{r_0}\right)\sin\phi_0, r_0\cos\left(\theta_0 + \frac{s}{r_0}\right)\right) \\
\boldsymbol{\gamma}_3(s) &= \left(r_0\sin\theta_0\cos\left(\phi_0 + \frac{s}{r_0\sin\theta_0}\right), r_0\sin\theta_0\sin\left(\phi_0 + \frac{s}{r_0\sin\theta_0}\right), r_0\cos\theta_0\right).
\end{aligned}$$

The vectors $\{\mathbf{r}, \boldsymbol{\theta}, \boldsymbol{\phi}\}$ are seen to give an orthonormal basis for $\mathbf{R}^3$ for any values of θ_0 and ϕ_0.

Returning to our general picture, let us now assume that (as for the case of spherical and – it can be shewn – cylindrical coordinates) the vectors $\mathbf{y}_j$ are all mutually orthogonal (and hence orthonormal since they have unit length by construction). Then we have simply

$$\nabla f(\mathbf{x}_0) = (\mathbf{y}_1 \cdot \nabla f(\mathbf{x}_0))\,\mathbf{y}_1 + \cdots + (\mathbf{y}_n \cdot \nabla f(\mathbf{x}_0))\,\mathbf{y}_n.$$

Now by our work above, we have (since by the definition of arclength, we have $\frac{ds}{dt} = |\boldsymbol{\gamma}_j'|$, so $\frac{dt}{ds} = \frac{1}{|\boldsymbol{\gamma}_j'|}$)

$$\begin{aligned}
\mathbf{y}_j \cdot \nabla f(\mathbf{x}_0) &= \frac{d}{ds}\left(f(\boldsymbol{\gamma}_j(s))\right)\bigg|_{s=0} \\
&= \frac{d}{dt}\left(f(\boldsymbol{\gamma}_j(t))\right)\bigg|_{t=0}\frac{dt}{ds}\bigg|_{s=0} = \frac{1}{|\boldsymbol{\gamma}_j'(0)|}\frac{d}{dt}\left(f(\boldsymbol{\gamma}_j(t))\right)\bigg|_{t=0} \\
&= \frac{1}{|\boldsymbol{\gamma}_j'(0)|}\sum_{i=1}^n \frac{\partial f}{\partial x^i}\frac{d\boldsymbol{\gamma}_j^i}{dt}\bigg|_{t=0} \\
&= \frac{1}{|\boldsymbol{\gamma}_j'(0)|}\sum_{i=1}^n \frac{\partial f}{\partial x^i}\frac{\partial x^i}{\partial y^j}\bigg|_{\mathbf{x}=\mathbf{x}_0} \\
&= \frac{1}{|\boldsymbol{\gamma}_j'(0)|}\frac{\partial f}{\partial y^j}\bigg|_{(y_0^i)}.
\end{aligned}$$

Applying this formula to the special case of spherical coordinates, we see first of all that (the derivatives are with respect to t, not s)

$$|\gamma_1'(0)| = 1, \qquad |\gamma_2'(0)| = r_0, \qquad |\gamma_3'(0)| = r_0 \sin\theta_0.$$

Thus we obtain

$$\begin{aligned}
\mathbf{y}_1 \cdot \nabla f(\mathbf{x}_0) &= \mathbf{r} \cdot \nabla f(\mathbf{x}_0) = \frac{\partial f}{\partial r}, \\
\mathbf{y}_2 \cdot \nabla f(\mathbf{x}_0) &= \boldsymbol{\theta} \cdot \nabla f(\mathbf{x}_0) = \frac{1}{r}\frac{\partial f}{\partial \theta}, \\
\mathbf{y}_3 \cdot \nabla f(\mathbf{x}_0) &= \boldsymbol{\phi} \cdot \nabla f(\mathbf{x}_0) = \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi},
\end{aligned}$$

where all quantities are to be evaluated at the point (r_0, θ_0, ϕ_0). Thus we have finally

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\boldsymbol{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\boldsymbol{\phi}.$$

Similarly, in cylindrical coordinates we have the three curves and unit vectors

$$\begin{aligned}
\gamma_1(t) &= ((\rho_0 + t)\cos\phi_0, (\rho_0 + t)\sin\phi_0, z) & \boldsymbol{\rho} &= \cos\phi_0 \mathbf{i} + \sin\phi_0 \mathbf{j} \\
\gamma_2(t) &= (\rho_0 \cos(\phi_0 + t), \rho_0 \sin(\phi_0 + t), z) & \boldsymbol{\phi} &= -\sin\phi_0 \mathbf{i} + \cos\phi_0 \mathbf{j} \\
\gamma_3(t) &= (\rho_0 \cos\phi_0, \rho_0 \sin\phi_0, z + t) & \mathbf{z} &= \mathbf{k}
\end{aligned}$$

and

$$|\gamma_1'(0)| = 1, \qquad |\gamma_2'(0)| = \rho_0, \qquad |\gamma_3'(0)| = 1,$$

so that

$$\nabla f = \frac{\partial f}{\partial \rho}\boldsymbol{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\boldsymbol{\phi} + \frac{\partial f}{\partial z}\mathbf{k}.$$

DIVERGENCE. For this section we shall work exclusively in $\mathbf{R}^3$. Recall that the *divergence* of a vector field $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$ in $\mathbf{R}^3$ is defined by

$$\operatorname{div}\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z},$$

and that we have the *divergence theorem*

$$\iiint_V \operatorname{div}\mathbf{F}\,dV = \iint_{\partial V} \mathbf{F}\cdot\mathbf{n}\,dS,$$

where $\mathbf{n}$ represents the outwards unit normal to the boundary ∂V of V.

We also note for future reference that, given a general coordinate system $\{y^j\}$ as above, the *area element* in a surface of constant coordinate y^j is given by

$$A_j := |\gamma_i' \times \gamma_k'|,$$

where i and k are the two elements of $\{1, 2, 3\}$ not equal to j. Thus, for example, in the case of spherical coordinates (recalling the formula $|\mathbf{A}\times\mathbf{B}| = |A||B|\sin\theta_{\mathbf{AB}}$, where $\theta_{\mathbf{AB}}$ is the angle between $\mathbf{A}$ and $\mathbf{B}$, and that the vectors γ_j' are all mutually orthogonal so $\sin\theta_{\gamma_i'\gamma_k'} = 1$ for all i and k, so that $|\gamma_i' \times \gamma_j'| = |\gamma_i'||\gamma_j'|$; this formula makes sense when we consider that we are taking the area of a small rectangle whose sides have length $|\gamma_i'|$ and $|\gamma_j'|$), the area elements in surfaces of constant r, θ, and ϕ are given respectively by

$$\begin{aligned}
|\gamma_2' \times \gamma_3'| &= r^2|\cos\theta\cos\phi\mathbf{i} + \cos\theta\sin\phi\mathbf{j} - \sin\theta\mathbf{k}||-\sin\theta\sin\phi\mathbf{i} + \sin\theta\cos\phi\mathbf{j}| \\
&= r^2\sin\theta, \\
|\gamma_1' \times \gamma_3'| &= |\sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k}||-r\sin\theta\sin\phi\mathbf{i} + r\sin\theta\cos\phi\mathbf{j}| \\
&= r\sin\theta, \\
|\gamma_1' \times \gamma_2'| &= |\sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k}||r\cos\theta\cos\phi\mathbf{i} + r\cos\theta\sin\phi\mathbf{j} - r\sin\theta\mathbf{k}| \\
&= r.
\end{aligned}$$

Let us now return to the case of a general coordinate system, but still assume it to be orthogonal (meaning that the vectors $\mathbf{y}_j$ are mutually orthogonal at all points of $\mathbf{R}^3$), pick some point $\mathbf{x}_0 \in \mathbf{R}^3$ with coordinates (y_0^i), and apply the divergence theorem to the small curvilinear cube given by

$$V = [y_0^1, y_0^1 + \Delta y^1] \times [y_0^2, y_0^2 + \Delta y^2] \times [y_0^3, y_0^3 + \Delta y^3].$$

Then, by the change-of-variables formula and the mean value theorem for integrals, there will be some point (y_*^i) in this cube such that

$$\iiint_V \operatorname{div} \mathbf{F} dV = \operatorname{div} \mathbf{F}(y_*^i) J \Delta y^1 \Delta y^2 \Delta y^3,$$

where J is the Jacobian of the coordinate transformation $\mathbf{x} \mapsto \mathbf{y}$; we note that $J = |\gamma_1' \cdot (\gamma_2' \times \gamma_3')| = |\gamma_1'||\gamma_2'||\gamma_3'|$, since the vectors are all orthogonal.

Let us now consider the right-hand side of the divergence theorem. The cube given above has evidently six faces; these can be grouped into three pairs, the treatment of each of which is analogous. Let us work with the pair

$$\{y_0^1\} \times [y_0^2, y_0^2 + \Delta y^2] \times [y_0^3, y_0^3 + \Delta y^3] \cup \{y_0^1 + \Delta y^1\} \times [y_0^2, y_0^2 + \Delta y^2] \times [y_0^3, y_0^3 + \Delta y^3].$$

The unit normal vector on the second part of this pair will simply be the vector $\mathbf{y}_1$, while that on the first will be (since we need the *outer* normal in the divergence theorem) $-\mathbf{y}_1$; thus the integral on the right-hand side of the divergence theorem corresponding to these two surfaces is equal to (we let $F^j = \mathbf{y}^j \cdot \mathbf{F}$)

$$\begin{aligned}
&\int_{y_0^2}^{y_0^2+\Delta y^2} \int_{y_0^3}^{y_0^3+\Delta y^3} F^1(y_0^1 + \Delta y^1, y^2, y^3) A_1(y_0^1 + \Delta y^1, y^2, y^3) dy^3 dy^2 \\
&\qquad - \int_{y_0^2}^{y_0^2+\Delta y^2} \int_{y_0^3}^{y_0^3+\Delta y^3} F^1(y_0^1, y^2, y^3) A_1(y_0^1, y^2, y^3) dy^3 dy^2 \\
&= \int_{y_0^2}^{y_0^2+\Delta y^2} \int_{y_0^3}^{y_0^3+\Delta y^3} \left(F^1 A_1\right)(y_0^1 + \Delta y^1, y^2, y^3) - \left(F^1 A_1\right)(y_0^1, y^2, y^3) dy^2 dy^3 \\
&= \int_{y_0^2}^{y_0^2+\Delta y^2} \int_{y_0^3}^{y_0^3+\Delta y^3} \left.\frac{\partial \left(F^1 A_1\right)}{\partial y^1}\right|_{(y_0^1, y^2, y^3)} \Delta y^1 + o(\Delta y^1) dy^2 dy^3 \\
&= \left(\left.\frac{\partial \left(F^1 A_1\right)}{\partial y^1}\right|_{(y_0^1, y_*^2, y_*^3)} \Delta y^1 + o(\Delta y^1) \right) \Delta y^2 \Delta y^3,
\end{aligned}$$

where $o(h)$ represents a quantity which satisfies

$$\lim_{h \to 0} \frac{o(h)}{h} = 0,$$

and we have again used the mean value theorem for integrals. (Here, and below, in order to keep the notation from becoming too cumbersome we shall use (y_*^i) to denote any point that lies in the above cube; it may represent multiple different points on the same line. This will not ultimately cause any troubles since we will take a limit which forces $(y_*^i) \to (y_0^i)$ at the end.) The other two pairs are treated similarly, giving rise finally to the equation

$$\begin{aligned}
\operatorname{div} \mathbf{F}(y_*^i) J \Delta y^1 \Delta y^2 \Delta y^3 = &\left(\left.\frac{\partial \left(F^1 A_1\right)}{\partial y^1}\right|_{(y_0^1, y_*^2, y_*^3)} \Delta y^1 + o(\Delta y^1) \right) \Delta y^2 \Delta y^3 \\
&+ \left(\left.\frac{\partial \left(F^2 A_2\right)}{\partial y^2}\right|_{(y_*^1, y_0^2, y_*^3)} \Delta y^2 + o(\Delta y^2) \right) \Delta y^1 \Delta y^3 \\
&+ \left(\left.\frac{\partial \left(F^3 A_3\right)}{\partial y^3}\right|_{(y_*^1, y_*^2, y_0^3)} \Delta y^3 + o(\Delta y^3) \right) \Delta y^1 \Delta y^2.
\end{aligned}$$

If we now divide through by $J\Delta y^1\Delta y^2\Delta y^3$ and take the limit as $\Delta y^1, \Delta y^2, \Delta y^3 \to 0$,[1] we obtain finally the expression (since all points (y^i_*) must go to (y^i_0) in this limit)

$$\operatorname{div}\mathbf{F} = \frac{1}{J}\left(\frac{\partial\left(F^1A_1\right)}{\partial y^1} + \frac{\partial\left(F^2A_2\right)}{\partial y^2} + \frac{\partial\left(F^3A_3\right)}{\partial y^3}\right).$$

In particular, in spherical coordinates we have

$$J = r^2\sin\theta, \qquad A_1 = r^2\sin\theta, \qquad A_2 = r\sin\theta, \qquad A_3 = r,$$

whence we obtain (writing $\mathbf{F} = F_r\mathbf{r} + F_\theta\boldsymbol{\theta} + F_\phi\boldsymbol{\phi}$)

$$\begin{aligned}\operatorname{div}\mathbf{F} &= \frac{1}{r^2\sin\theta}\left(\frac{\partial\left(r^2\sin\theta F_r\right)}{\partial r} + \frac{\partial\left(r\sin\theta F_\theta\right)}{\partial\theta} + \frac{\partial\left(rF_\phi\right)}{\partial\phi}\right)\\ &= \frac{\partial F_r}{\partial r} + \frac{2}{r}F_r + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta} + \frac{1}{r}\cot\theta F_\theta + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}.\end{aligned}$$

Similarly, for cylindrical coordinates we have the area elements

$$\begin{aligned}A_1 &= |-\rho\sin\phi\mathbf{i} + \rho\cos\phi\mathbf{j}||\mathbf{k}| = \rho,\\ A_2 &= |\cos\phi\mathbf{i} + \sin\phi\mathbf{j}||\mathbf{k}| = 1,\\ A_3 &= |\cos\phi\mathbf{i} + \sin\phi\mathbf{j}||-\rho\sin\phi\mathbf{i} + \rho\cos\phi\mathbf{j}| = \rho,\end{aligned}$$

while $J = r$; thus we have the formula (writing $\mathbf{F} = F_\rho\boldsymbol{\rho} + F_\phi\boldsymbol{\phi} + F_z\mathbf{k}$)

$$\begin{aligned}\operatorname{div}\mathbf{F} &= \frac{1}{\rho}\left(\frac{\partial\left(\rho F_\rho\right)}{\partial\rho} + \frac{\partial F_\phi}{\partial\phi} + \frac{\partial\left(\rho F_z\right)}{\partial z}\right)\\ &= \frac{\partial F_\rho}{\partial\rho} + \frac{1}{\rho}F_\rho + \frac{1}{\rho}\frac{\partial F_\phi}{\partial\phi} + \frac{\partial F_z}{\partial z}.\end{aligned}$$

Finally, putting all of this together with the expressions for the gradients derived above gives the following expressions for the Laplacian in spherical and cylindrical coordinates:

$$\begin{aligned}\nabla^2 u &= \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2},\\ \nabla^2 u &= \frac{\partial^2 u}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\phi^2} + \frac{\partial^2 u}{\partial z^2}.\end{aligned}$$

SEPARATION OF VARIABLES IN SPHERICAL COORDINATES. Consider now Laplace's equation in spherical coordinates,

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0. \tag{2}$$

As we did when treating Laplace's equation in rectangular coordinates, we begin by seeking simple solutions of the form

$$u = R(r)\Theta(\theta)\Phi(\phi),$$

[1]Note that there is one other subtle point which must be dealt with here, namely whether the quantities $\frac{o(\Delta y^1)}{\Delta y^1}$ etc. go to zero uniformly in the other Δy^i. They will if we assume that the vector field $\mathbf{F}$ possesses continuous second-order derivatives.

in the hopes that the general solution can be expressed in a series of such solutions. Substituting this into equation (2) and dividing by u, we obtain (here prime denotes differentiation with respect to the whatever single variable the function depends on; e.g., $R' = \frac{dR}{dr}$)

$$\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \frac{1}{r^2}\left(\frac{\Theta''}{\Theta} + \cot\theta\frac{\Theta'}{\Theta} + \frac{1}{\sin^2\theta}\frac{\Phi''}{\Phi}\right) = 0.$$

Now we see that of all the terms on the left-hand side, only $\frac{\Phi''}{\Phi}$ depends on ϕ; hence it must be constant. (Somewhat more explicitly, note that we may solve the above equation for $\frac{\Phi''}{\Phi}$, obtaining

$$\frac{\Phi''}{\Phi} = -\sin^2\theta\left(r^2\frac{R''}{R} + 2r\frac{R'}{R} + \frac{\Theta''}{\Theta} + \cot\theta\frac{\Theta'}{\Theta}\right);$$

now the right-hand side of the above expression does not depend on ϕ, and hence neither can the left-hand side, i.e., $\frac{\Phi''}{\Phi}$ is constant, as claimed.) We would like to know something about this constant before proceeding further. Suppose that we are interested in solving Laplace's equation on a ball (the interior of a sphere): then the solution must be valid, continuous, and single-valued for all values of the angle ϕ. Since increasing ϕ by 2π leaves us at the same point, out solution must be periodic in ϕ with angle 2π. Since Φ is the only part of the solution depending on ϕ, this means that Φ must itself be periodic with period 2π. Now we know that if $\frac{\Phi''}{\Phi}$ is positive, then Φ will be a linear combination of sinh and cosh, and hence will *not* be periodic; thus $\frac{\Phi''}{\Phi}$ must be zero or negative. If it is zero, then it must be of the form $a + b\phi$; again, ϕ is not periodic, and hence we must have $b = 0$, i.e., in this case Φ must be a constant. (This corresponds to what is called an *azimuthally symmetric* solution; we shall have more to say about this when we discuss Legendre's equation and Legendre polynomials shortly.) Otherwise, $\frac{\Phi''}{\Phi}$ must be negative, and we may write it as $-m^2$ for some positive real number m. (Choosing $m > 0$ is simply a convention; we could as well have chosen $m < 0$; but we cannot have both. Here we choose $m > 0$.) Thus $\Phi'' = -m^2\Phi$, which has as a general solution $\Phi_m = a_m\cos m\phi + b_m\sin m\phi$. Since Φ_m must have period 2π (general periodicity is not enough), we must actually have $m \in Z$. Thus the ϕ dependence of our solution will be of the form $a_m\cos m\phi + b_m\sin m\phi$ (note that we could also have used the complex basis $e^{im\phi}$).[2]

Substituting $\frac{\Phi''}{\Phi} = -m^2$ back into Laplace's equation, and multiplying by r^2, we obtain

$$r^2\frac{R''}{R} + 2r\frac{R'}{R} + \left(\frac{\Theta''}{\Theta} + \cot\theta\frac{\Theta'}{\Theta} - \frac{m^2}{\sin^2\theta}\right) = 0.$$

Again, the first of these two terms depends only on R, and the second depends only on Θ, which means (as with Φ) that each of them must be constant. Let us let α[3] denote the term in parentheses, so that we obtain for R the equation

$$r^2\frac{R''}{R} + 2r\frac{R'}{R} = -\alpha,$$

or

$$r^2R'' + 2rR' + \alpha R = 0.$$

[2]As hinted above, and mentioned in somewhat greater detail in class, this form for Φ is contingent on the region over which we are solving containing a full range of angles ϕ. Should we be solving only on a wedge, for example, then not only would we no longer necessarily have $m \in Z$, we might actually need to consider also the exponential solutions for Φ – at least in principle. In this case, we would need boundary conditions on the constant-ϕ boundaries, much as we have boundary conditions on the constant-y and constant-x boundaries in the problems we have done in rectangular coordinates. For the moment, though, to keep the discussion simple, we shall stick with this form for Φ.

[3]It would be more natural to denote this constant by $-\alpha$, but since the author was careless and denoted it by α in the lecture, it seems prudent to keep that convention here. At any rate, as noted in lecture and as will be pointed out shortly, α itself is not really the fundamental quantity; ℓ is much more fundamental.

The form of this equation suggests that it should possess power-law solutions; thus let us try to find solutions of the form $R = r^\ell$, for some ℓ (which at present we know nothing about). Substituting this expression in for R, we obtain

$$\begin{aligned} r^2\left(\ell(\ell-1)r^{\ell-2}\right) + 2r\left(\ell r^{\ell-1}\right) + \alpha r^\ell &= \ell(\ell-1)r^\ell + 2\ell r^\ell + \alpha r^\ell \\ &= \left[\ell(\ell+1)+\alpha\right]r^\ell = 0, \end{aligned}$$

from which we see that ℓ must satisfy the equation $\ell(\ell+1) = -\alpha$. This is a quadratic equation with solutions

$$\ell = -\frac{1}{2} \pm \frac{1}{2}\left(1-4\alpha\right)^{\frac{1}{2}}.$$

(Note that these may be complex.) From this we obtain also the result that if ℓ is one solution to $\ell(\ell+1) = -\alpha$, then $-(\ell+1)$ is the other solution. Thus in general we have the solution

$$R_\ell = a_\ell r^\ell + b_\ell r^{-(\ell+1)}.$$

Repeated roots occur when $\alpha = \frac{1}{4}$; and if $\alpha > \frac{1}{4}$ the roots will be complex: while the expressions r^ℓ and $r^{-(\ell+1)}$ can still be defined in this case, they are not as simple. For reasons which shall become apparent when we study Legendre's equation in a moment, we are interested mostly in cases in which ℓ is a nonnegative integer. Thus (as with our choice for m above) we shall for the moment restrict to this case. Thus we consider only α which are of the form $-\ell(\ell+1)$ for some $\ell \in \mathbf{Z}$, $\ell \geq 0$. (It is because of this that we said above that ℓ is more fundamental than α, so that our use of α instead of $-\alpha$ was not that important.)

Having solved the equations for Φ and R, let us now treat the equation for Θ. This is the most interesting of them all and will introduce us to the field of *orthogonal polynomials* through the so-called Legendre polynomials.

Setting $\alpha = -\ell(\ell+1)$, we see that we obtain for Θ the equation

$$\Theta'' + \cot\theta\Theta' + \left(\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right)\Theta = 0. \tag{3}$$

Unfortunately, as it stands there is no clear way to approach this equation, since while it is a second-order linear ordinary differential equation it has variable coefficients. It turns out to be useful to make the change of variables $x = \cos\theta$ (here x does *not* refer to the Cartesian coordinate corresponding to the spherical coordinate system we are using – that would be $r\sin\theta\cos\phi$); note that this implies that $x \in [-1,1]$. For this change of variables, the chain rule gives (for some function f)

$$\begin{aligned} \frac{df}{d\theta} &= \frac{df}{dx}\frac{dx}{d\theta} = -\sin\theta\frac{df}{dx}, \\ \frac{d^2f}{d\theta^2} &= \frac{d}{d\theta}\left(-\sin\theta\frac{df}{dx}\right) \\ &= -\cos\theta\frac{df}{dx} - \sin\theta\left(-\sin\theta\frac{d^2f}{dx^2}\right) = -x\frac{df}{dx} + (1-x^2)\frac{d^2f}{dx^2}, \end{aligned}$$

whence we see that equation (3) becomes, letting $P(x)$ be the function of x corresponding to $\Theta(\theta)$ (and since $\cot\theta\sin\theta = \cos\theta = x$ in the second term in that equation)

$$(1-x^2)\frac{d^2P}{dx^2} - x\frac{dP}{dx} - x\frac{dP}{dx} + \left(\ell(\ell+1) - \frac{m^2}{1-x^2}\right)P = (1-x^2)P'' - 2xP' + \left(\ell(\ell+1) - \frac{m^2}{1-x^2}\right)P = 0.$$

This equation is called *Legendre's equation*, and the solutions for nonnegative integers ℓ and m are called the *associated Legendre functions*. Since $x = \cos\theta \in [-1,1]$, it is an equation on $[-1,1]$.

Let us consider the special case $m = 0$; in this case there is no ϕ dependence and our solution is azimuthally symmetric. The equation in this case is simply

$$(1-x^2)P'' - 2xP' + \ell(\ell+1)P = 0. \tag{4}$$

We shall look for a solution P which has a power series expansion around $x = 0$; in other words, we look for a solution

$$P = \sum_{n=0}^{\infty} a_n x^n.$$

Substituting this expression in to the above equation, we obtain

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} (1-x^2) a_n n(n-1) x^{n-2} - 2xna_n x^{n-1} + \ell(\ell+1) a_n x^n \\ &= \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - a_n n(n-1)x^n - 2na_n x^n + \ell(\ell+1)a_n x^n \\ &= \sum_{n=0}^{\infty} \left(a_{n+2}(n+2)(n+1) + a_n \left(\ell(\ell+1) - n(n+1)\right)\right) x^n, \end{aligned}$$

from which we obtain the recurrence relation

$$a_{n+2} = a_n \frac{n(n+1) - \ell(\ell+1)}{(n+2)(n+1)}.$$

We see that this will determine all even coefficients a_{2k} given a_0, and all odd coefficients a_{2k+1} given a_1; since we started with a second-order differential equation, it is natural that we have two undetermined coefficients. (Another way of looking at it is to think of a_0 and a_1 as being the coefficients in the linear combination giving the general solution to the equation.) Moreover, if $a_0 = 0$, then all even coefficients will vanish, and if $a_1 = 0$ then all odd coefficients will vanish.

We note something else about this recurrence relation: If $n(n+1) = \ell(\ell+1)$ for some n, then a_{n+2} and hence a_{n+2k} for all $k > 0$ will vanish. This means that if $n(n+1) = \ell(\ell+1)$ for some odd integer n, then there will be only finitely many odd-power terms in the power series, while if $n(n+1) = \ell(\ell+1)$ for some even integer n there will be only finitely many even-power terms in the power series. In either case, by requiring the terms of opposite valence to vanish (i.e., setting $a_0 = 0$ in the first case and $a_1 = 0$ in the second case), we obtain power series solutions which are *finite* – which is to say, polynomial solutions. These are called the *Legendre polynomials*.

Let us be more specific. Suppose that $\ell = 2k$ for some $k \in \mathbf{Z}$, $k \geq 0$, and let $a_1 = 0$. Then, as noted above, all odd coefficients in the series will vanish. Moreover,

$$a_{2k+2} = a_{2k} \frac{2k(2k+1) - \ell(\ell+1)}{(2k+2)(2k+1)} = a_{2k} \frac{2k(2k+1) - 2k(2k+1)}{(2k+2)(2k+1)} = 0,$$

and thus $a_{2k+2j} = 0$ for all $j \in \mathbf{Z}$, $j > 0$. Since all odd-order coefficients vanish, the power series will truncate and we will be left with a polynomial of degree $2k = \ell$.

Similarly, suppose that $\ell = 2k+1$ for some $k \in \mathbf{Z}$, $k \geq 0$, and let now $a_0 = 0$. Then all even terms vanish; moreover, as before,

$$a_{2k+3} = a_{2k+1} \frac{(2k+1)(2k+2) - \ell(\ell+1)}{(2k+3)(2k+2)} = a_{2k+1} \frac{(2k+1)(2k+2) - (2k+1)(2k+2)}{(2k+3)(2k+2)} = 0,$$

so $a_{2k+1+2j} = 0$ for all $j \in \mathbf{Z}$, $j > 0$, and our power series truncates to give a polynomial of order $2k+1 = \ell$.

Thus we see that whenever ℓ is a nonnegative integer, equation (4) will have a solution which is a polynomial of degree ℓ. It is determined up to an overall multiplicative factor. We denote by $P_\ell(x)$ the polynomial satisfying (4) and satisfying also $P_\ell(1) = 1$; this will fix the value of a_0 (ℓ even) or a_1 (ℓ odd), which we left open above. $P_\ell(x)$ is called the *ℓth Legendre polynomial*, or the *Legendre polynomial of degree ℓ*.[4]

[4]Since our original equation was second-order, even in the case where ℓ is a nonnegative integer it will possess another solution linearly independent of $P_\ell(x)$; this would correspond to letting the other one of a_0 or a_1 equal something nonzero. Since it turns out that the set of Legendre polynomials is complete on the interval $[-1, 1]$, they are sufficient for our purposes at the moment.

EXAMPLES. Let us compute the first few Legendre polynomials. If $\ell = 0$, we seek a polynomial of degree 0, i.e., a constant polynomial, with $P_0(1) = 1$; thus $P_0(x) = 1$ for all x. If $\ell = 1$, then we set $a_0 = 0$ and leave a_1 undetermined for the moment; but then $a_3 = 0$, so $P_1(x) = a_1 x$ and the normalisation condition $P_1(1) = 1$ implies that $a_1 = 1$.

The case $\ell = 2$ is a bit more interesting. In this case we set $a_1 = 1$ and leave a_0 undetermined; then we have

$$a_2 = a_0 \frac{0(0+1) - 2(2+1)}{(0+2)(0+1)} = -3a_0,$$

while a_4 and all higher-order coefficients vanish. Thus $P_2(x) = -3a_0 x^2 + a_0$, so $P_2(1) = -2a_0 = 1$ forces $a_0 = -\frac{1}{2}$ and $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$.

EXAMPLES OF SOLUTIONS TO LAPLACE'S EQUATION. Let us see how all of these results may be pulled together to give some simple solutions to Laplace's equation on the unit sphere.

(a) Solve the boundary-value problem on the unit boll $\{(r, \theta, \phi) | r < 1\}$:

$$\nabla^2 u = 0, \qquad u|_{r=1} = 1.$$

Since the boundary data and the region are both spherically symmetric, we anticipate that the solution will be as well, meaning that we expect a solution depending only on r; this is equivalent to looking for a separated solution with Θ and Φ both constant, which means (in the context of what we have just done) that $m = \ell = 0$. In this case, the equation for R becomes simply

$$r^2 R'' + 2rR' = 0,$$

and by our previous work this has general solution $R = a + \frac{b}{r}$, and this will also be the form of our solution u. Since we wish u to satisfy $\nabla^2 u = 0$ everywhere on the interior of the unit sphere, u must in particular be continuous and finite there, and thus we must have $b = 0$, so $u = a$ is just a constant. The boundary condition then gives $a = 1$, so the solution to this boundary-value problem is simply $u = 1$. (We could also have obtained this by inspection.)

(b) Solve the boundary-value problem on the set $\{(r, \theta, \phi) | 1 < r < 2\}$:

$$\nabla^2 u = 0, \qquad u|_{r=1} = 1, \quad u|_{r=2} = 0.$$

In this case we still have a spherically symmetric region and spherically symmetric boundary data, so we expect to obtain a spherically symmetric solution. By our work in part (a), we see immediately that we must have $u = a + \frac{b}{r}$ for some constants a, b. In this case we can no longer immediately set $b = 0$ since the point $r = 0$ (which is where the second term goes to infinity) is not in the region where we require $\nabla^2 u = 0$. This allows us to fit both boundary conditions, as follows. We have

$$\begin{aligned} u|_{r=1} &= a + b = 1 \\ u|_{r+2} &= a + \frac{b}{2} = 0, \end{aligned}$$

from which we see easily that $b = 2$, $a = -1$, so $u = -1 + \frac{2}{r}$ is the solution to the boundary-value problem.

(c) Solve the boundary-value problem on the unit ball:

$$\nabla^2 u = 0, \qquad u|_{r=1} = \cos\theta.$$

In this case we no longer have spherical symmetry, though we do have azimuthal symmetry, meaning that our solution will not depend on ϕ. In general, our approach to solving this type of problem is very similar to our approach for solving boundary-value problems on a square: we suppose that our solution can be written as a series of separated solutions, in this case

$$u(r, \theta, \phi) = \sum_{\ell=0}^{\infty} \left(a_\ell r^\ell + b_\ell r^{-(\ell+1)} \right) P_\ell(\cos\theta);$$

in the present case, as in (a), since we wish u to satisfy Laplace's equation on the unit ball, we must set all of the b_ℓ to zero (this is similar to how we used the boundary conditions to require that the coefficients of all of the cosine terms vanished when we solved Laplace's equation on the unit square, although the reason is different). We then apply the remaining boundary condition:

$$u(1, \theta, \phi) = \sum_{\ell=0}^{\infty} a_\ell P_\ell(\cos\theta) = \cos\theta,$$

and try to determine a_ℓ. We shall see soon that $\{P_\ell(x) | \ell \in \mathbf{Z}, \ell \geq 0\}$ forms an orthogonal set on $[-1, 1]$; it is also complete (though we shall not prove this at present), and thus for any reasonable boundary data $u(1, \theta)$ it will always be possible to find coefficients a_ℓ satisfying the above equation – and moreover these coefficients will be unique. At present it is sufficient to note that $\cos\theta = P_1(\cos\theta)$, so that we may take simply $a_1 = 1$, $a_\ell = 0$, $\ell \neq 1$ (note that this a_1 is completely different from the a_1 we had above when we investigated the power series representation of P_ℓ!). The final solution is then simply $u = rP_1(\cos\theta) = r\cos\theta = z$.

Another way of looking at the above description of our method is as follows. Suppose that we are solving on the unit ball and given boundary data $P_\ell(\cos\theta)$ on the unit sphere. Then we know that the corresponding radial solution is $ar^\ell + br^{-(\ell+1)}$, but we reject the second term (i.e., set $b = 0$) since this term is not continuous on the unit ball; thus our solution must be of the form $ar^\ell P_\ell(\cos\theta)$, and since our boundary data is $P_\ell(\cos\theta)$ on the unit ball, we must have $a = 1$, and our solution is $r^\ell P_\ell(\cos\theta)$. (Were we given the same boundary data, but on the ball $\{(r, \theta, \phi) | r \leq r_0\}$, then we would need $ar_0^\ell = 1$, so we would set $a = r_0^{-\ell}$ and our solution would be $\left(\frac{r}{r_0}\right)^\ell P_\ell(\cos\theta)$.) If our boundary data is a linear combination (or a series) of P_ℓ for different ℓ, then this method may be applied to each term in the linear combination, and then sum the results to get the full solution. In the case where our boundary data is a series in the P_ℓ, we must use methods of orthogonal functions to determine the coefficients, as we did when solving Laplace's equation in rectangular coordinates. We shall discuss this in more detail later.

The moral of the story is: boundary data $P_\ell(\cos\theta)$ gives rise to a solution of the form

$$\left(ar^\ell + br^{-(\ell+1)}\right) P_\ell(\cos\theta),$$

with a and b to be determined from the other requirements in the problem, and general boundary data may be treated by linearity. This is analogous to how the initial data $\sin kx$ leads to a solution $\sin kx e^{-k^2 Dt}$ to the heat equation, as we discussed in the first week of class, or to how boundary data $\sin n\pi x$ leads to a solution $\sin n\pi x\, (a\sinh n\pi y + b\cosh n\pi y)$ to Laplace's equation on the unit square.

For more complicated problems, such as those on Homework 4, variants and combinations of the above methods may be used.

APM 346 (Summer 2019), Homework 4 solutions.

APM 346, Homework 4. Due Monday, June 3, at 6.00 AM EDT. To be marked completed/not completed.

Consider the following boundary-value problem on $[0,1]\times[0,1]$:

$$\nabla^2 u = 0 \quad \text{on} \quad (0,1)\times(0,1), \qquad u(0,y)=0, \qquad u_x(1,y) = -u(1,y),$$
$$u(x,0) = \sin n\pi x, \qquad u(x,1) = \cos n\pi x,$$

where $n \in \mathbf{Z}$, $n > 0$ is some fixed positive integer.

1. Determine all separated solutions satisfying the homogeneous boundary conditions (these are the boundary conditions on $x = 0$ and $x = 1$ above).

We are looking for solutions of the form $u(x,y) = X(x)Y(y)$; substituting this in to Laplace's equation gives as usual

$$X''(x)Y(y) + X(x)Y''(y) = 0,$$

whence dividing by u gives

$$\frac{X''}{X} + \frac{Y''}{Y} = 0.$$

Since the first term depends only on x, and the second term only on y, the must individually be constant; since the data on the top and bottom edges of the boundary is oscillatory in x, we take X to be the oscillatory solution and Y to be the exponential one. Thus we have the equations

$$X'' = -\lambda^2 X, \qquad Y'' = \lambda^2 Y,$$

where we may take $\lambda > 0$. The boundary conditions on the left and right sides of the boundary give

$$u(0,y) = X(0)Y(y) = 0,$$
$$\partial_x u(1,y) = X'(1)Y(y) = -X(1)Y(y),$$

and since we cannot have Y identically zero we must have $X(0) = 0$, $X'(1) = -X(1)$. Thus X must satisfy the ordinary differential equation we have studied in Homework 2 and Homework 3; this means that we may take (we shall write the arbitrary constants in Y, as we did in the boundary-value problem we did in lecture) $X = \sin \lambda x$, where λ satisfies $\lambda = -\tan\lambda$. We must also have $Y = a_\lambda \sinh \lambda y + b_\lambda \cosh \lambda y$.

2. Assuming that the functions of x appearing in the separated solutions in 1 form a complete set on $[0,1]$, write out the general solution to $\nabla^2 u = 0$ satisfying the first three boundary conditions above.

By Homework 3 we know that the set $\{\sin\lambda x | \lambda = -\tan\lambda\}$ is orthogonal on $[0,1]$, and we now assume (per the statement of the problem) that it is complete. This means that we can write the whole solution as a series in the separated solutions, i.e. (letting $\{\lambda_k\}_{k=1}^\infty$ denote the set of solutions to $\lambda = -\tan\lambda$ – which is easily seen to be discrete –)

$$u(x,y) = \sum_{k=1}^\infty (a_k \sinh \lambda_k y + b_k \cosh \lambda_k y) \sin \lambda_k x,$$

whence

$$\sin n\pi x = u(x,0) = \sum_{k=1}^\infty b_k \sin \lambda_k x,$$

and by our general results about expansions in complete orthogonal sets,

$$b_k = \frac{\int_0^1 \sin n\pi x \sin \lambda_k x \, dx}{\int_0^1 \sin^2 \lambda_k x \, dx} = \frac{\frac{1}{2}\left[\frac{1}{n\pi - \lambda_k}\sin\left(n\pi - \lambda_k\right) - \frac{1}{n\pi+\lambda_k}\sin\left(n\pi + \lambda_k\right)\right]}{\frac{1}{2} - \frac{\sin 2\lambda_k}{4\lambda_k}}$$
$$= \frac{4n\pi(-1)^{n+1}\sin\lambda_k}{(n^2\pi^2 - \lambda_k^2)\left(2 - \frac{\sin 2\lambda_k}{\lambda_k}\right)},$$

so that the general solution satisfying the first three boundary conditions is

$$u(x,y) = \sum_{k=1}^{\infty} \left(a_k \sinh \lambda_k y + \frac{4n\pi(-1)^{n+1} \sin \lambda_k}{(n^2\pi^2 - \lambda_k^2)\left(2 - \frac{\sin 2\lambda_k}{\lambda_k}\right)} \cosh \lambda_k y \right) \sin \lambda_k x.$$

3. Finally, determine the unique solution to the full boundary-value problem.
Finally, we must have (writing b_k for the coefficient just given, for simplicity)

$$\cos n\pi x = u(x,1) = \sum_{k=1}^{\infty} (a_k \sinh \lambda_k + b_k \cosh \lambda_k) \sin \lambda_k x,$$

whence as in 2 we have, by our general results about expansions in complete sets of orthogonal functions,

$$\begin{aligned}
a_k \sinh \lambda_k + b_k \cosh \lambda_k &= \frac{4}{2 - \frac{\sin 2\lambda_k}{\lambda_k}} \left(\int_0^1 \cos n\pi x \sin \lambda_k x \, dx \right) \\
&= -\frac{2}{2 - \frac{\sin 2\lambda_k}{\lambda_k}} \left(\frac{1}{n\pi + \lambda_k} (\cos(\lambda_k + n\pi) - 1) + \frac{1}{\lambda_k - n\pi} (\cos(\lambda_k - n\pi) - 1) \right) \\
&= -\frac{2}{2 - \frac{\sin 2\lambda_k}{\lambda_k}} ((-1)^n \cos \lambda_k - 1) \frac{2\lambda_k}{\lambda_k^2 - \pi^2 n^2},
\end{aligned}$$

whence using the result from 2 and solving for a_k, we obtain

$$\begin{aligned}
a_k &= -\coth \lambda_k \left(\frac{4n\pi(-1)^{n+1} \sin \lambda_k}{(n^2\pi^2 - \lambda_k^2)\left(2 - \frac{\sin 2\lambda_k}{\lambda_k}\right)} \right) - \frac{2}{\left(2 - \frac{\sin 2\lambda_k}{\lambda_k}\right) \sinh \lambda_k} \frac{2\lambda_k ((-1)^n \cos \lambda_k - 1)}{\lambda_k^2 - \pi^2 n^2} \\
&= \frac{4}{\left(2 - \frac{\sin 2\lambda_k}{\lambda_k}\right)(n^2\pi^2 - \lambda_k^2) \sinh \lambda_k} \left(n\pi(-1)^n \sin \lambda_k \cosh \lambda_k + \lambda_k ((-1)^n \cos \lambda_k - 1) \right),
\end{aligned}$$

and the final solution is given by the wonderful and marvelous expression

$$\begin{aligned}
u(x,y) = \sum_{k=1}^{\infty} \Bigg(& \frac{4\left(n\pi(-1)^n \sin \lambda_k \cosh \lambda_k + \lambda_k ((-1)^n \cos \lambda_k - 1)\right)}{\left(2 - \frac{\sin 2\lambda_k}{\lambda_k}\right)(n^2\pi^2 - \lambda_k^2) \sinh \lambda_k} \sinh \lambda_k y \\
& + \frac{4n\pi(-1)^{n+1} \sin \lambda_k}{(n^2\pi^2 - \lambda_k^2)\left(2 - \frac{\sin 2\lambda_k}{\lambda_k}\right)} \cosh \lambda_k y \Bigg) \sin \lambda_k x.
\end{aligned}$$

[Note. In 2 and 3, if you wish to use orthogonality of a certain set of functions, you must say how you know it is orthogonal (for example, by citing a specific result you have seen earlier in the course, or by giving a proof).
As noted above, orthogonality follows from Homework 3.]

The next two problems deal with Laplace's equation in spherical coordinates.

4. Consider the boundary-value problem on the region given by $\{(r, \theta, \phi) | 1 \le r \le 2\}$:

$$\nabla^2 u = 0, \quad 1 < r < 2, \quad u(r = 1) = 1, u_r(r = 2) = -u(r = 2).$$

Using our work with the Laplace equation in class, find the solution to this problem. [Hint: it depends only on r, not on θ or ϕ.]

Since the boundary data depends only on r, we posit a solution of the form $u = u(r)$; substituting this into our expression for Laplace's equation in spherical coordinates, we see that u must satisfy

$$u'' + \frac{2}{r}u' = 0,$$

and by our work with Laplace's equation we see that we are looking for a separated solution with $\ell = m = 0$, which means that it must be of the form $u = a + \frac{b}{r}$ for some constants a and b. (This can also be obtained directly from the above equation, without recourse to our more general work in class, of course.) The boundary conditions then give

$$\begin{aligned} u(1) &= a + b = 1, \\ u_r(2) &= -\frac{b}{r^2}\bigg|_{r=2} = -\frac{b}{4} = -u(2) = -a - \frac{b}{2}, \end{aligned}$$

whence we see that $b = -4a$, $a = -\frac{1}{3}$, $b = \frac{4}{3}$. Thus the solution is $u = -\frac{1}{3} + \frac{4}{3r}$.

5. Consider the same problem as in 4, but with the second boundary condition replaced by $u(r = 2) = \cos\theta$. Find the solution to this problem. [Hint: it can be written as a sum of two separated solutions.]

By the hint, we are looking for a solution which is a sum of two separated solutions. Now on the inner boundary we have simply $u = 1$, which does not depend on either θ or ϕ and thus (as in fact we saw in 4) looks like the kind of condition which can be fit with a solution having $m = \ell = 0$, while on the outer boundary we have $u = \cos\theta = P_1(\cos\theta)$, which can be fit with a solution having $m = 0$, $\ell = 1$. Thus we look for a solution of the form

$$u(r, \theta) = a_0 + \frac{b_0}{r} + \left(a_1 r + \frac{b_1}{r^2}\right)\cos\theta$$

(remember that the general separated solution to Laplace's equation with $m = 0$ is $\left(ar^\ell + \frac{b}{r^{\ell+1}}\right)P_\ell(\cos\theta)$). This expression satisfies Laplace's equation on the given region (note that the region does not contain the origin) by construction, so we need only determine the coefficients from the boundary conditions. On the inner boundary we have

$$\begin{aligned} u(1, \theta) &= a_0 + b_0 + (a_1 + b_1)\cos\theta \\ &= (a_0 + b_0)P_0(\cos\theta) + (a_1 + b_1)\,P_1(\cos\theta) = 1, \end{aligned}$$

whence we see that (since the $P_n(x)$ form an orthogonal set on $[-1, 1]$ and hence are linearly independent) we must have $a_0 + b_0 = 1$, $a_1 + b_1 = 0$. Similarly, on the outer boundary we have

$$\begin{aligned} u(2, \theta) &= a_0 + \frac{1}{2}b_0 + \left(2a_1 + \frac{1}{4}b_1\right)\cos\theta \\ &= \left(a_0 + \frac{1}{2}b_0\right)P_0(\cos\theta) + \left(2a_1 + \frac{1}{4}b_1\right)P_1(\cos\theta) = P_1(\cos\theta), \end{aligned}$$

whence we have by the same logic that $a_0 + \frac{1}{2}b_0 = 0$, $2a_1 + \frac{1}{4}b_1 = 1$. Putting all of these equations together, we see that we have $a_0 = -1$, $b_0 = 2$, $a_1 = \frac{4}{7}$, $b_1 = -\frac{4}{7}$, so that the full solution is

$$u(r, \theta) = -1 + \frac{2}{r} + \left(\frac{4}{7}r - \frac{4}{7r^2}\right)\cos\theta.$$

(Those of you who have studied electrodynamics may recognise the last term – $\frac{\cos\theta}{r^2}$ – as being the electrostatic potential of an electric dipole.)

Summary:
- We deduce additional properties of the Legendre polynomials introduced last week which enable us to use them to solve boundary-value problems, and give a few examples.
- We then introduce the *associated Legendre functions*, give some of their properties, and indicate how they are combined with the functions $\cos m\phi$ and $\sin m\phi$ which we saw last week to give a complete orthonormal set on a sphere.
- We then indicate how all of this is used to solve general (nonsymmetric) boundary-value problems for Laplace's equation on a sphere.

LEGENDRE POLYNOMIALS. Recall that the Legendre polynomials were defined last time as solutions to the differential equation

$$\left(1-x^2\right)\frac{d^2P}{dx^2}-2x\frac{dP}{dx}+\ell(\ell+1)P=0$$

as follows: seeking a power series solution $P=\sum_{n=0}^{\infty}a_nx^n$ of the above equation results in the recurrence relation for the coefficients

$$a_{n+2}=a_n\frac{n(n+1)-\ell(\ell+1)}{(n+2)(n+1)}.$$

If $\ell\in\mathbf{Z}$, $\ell\geq 0$, this says that all a_n for n of the same parity[1] as ℓ must eventually vanish. We then define the degree-ℓ *Legendre polynomial* as follows: If ℓ is even, let P_ℓ be the above power series solution with $a_1=0$ and a_0 chosen[2] so that $P_\ell(1)=1$; if ℓ is odd, let P_ℓ be the above power series solution with $a_0=0$ and a_1 chosen so that $P_\ell(1)=1$. Then we note the following properties:
- If ℓ is even, then P_ℓ is a sum of even powers of x and is hence an even function; if ℓ is odd, then P_ℓ is a sum of odd powers of x and is hence an odd function.
- From this, $P_\ell(-1)=(-1)^\ell$, and $P_\ell(0)=0$ if ℓ is odd. ($P_e\ell(0)$ for *even* ℓ will be found below.)

Next we shall derive some results about the Legendre polynomials which are very useful in manipulating them. The proofs (except where noted) may be omitted without loss of continuity. Some of them are more advanced than the general level of this course.

The first result, while it may look strange at first,[3] is very useful (see the appendix for a similar result for the trigonometric functions and its use):

PROPOSITION. For $x\in[-1,1]$ and $|h|<\frac{1}{4}$,

$$\left(1-2xh+h^2\right)^{-\frac{1}{2}}=\sum_{\ell=0}^{\infty}h^\ell P_\ell(x).$$

Proof. We note first that

$$1-2|x||h|+h^2\leq 1-2xh+h^2\leq 1+2|x||h|+h^2,$$

and since $0\leq|x|\leq 1$, this shows that

$$(1-|h|)^2\leq 1-2xh+h^2\leq(1+|h|)^2,$$

so in particular $1-2xh+h^2\geq\frac{9}{16}>0$ and the function above is well-defined, and also $2xh-h^2\leq\frac{7}{16}<\frac{1}{2}$ and $2xh-h^2\geq 1-(1+|h|)^2>-\frac{9}{16}>-1$; this last implies that for any fixed x in $[-1,1]$ the above function can be expanded using the general binomial expansion theorem (exercise: prove this!)

$$(1+h)^\alpha=\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}h^n,$$

[1] By 'parity' we mean the property of a nonnegative integer according to which it is odd or even; in other words, two nonnegative integers are of the same parity if they are either both odd or both even.

[2] Note that a_0 is effectively an overall multiplicative constant in this case since we have only even-order terms.

[3] Some readers may note the relationship this result bears to the multipole expansion in electrostatics. This is not a coincidence!

where $|h| < 1$ and $(\alpha)_0 = 1$, $(\alpha)_{n+1} = (\alpha)_n \cdot (\alpha - n)$ (so that, for example, $(n)_n = n!$). In other words, for fixed $x \in [-1, 1]$ we may write

$$\left(1 - 2xh + h^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{1}{2}\right)_n (-1)^n \left(2xh - h^2\right)^n ;$$

by the standard theory of power series, the series on the left is uniformly and absolutely convergent also for $x \in [-1, 1]$, $|h| < \frac{1}{4}$, which means that we may reorder the terms as we wish. Doing so, we see easily that we get an expansion of the form

$$\left(1 - 2xh + h^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} H_n(x) h^n,$$

where $H_n(x)$ is a polynomial in x. The fact that the above series converges uniformly in h for fixed x means that we may differentiate termwise with respect to h; since the original series may also be written in the form $\sum_{n=0}^{\infty} X(h)x^n$ for some polynomial X of h, and this series will also converge uniformly in x for fixed h,[4] we may also differentiate with respect to x termwise. We shall show that $H_n(x) = P_n(x)$, which will establish the result in the proposition, by showing that it satisfies Legendre's equation and has the correct normalisation. (By the foregoing, Legendre's equation has, up to normalisation, at most one polynomial solution for each ℓ.)

To do this, let $s(x, h) = \left(1 - 2xh + h^2\right)^{-\frac{1}{2}}$, and note that

$$\frac{\partial s}{\partial x} = h \left(1 - 2xh + h^2\right)^{-\frac{3}{2}} = hs^3, \qquad \frac{\partial^2 s}{\partial x^2} = 3h^2 s^5.$$

Similarly, note that

$$\frac{\partial}{\partial h} (hs) = s + h(x - h)s^3, \qquad \frac{\partial^2}{\partial h^2} (hs) = (x - h)s^3 + 3h(x - h)^2 s^5 + (x - 2h)s^3,$$

whence we see that

$$\begin{aligned} (1 - x^2)\frac{\partial^2 s}{\partial x^2} - 2x\frac{\partial s}{\partial x} &= (1 - x^2) \cdot 3h^2 s^5 - 2xhs^3 = s^5 \left(3h^2(1 - x^2) - 2xh(1 - 2xh + h^2)\right) \\ &= s^5 \left(3h^2 + h^2x^2 - 2xh - 2xh^3\right) \end{aligned}$$

,

while

$$\begin{aligned} \frac{\partial^2}{\partial h^2} (hs) &= s^5 \left(3h(x - h)^2 + (2x - 3h)(1 - 2xh + h^2)\right) \\ &= s^5 \left(3h(h^2 - 2xh + x^2) + 2x(1 - 2xh + h^2) - 3h(1 - 2xh + h^2)\right) \\ &= s^5 \left(3hx^2 + 2x - 4x^2h + 2h^2x - 3h\right) \\ &= s^5 \left(-hx^2 + 2x + 2h^2x - 3h\right), \end{aligned}$$

whence we see that

$$-h\frac{\partial^2}{\partial h^2} (hs) = (1 - x^2)\frac{\partial^2 s}{\partial x^2} - 2x\frac{\partial s}{\partial x}.$$

But by the power series expansion, we have

$$-h\frac{\partial^2}{\partial h^2} (hs) = -h\sum_{n=0}^{\infty} H_n(x)n(n + 1)h^{n-1} = -\sum_{n=0}^{\infty} n(n + 1)H_n(x)h^n,$$

[4] Note that we can allow x to lie in an open interval slightly larger than $[-1, 1]$ without changing the foregoing arguments, since $|h| < \frac{1}{4}$ kept us well inside the radius of convergence of the binomial expansion theorem; this justifies the uniform convergence just claimed.

whence we see that H_n is a polynomial satisfying Legendre's equation, as claimed. To check its normalisation, set $x = 1$ in s; then we have

$$\sum_{n=0}^{\infty} H_n(1)h^n = \left(1 - 2h + h^2\right)^{-\frac{1}{2}} = \frac{1}{1-h},$$

which implies that $H_n(1) = 1$ for all n. Thus we must have $H_n(x) = P_n(x)$ for $x \in [-1, 1]$, as claimed.QED.

The function s in this result is called the *generating function* for the Legendre polynomials. From this result the five identities given in class can be easily derived, as follows.

PROPOSITION. The Legendre polynomials satisfy the following identities (where $n \in \mathbf{Z}$, $n \geq 0$, and we set $P_{-1} = 0$):

1. $(n+1)P_{n+1} - (2n+1)xP_n + nP_{n-1} = 0$.
2. $P'_{n+1} - 2xP'_n + P'_{n-1} = P_n$.
3. $xP'_n - P'_{n-1} = nP_n$.
4. $P'_{n+1} - P'_{n-1} = (2n+1)P_n$.
5. $(1-x^2)P'_n = nP_{n-1} - nxP_n$.

Proof. We note that (letting, as above, $s = \left(1 - 2xh + h^2\right)^{-\frac{1}{2}}$)

$$\begin{aligned}\frac{\partial s}{\partial h} &= (x-h)s^3 = (x-h)(1 - 2xh + h^2)^{-1}s \\ &= \sum_{n=0}^{\infty} nP_n(x)h^{n-1},\end{aligned}$$

whence we see that

$$\begin{aligned}(x-h)\sum_{n=0}^{\infty} P_n(x)h^n &= \sum_{n=0}^{\infty} \left(xP_n(x) - P_{n-1}\right)h^n = (x-h)s \\ &= (1 - 2xh + h^2)\sum_{n=0}^{\infty} nP_n(x)h^{n-1} \\ &= \sum_{n=0}^{\infty} \left((n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x)\right)h^n\end{aligned}$$

(recall our convention that $P_{-1} = 0$; this was used twice in the above calculation) from which we obtain

$$\begin{gathered}xP_n - P_{n-1} = (n+1)P_{n+1} - 2xnP_n + (n-1)P_{n-1} \\ (n+1)P_{n+1} - (2n+1)xP_n + nP_{n-1} = 0,\end{gathered}$$

proving the first identity. Similarly, differentiating s with respect to x gives

$$\begin{aligned}\frac{\partial s}{\partial x} &= hs^3 = h(1 - 2xh + h^2)^{-1}s \\ &= \sum_{n=0}^{\infty} P'_n h^n,\end{aligned}$$

whence we have (noting that this last series has no $n = 0$ term since $P'_0 = 0$, so that we may advance its index by 1)

$$\begin{aligned}hs = \sum_{n=0}^{\infty} P_n h^{n+1} &= (1 - 2xh + h^2)\sum_{n=0}^{\infty} P'_{n+1}h^{n+1} \\ &= \sum_{n=0}^{\infty} \left(P'_{n+1} - 2xP'_n + P'_{n-1}\right)h^{n+1},\end{aligned}$$

from which the second identity easily follows. Now multiply the second identity by $n+1$ and subtract it from the derivative of the first identity to obtain

$$-(n+1)P_n = (n+1)P'_{n+1} - (2n+1)xP'_n - (2n+1)P_n + nP'_{n-1} - \left((n+1)P'_{n+1} - (2n+2)xP'_n + (n+1)P'_{n-1}\right)$$
$$xP'_n - (2n+1)P_n - P'_{n-1} = -(n+1)P_n$$
$$xP'_n - P'_{n-1} = nP_n,$$

which is the third identity. Adding twice the third identity to the second identity gives the fourth identity. Finally, to obtain the fifth identity, note that adding the second and fourth identities gives $2P'_{n+1} - 2xP'_n = 2(n+1)P_n$, which, upon dividing by 2 and replacing $n+1$ by n, gives

$$P'_n = xP'_{n-1} + nP_{n-1},$$

and thus, from the third identity,

$$(1-x^2)P'_n = P'_n - x\left(P'_{n-1} + nP_n\right) = -nxP_n + nP_{n-1},$$

which is exactly the fifth identity. QED.

We also have *the Rodrigues formula:*

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

We shall not prove this at the moment (it can be proved by expanding out $(x^2-1)^n$ using the standard binomial expansion theorem, differentiating term-by-term, showing that the resulting coefficients of the powers of x satisfy the same recursion relation as the coefficients for the Legendre polynomials, and then checking the normalisation at 1).

From the Rodrigues formula we may deduce the orthogonality property of the Legendre polynomials:

PROPOSITION. We have
$$\int_{-1}^{1} P_\ell(x)P_{\ell'}(x)\,dx = \begin{cases} 0, & \ell \neq \ell' \\ \frac{2}{2\ell+1}, & \ell = \ell' \end{cases}.$$

Proof. Suppose that $\ell \geq \ell'$; then we have, applying the Rodrigues formula and integrating by parts (it can be shewn that the boundary terms all vanish)

$$\begin{aligned}\int_{-1}^{1} P_\ell(x)P_{\ell'}(x)\,dx &= \frac{1}{2^\ell \ell!}\frac{1}{2^{\ell'}\ell'!}\int_{-1}^{1}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell \frac{d^{\ell'}}{dx^{\ell'}}(x^2-1)^{\ell'}\,dx \\ &= -\frac{1}{2^\ell \ell!}\frac{1}{2^{\ell'}\ell'!}\int_{-1}^{1}\frac{d^{\ell-1}}{dx^{\ell-1}}(x^2-1)^\ell \frac{d^{\ell'+1}}{dx^{\ell'+1}}(x^2-1)^{\ell'}\,dx \\ &= (-1)^\ell\frac{1}{2^\ell \ell!}\frac{1}{2^{\ell'}\ell'!}\int_{-1}^{1}(x^2-1)^\ell \frac{d^{\ell'+\ell}}{dx^{\ell'+\ell}}(x^2-1)^{\ell'}\,dx.\end{aligned}$$

Now if $\ell' \neq \ell$, then since $\ell \geq \ell'$ we must have $\ell > \ell'$; thus $\ell' + ell > 2\ell'$, but since $(x^2-1)^{\ell'}$ is a polynomial of degree $2\ell'$ this implies that $\frac{d^{\ell'+\ell}}{dx^{\ell'+\ell}}(x^2-1)^{\ell'} = 0$ identically and the above integral must be zero, as claimed. If $\ell' = \ell$, then the foregoing derivative is simply the constant $(2\ell)!$; finishing the proof requires integrating $\int_{-1}^{1}(x^2-1)^\ell\,dx$, which requires the use of a trigonometric reduction formula and which we pass over for the time being. QED.

The foregoing shows that the set $\{P_\ell | \ell \in \mathbf{Z}, \quad \ell \geq 0\}$ is an orthogonal set on $[-1,1]$; it can be shewn that it is complete. Thus any (suitably nice; e.g., piecewise continuous) function f on the interval $[-1,1]$ can be expanded uniquely in a series

$$f(x) = \sum_{\ell=0}^{\infty} a_\ell P_\ell(x),$$

where by our work with general orthogonal complete sets at the start of the course we have the usual formula

$$a_\ell = \frac{(f, P_\ell)}{(P_\ell, P_\ell)} = \frac{2\ell+1}{2}(f, P_\ell).$$

We would now like to know how this helps us solve boundary-value problems. Suppose that we are asked to solve Laplace's equation in a spherical shell $a < r < b$, with azimuthally symmetric boundary data given on the inner and outer spheres $r = a$ and $r = b$. On the interior region, the solution will be given by the general expression

$$u(r, \theta) = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta) \left(a_\ell r^\ell + b_\ell r^{-\ell-1}\right).$$

It is instructive to compare this to the general expression

$$u(x, y) = \sum_{n=1}^{\infty} \sin n\pi x \, (a_n \sinh n\pi y + b_n \cosh n\pi y)$$

which we obtained for the solution to Laplace's equation in the unit square with homogeneous Dirichlet data on the two vertical boundaries (i.e., $u(0, y) = u(1, y) = 0$). As we did on the unit square, we now apply the boundary conditions on the inner and outer spheres to fix the coefficients a_ℓ and b_ℓ. More concretely, suppose that we are given $u(a, \theta) = f(\theta)$, $u(b, \theta) = g(\theta)$; then we may write

$$\begin{aligned} f(\theta) &= \sum_{\ell=0}^{\infty} P_\ell(\cos\theta) \left(a_\ell a^\ell + b_\ell a^{-\ell-1}\right), \\ g(\theta) &= \sum_{\ell=0}^{\infty} P_\ell(\cos\theta) \left(a_\ell b^\ell + b_\ell b^{-\ell-1}\right). \end{aligned}$$

We would like to use the fact that P_ℓ is complete on $[-1, 1]$ in order to evaluate the coefficients of the above series. This requires a little bit of work though since the series above are in terms of $P_\ell(\cos\theta)$ while the functions f and g are given in terms of θ itself. Note that since $\theta \in [0, \pi]$, we have $\cos\theta \in [-1, 1]$, so that (letting $x = \cos\theta$ as before) we have $\cos^{-1} x = \theta \in [0, \pi]$ for $x \in [-1, 1]$; this implies that we may expand $f(\theta) = f(\cos^{-1} x)$ and $g(\theta) = g(\cos^{-1} x)$ in series of $P_\ell(x) = P_\ell(\cos\theta)$ using the above formula. To do this, we calculate

$$\begin{aligned} (f \circ \cos^{-1}, P_\ell) &= \int_{-1}^{1} f(\cos^{-1}(x)) P_\ell(x) \, dx \\ &= \int_0^\pi f(\theta) P_\ell(\cos\theta) \sin\theta \, d\theta, \end{aligned}$$

where we have changed variables in the integral in the last step. This shows that in an expansion of the form

$$f(\theta) = \sum_{\ell=0}^{\infty} c_\ell P_\ell(\cos\theta),$$

we have

$$c_\ell = \frac{2\ell+1}{2} \int_0^\pi f(\theta) P_\ell(\cos\theta) \sin\theta \, d\theta.$$

Another way of saying this is that the functions $P_\ell(\cos\theta)$ are orthogonal on the interval $[0, \pi]$ with respect to the inner product

$$(f, g) = \int_0^\pi f(\theta) \overline{g(\theta)} \sin\theta \, d\theta.$$

Thus we obtain finally the system of equations

$$\begin{aligned} a_\ell a^\ell + b_\ell a^{-\ell-1} &= \frac{2\ell+1}{2}\int_0^\pi f(\theta)P_\ell(\cos\theta)\sin\theta\,d\theta, \\ a_\ell b^\ell + b_\ell b^{-\ell-1} &= \frac{2\ell+1}{2}\int_0^\pi g(\theta)P_\ell(\cos\theta)\sin\theta\,d\theta, \end{aligned}$$

which when solved will give us a_ℓ and b_ℓ for all ℓ, from which the solution to the desired problem follows.

Other problems (for example, when the boundary data involves the derivatives u_r) can be solved in a similar way.

It is worthwhile pausing for a moment to note a general pattern here which will come up again in the future: a given polynomial $P_\ell(\cos\theta)$ on the boundary will give a solution varying like $P_\ell(\cos\theta)r^\ell$ or $P_\ell(\cos\theta)r^{-\ell-1}$ in the interior, and the 'amount' of a certain Legendre polynomial P_ℓ in given boundary data determines exactly the 'amount' of that polynomial in the final solution. In other words, if we think of the coefficients in the expansions of f and g above in terms of $P_\ell(\cos\theta)$ as being knobs we can turn, then it is as if each knob corresponds to a particular type of behaviour of the full solution, and fixing boundary data is equivalent to fixing the position of each knob. The specific way in which the knobs control the solution is determined by solving the equations above.

FULL SOLUTIONS TO LAPLACE'S EQUATION. Let us now consider the problem of finding solutions to Laplace's equation in the absence of azimuthal symmetry. Recalling our results from applying separation of variables to Laplace's equation in spherical symmetry, we see that in this case Legendre's equation is replaced by the equation (still writing $x = \cos\theta$)

$$\left(1-x^2\right)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + \left(\ell(\ell+1) - \frac{m^2}{1-x^2}\right)P = 0,$$

while if $P_{\ell m}$ is a solution to this equation, then the corresponding general separated solution to Laplace's equation is given by

$$P_{\ell m}(\cos\theta)\left(a_{\ell m}r^\ell + b_{\ell m}r^{-\ell-1}\right)\left(c_{\ell m}\cos m\phi + d_{\ell m}\sin m\phi\right).$$

Solutions to this equation may be found by the following trick. We differentiate Legendre's equation m times:

$$\begin{aligned} \left(1-x^2\right)P''' - 4xP'' + \left(\ell(\ell+1) - 2\right)P' &= 0 \\ \left(1-x^2\right)P^{(4)} - 6xP''' + \left(\ell(\ell+1) - 2 - 4\right)P'' &= 0 \\ \left(1-x^2\right)P^{(5)} - 8xP^{(4)} + \left(\ell(\ell+1) - 2 - 4 - 6\right)P'' &= 0 \\ &\vdots \\ \left(1-x^2\right)P^{(m+2)} - 2(m+1)xP^{(m+1)} + \left(\ell(\ell+1) - m(m+1)\right)P^{(m)} &= 0, \end{aligned}$$

since $2+4+6+8+\cdots+2m = 2\frac{m(m+1)}{2} = m(m+1)$. Thus

$$\begin{aligned} &\frac{d}{dx}\left(\left(1-x^2\right)\frac{d}{dx}\left(\left(1-x^2\right)^{\frac{m}{2}}P^{(m)}\right)\right) \\ &\quad= \frac{d}{dx}\left[\left(1-x^2\right)^{\frac{m}{2}+1}P^{(m+1)} - mx\left(1-x^2\right)^{\frac{m}{2}}P^{(m)}\right] \\ &\quad= \left(1-x^2\right)^{\frac{m}{2}+1}P^{(m+2)} - m\left(1-x^2\right)^{\frac{1}{m}}P^{(m)} + m^2x^2\left(1-x^2\right)^{\frac{m}{2}-1}P^{(m)} \\ &\qquad\qquad - (m+m+2)\left(1-x^2\right)^{\frac{m}{2}}P^{(m+1)} \\ &\quad= \left(1-x^2\right)^{\frac{m}{2}}\left[\left(1-x^2\right)P^{(m+2)} - 2(m+1)xP^{(m+1)} - \left(m - \frac{m^2x^2}{1-x^2}\right)P^{(m)}\right] \\ &\quad= \left(1-x^2\right)^{\frac{m}{2}}\left[m(m+1) - \ell(\ell+1) - m + \frac{m^2x^2}{1-x^2}\right]P^{(m)} = \left[\frac{m^2}{1-x^2} - \ell(\ell+1)\right]\left(1-x^2\right)^{\frac{m}{2}}P^{(m)}; \end{aligned}$$

comparison with the equation we are trying to solve shows that it has the solution

$$P_{\ell,m}(x) = (1-x^2)^{\frac{m}{2}} P_\ell^{(m)}(x).$$

We call these the *associated Legendre functions*. The corresponding general separated solution to Laplace's equation is (as given above)

$$P_{\ell m}(\cos\theta)\left(a_{\ell m} r^\ell + b_{\ell m} r^{-\ell-1}\right)\left(c_{\ell m}\cos m\phi + d_{\ell m}\sin m\phi\right).$$

It is worthwhile to pause for a moment to consider what these functions look like, and what the corresponding solutions to Laplace's equation look like. First, since P_ℓ is a polynomial of degree ℓ, we see that $P_\ell^{(m)}$ will vanish if $m > \ell$; thus we require that $m \leq \ell$.[5] Further, $P_{\ell,0} = P_\ell$ is just the ordinary Legendre polynomial. The first few additional associated Legendre functions may be calculated as follows. It is instructive to evaluate them at $\cos\theta$ (though we must bear carefully in mind that the derivatives in $P_\ell^{(m)}$ are with respect to x, not θ!), noting that since $\theta \in [0,\pi]$, $(1-\cos^2\theta)^{\frac{1}{2}} = (\sin^2\theta)^{\frac{1}{2}} = |\sin\theta| = \sin\theta$.

$$\begin{aligned}
P_{1,1}(\cos\theta) &= \sin\theta \\
P_{2,1}(\cos\theta) &= 3\sin\theta\cos\theta \\
P_{2,2}(\cos\theta) &= 3\sin^2\theta.
\end{aligned}$$

Thus we see that increasing m by one essentially trades a $\cos\theta$ for a $\sin\theta$. Now the corresponding solutions to Laplace's equation on a ball (meaning that we disregard the $r^{-\ell-1}$ solutions) are of the form

$$\begin{aligned}
P_{1,1}(\cos\theta) r\cos\phi &= \sin\theta r\cos\phi = r\sin\theta\cos\phi = x \\
P_{2,1}(\cos\theta) r^2\cos\phi &= 3r^2\sin\theta\cos\theta\cos\phi = 3xz \\
P_{2,2}(\cos\theta) r^2\cos 2\phi &= 3r^2\sin^2\theta\left(\cos^2\phi - \sin^2\phi\right) = 3\left(x^2 - y^2\right),
\end{aligned}$$

with similar expressions obtaining if the $\sin\phi$ and $\sin 2\phi$ solutions are used instead. The polynomials of the form $P_{\ell,m} r^\ell \cos m\phi$ and $P_{\ell,m} r^\ell \sin m\phi$,[6] which are all solutions to Laplace's equation, are called *harmonic polynomials*. As we shall see shortly, the set of all products of the form $P_{\ell,m}\cos m\phi$, $P_{\ell,m}\sin m\phi$ forms a complete orthogonal set over the sphere; since on a sphere r is a constant, this implies that given any polynomial on $\mathbf{R}^n$ and a sphere of radius $r = a$ centred at the origin, there will be a harmonic polynomial which agrees with the given polynomial on the sphere. This harmonic polynomial will then be the solution to Laplace's equation with boundary data equal to the given polynomial. This is an interesting branch of mathematics but we shall not explore it in detail here.

We would now like to see how we can use the associated Legendre functions to solve boundary-value problems for Laplace's equation on a sphere. First, we note that the $P_{\ell,m}$, for constant m, form an orthogonal set; in particular, we have the following result.

PROPOSITION. Let $m \in \mathbf{Z}$, $m \geq 0$, $\ell_1, \ell_2 \in \mathbf{Z}$, $\ell_1, \ell_2 \geq m$, $\ell_1 \neq \ell_2$. Then

$$\int_{-1}^{1} P_{\ell_1,m} P_{\ell_2,m}\, dx = 0.$$

[NOTE. By the same logic as we used above for the Legendre polynomials, in terms of θ the above orthogonality result becomes

$$\int_{-1}^{1} P_{\ell_1,m}(\cos\theta) P_{\ell_2,m}(\cos\theta)\sin\theta\, dx = 0.]$$

[5]This requirement might be familiar to those of you who have studied the theory of angular momentum in quantum mechanics. What we are building here are parts of the angular momentum eigenfunctions in the position representation.

[6]These may be handled simultaneously by using the complex form $P_{\ell,m} r^\ell e^{im\phi}$, in the which case the general form becomes more transparent.

Proof. We use a general method (which can also be applied to the Legendre polynomials themselves), by showing that the differential operator

$$f \mapsto Lf := \frac{d}{dx}\left(\left(1-x^2\right)\frac{df}{dx}\right)$$

is what is called *self-adjoint*, i.e., that if $f, g \in C^2$, then $(Lf, g) = (f, Lg)$. This may be shewn as follows:

$$\begin{aligned}(Lf, g) &= \int_{-1}^{1} [Lf](x) \cdot \overline{g(x)}\, dx = \int_{-1}^{1} \frac{d}{dx}\left(\left(1-x^2\right)\frac{df}{dx}\right)\overline{g(x)}\, dx \\ &= \left(1-x^2\right)\frac{df}{dx}\overline{g(x)}\Big|_{-1}^{1} - \int_{-1}^{1}\left(1-x^2\right)\frac{df}{dx}\overline{\frac{dg}{dx}}\, dx \\ &= -\, f(x)\left(1-x^2\right)\overline{\frac{dg}{dx}}\Big|_{-1}^{1} + \int_{-1}^{1} f(x)\overline{\frac{d}{dx}\left(\left(1-x^2\right)\frac{dg}{dx}\right)}\, dx \\ &= (f, Lg),\end{aligned}$$

as claimed. Now let $m \in \mathbf{Z}$, $m \geq 0$. It is clear that the operator $f \mapsto Mf := \frac{m^2}{1-x^2} f$ also satisfies this property, and hence so does the difference $L' = L - M$. Now let $P_{\ell_1,m}$ and $P_{\ell_2,m}$ be as in the statement of the proposition. Then we have $L'P_{\ell_1,m} = -\ell_1(\ell_1+1)P_{\ell_1,m}$ and $L'P_{\ell_2,m} = -\ell_2(\ell_2+1)P_{\ell_2,m}$; thus

$$\begin{aligned}(L'P_{\ell_1,m}, P_{\ell_2,m}) &= -\ell_1(\ell_1+1)(P_{\ell_1,m}, P_{\ell_2,m}) \\ &= (P_{\ell_1,m}, L'P_{\ell_2,m}) = -\ell_2(\ell_2+1)(P_{\ell_1,m}, P_{\ell_2,m}),\end{aligned}$$

whence $(P_{\ell_1,m}, P_{\ell_2,m}) = 0$ since $\ell_1 \neq \ell_2$. QED.

It may also be shewn that (see (4.2.25) in the textbook)

$$(P_{\ell,m}, P_{\ell,m}) = \frac{(\ell+m)!}{(\ell-m)!}\frac{2}{2\ell+1}. \tag{1}$$

We now recall that the set

$$\{1, \cos m\phi, \sin m\phi | m \in \mathbf{Z}, m \geq 1\}$$

is a complete orthogonal set on $[0, 2\pi]$ (while we may not have explicitly shewn this earlier, it follows readily from what we have done). We claim that this, together with the above proposition, implies that

$$\{P_\ell(\cos\theta)\} \cup \{P_{\ell,m}(\cos\theta)\cos m\phi, P_{\ell,m}(\cos\theta)\sin m\phi | m \in \mathbf{Z}, m \geq 1\}$$

is a complete orthogonal set on the sphere (i.e., on the set $[0, \pi] \times [0, 2\pi]$ with respect to the inner product

$$(f(\theta,\phi), g(\theta,\phi)) = \int_0^\pi \int_0^{2\pi} f(\theta, phi)\overline{g(\theta,\phi)}\sin\theta\, d\phi\, d\theta.$$

The orthogonality is simple to shew. We note first of all that if $f(\theta,\phi) = f_1(\theta)f_2(\phi)$ and $g(\theta, phi) = g_1(\theta)g_2(\phi)$, then the above inner product decomposes as follows:

$$(f, g) = \int_0^\pi f_1(\theta)\overline{g_1(\theta)}\sin\theta\, d\theta \int_0^{2\pi} f_2(\phi)\overline{g_2(\phi)}\, d\phi;$$

in other words, it is simply the product of the inner products we have been using for functions of θ and ϕ.[7] Thus

$$(P_{\ell,m}\cos m\phi, P_{\ell',m'}\cos m'\phi) = (P_{\ell,m}, P_{\ell',m'})(\cos m\phi, \cos m'\phi) = 0$$

[7] This is related to the fact that the space of functions on the sphere can be viewed as the tensor product (in an appropriate sense: one needs to somehow close off the algebraic tensor product in an appropriate topology, such as that given by the product inner product indicated above) of the spaces of functions on $[0, 2\pi]$ and $[0, \pi]$.

if $m \neq m'$, while if $m = m'$ it will be zero unless $\ell = \ell'$, in the which case it will be given by the normalisation formula (1) above. An analogous result clearly holds if both of the cosine terms are replaced by sine, while if one is replaced by sine then all of the inner products will be zero regardless of m. This shews that the set above is an orthogonal set, as desired.

To see that it is complete (assuming that $\{P_{\ell,m}\}$ and the sine/cosine basis are), we may proceed as follows. Let $f(\theta,\phi)$ be any suitable (e.g., piecewise continuous) function on the sphere. Then for each fixed θ we may expand it in a series of sines and cosines, the coefficients of which will however depend on θ; in other words, we may write[8]

$$f(\theta,\phi) = \sum_{m=0}^{\infty} c_m(\theta)\cos m\phi + d_m(\theta)\sin m\phi;$$

now since the $P_{\ell,m}$, for each m, form a complete set on $[0,\pi]$, we may further expand each of the coefficients c_m, d_m, obtaining

$$\begin{aligned} c_m(\theta) &= \sum_{\ell=m}^{\infty} c_{\ell,m} P_{\ell,m}(\cos\theta), \\ d_m(\theta) &= \sum_{\ell=m}^{\infty} d_{\ell,m} P_{\ell,m}(\cos\theta). \end{aligned}$$

Thus we have finally

$$f(\theta,\phi) = \sum_{m=0}^{\infty}\sum_{\ell=m}^{\infty} P_{\ell,m}(\cos\theta)\left(c_{\ell,m}\cos m\phi + d_{\ell,m}\sin m\phi\right),$$

or, assuming that the series is sufficiently well-behaved that we may rearrange the order of the terms,

$$f(\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell} P_{\ell,m}(\cos\theta)\left(c_{\ell,m}\cos m\phi + d_{\ell,m}\sin m\phi\right),$$

where by the orthogonality result above

$$\begin{aligned} c_{\ell,m} &= \frac{(f, P_{\ell,m}(\cos\theta)\cos m\phi)}{(P_{\ell,m}(\cos\theta)\cos m\phi, P_{\ell,m}(\cos\theta)\cos m\phi)} \\ &= \frac{2\ell+1}{2\pi}\frac{(\ell-m)!}{(\ell+m)!}(f, P_{\ell,m}(\cos\theta)\cos m\phi), \end{aligned}$$

with an analogous formula for $d_{\ell,m}$.

Our procedure for solving general problems involving Laplace's equation on spherical shells is now clear: we start out with the general series representation

$$\sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell} P_{\ell m}(\cos\theta)\left(a_{\ell m} r^{\ell} + b_{\ell m} r^{-\ell-1}\right)\left(c_{\ell m}\cos m\phi + d_{\ell m}\sin m\phi\right),$$

and then apply the boundary conditions, using formulas like the above for $c_{\ell,m}$ to determine equations for the relevant coefficients (exactly as we did when solving Laplace's equation on a square), to determine all of the coefficients in the expansion. We then substitute back in to obtain the desired solution.

As we are well aware by now, this process in general produces very long expressions; also, the integrals arising for general boundary data can be very difficult to evaluate. Some simple examples can be done by exploiting the idea of harmonic polynomial mentioned earlier; we give one such example.

[8] Here and below, to avoid having to pull out the $m = 0$ term explicitly, we shall make the definition that d_0, etc., are all zero.

EXAMPLE. Solve Laplace's equation $\nabla^2 u = 0$ on the unit ball with boundary data $u|_{r=1} = \cos\theta \sin\theta \sin\phi + \sin^2\theta \sin\phi\cos\phi$.

First of all, we note that since we are solving on the interior of a sphere, our solution must be continuous at the origin, so the terms $r^{-\ell-1}$ cannot appear and we must have $b_{\ell,m} = 0$ for all ℓ, m; we may thus absorb the coefficients $a_{\ell,m}$ into the $c_{\ell,m}$ and $d_{\ell,m}$. We could proceed by rewriting the above boundary data as a linear combination of products of associated Legendre functions with functions $\cos m\phi$, $\sin m\phi$; this would give

$$\frac{1}{3}P_{2,1}\sin\phi + \frac{1}{6}P_{2,2}\sin 2\phi,$$

from which we would obtain the solution

$$\frac{1}{3}P_{2,1}r^2\sin\phi + \frac{1}{6}P_{2,2}r^2\sin 2\phi.$$

Alternatively, we may note that on $r = 1$ the above expression is equal to the polynomial

$$zy + xy,$$

which satisfies Laplace's equation everywhere through space; thus our solution is simply $u = zy + xy$, as can be verified from the first expression given above.

APM 346, Homework 5. Due Monday, June 10, at 6.00 AM EDT. To be marked completed/not completed.

1. Solve the following boundary-value problem on the region $\{(r,\theta,\phi)|1<r<2\}$:

$$\nabla^2 u = 0, \qquad u|_{r=2} = \begin{cases} 1, & 0 \le \theta < \frac{\pi}{2} \\ -1, & \frac{\pi}{2} < \theta \le \pi \end{cases}, \qquad u_r|_{r=1} = \begin{cases} 0, & 0 \le \theta < \frac{\pi}{2} \\ 1, & \frac{\pi}{2} < \theta \le \pi \end{cases}.$$

[Hint: use Legendre polynomial identities to calculate $\int_0^1 P_\ell(x)\,dx$ and $\int_{-1}^0 P_\ell(x)\,dx$.]

Since the boundary conditions are azimuthally symmetric (and since we are solving on a spherical shell, which is an azimuthally symmetric region) we may write the general solution to Laplace's equation as

$$u = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta)\left(a_\ell r^\ell + b_\ell r^{-\ell-1}\right).$$

Now at $r=2$ we have

$$u|_{r=2} = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta)\left(a_\ell 2^\ell + \frac{b_\ell}{2^{\ell+1}}\right) = \begin{cases} 1, & 0 \le \theta < \frac{\pi}{2} \\ -1, & \frac{\pi}{2} < \theta \le \pi \end{cases}.$$

If we write this in terms of $x = \cos\theta$,it comes

$$u|_{r=2} = \sum_{\ell=0}^{\infty} P_\ell(x)\left(a_\ell 2^\ell + \frac{b_\ell}{2^{\ell+1}}\right) = \begin{cases} -1, & -1 \le x < 0 \\ 1, & 0 < x \le 1 \end{cases};$$

thus the orthogonality and normalisation properties of the Legendre polynomials give

$$a_\ell 2^\ell + \frac{b_\ell}{2^{\ell+1}} = \frac{2\ell+1}{2}\left[\int_{-1}^0 -P_\ell(x)\,dx + \int_0^1 P_\ell(x)\,dx\right] = \frac{2\ell+1}{2}\left[\int_0^1 [P_\ell(x) - P_\ell(-x)]\,dx\right];$$

if ℓ is even this will vanish, since P_ℓ will be an even function, while if ℓ is odd we have

$$a_\ell 2^\ell + \frac{b_\ell}{2^{\ell+1}} = (2\ell+1)\int_0^1 P_\ell(x)\,dx.$$

Now we have the identity (from the lecture notes)

$$(2\ell+1)P_\ell(x) = P'_{\ell+1}(x) - P'_{\ell-1}(x);$$

thus for ℓ odd, say $\ell = 2k+1$,

$$a_\ell 2^\ell + \frac{b_\ell}{2^{\ell+1}} = [P_{\ell+1}(1) - P_{\ell-1}(1) - (P_{\ell+1}(0) - P_{\ell-1}(0))] = -\left(P_{\ell+1}(0) - P_{\ell-1}(0)\right);$$

we shall show how to calculate this last expression shortly.

The second boundary condition (again working in terms of $x = \cos\theta$) gives

$$u_r|_{r=1} = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta)\left(\ell a_\ell - (\ell+1)b_\ell\right) = \begin{cases} 1, & -1 \le x < 0 \\ 0, & 0 < x \le 1 \end{cases},$$

so

$$\ell a_\ell - (\ell+1)b_\ell = \frac{2\ell+1}{2}\int_{-1}^0 P_\ell(x)\,dx.$$

If $\ell = 0$ this is just $\frac{1}{2}$, while if $\ell > 0$ is even it is $\frac{2\ell+1}{4}\int_{-1}^{1} P_\ell(x)\,dx = 0$. If ℓ is odd, say again $\ell = 2k+1$, we obtain

$$\ell a_\ell - (\ell+1)b_\ell = \frac{1}{2}\left[P_{\ell+1}(0) - P_{\ell-1}(0) - (P_{\ell+1}(-1) - P_{\ell-1}(-1))\right] = \frac{1}{2}\left(P_{\ell+1}(0) - P_{\ell-1}(0)\right),$$

since $P_{\ell+1}(-1) = (-1)^{\ell+1} = (-1)^{\ell-1} = P_{\ell-1}(-1)$. Thus for $\ell = 2k+1$ we have the system

$$\begin{aligned} a_\ell 2^\ell + \frac{b_\ell}{2^{\ell+1}} &= -\left(P_{\ell+1}(0) - P_{\ell-1}(0)\right) \\ \ell a_\ell - (\ell+1)b_\ell &= \frac{1}{2}\left(P_{\ell+1}(0) - P_{\ell-1}(0)\right). \end{aligned}$$

This is a system of two linear equations in two unknowns and may be solved by a number of methods; perhaps the most systematic is to find the inverse of the coefficient matrix. We have the general formula (when the determinant $ad - bc$ is nonzero)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

which in the present case gives

$$\begin{pmatrix} 2^\ell & \frac{1}{2^{\ell+1}} \\ \ell & -(\ell+1) \end{pmatrix}^{-1} = \frac{1}{(\ell+1)2^\ell + \frac{\ell}{2^{\ell+1}}}\begin{pmatrix} \ell+1 & \frac{1}{2^{\ell+1}} \\ \ell & -2^\ell \end{pmatrix}.$$

Thus

$$\begin{aligned} \begin{pmatrix} a_\ell \\ b_\ell \end{pmatrix} &= \frac{1}{(\ell+1)2^\ell + \frac{\ell}{2^{\ell+1}}}\begin{pmatrix} \ell+1 & \frac{1}{2^{\ell+1}} \\ \ell & -2^\ell \end{pmatrix}\begin{pmatrix} -1 \\ \frac{1}{2} \end{pmatrix}\left(P_{\ell+1}(0) - P_{\ell-1}(0)\right) \\ &= \frac{2^{\ell+1}}{(\ell+1)2^{2\ell+2} + \ell}\begin{pmatrix} -(\ell+1) + \frac{1}{2^{\ell+2}} \\ -\ell - 2^{\ell-1} \end{pmatrix}\left(P_{\ell+1}(0) - P_{\ell-1}(0)\right). \end{aligned}$$

Now we have the general formula (this was discussed in lecture)

$$P_{2k}(0) = \frac{(-1)^k(2k-1)!!}{2^k k!};$$

thus

$$\begin{aligned} P_{2k+2}(0) - P_{2k}(0) &= \frac{(-1)^{k+1}(2k+1)!!}{2^{k+1}(k+1)!} - \frac{(-1)^k(2k-1)!!}{2^k k!} = \frac{(-1)^{k+1}(2k-1)!!}{2^k k!}\left(\frac{2k+1}{2k+2} + 1\right) \\ &= \frac{(-1)^{k+1}(2k-1)!!}{2^{k+1}(k+1)!}(4k+3), \end{aligned}$$

and since in the above formula we have $\ell = 2k+1$, we have finally the expression

$$\begin{pmatrix} a_{2k+1} \\ b_{2k+1} \end{pmatrix} = \frac{2^{2k+2}}{(2k+2)2^{4k+4} + 2k+1}\begin{pmatrix} -2k-2+\frac{1}{2^{2k+3}} \\ -2k-1-2^{2k} \end{pmatrix}\frac{(-1)^{k+1}(2k-1)!!}{2^{k+1}(k+1)!}(4k+3).$$

For ℓ even, $\ell > 0$, we have the system

$$\begin{aligned} a_\ell 2^\ell + \frac{b_\ell}{2^{\ell+1}} &= 0 \\ \ell a_\ell - (\ell+1)b_\ell &= 0, \end{aligned}$$

which implies that $a_\ell = b_\ell = 0$ in this case; while for $\ell = 0$ we have instead the system

$$\begin{aligned} a_0 + \frac{b_0}{2} &= 0 \\ -b_0 &= \frac{1}{2}, \end{aligned}$$

which implies that $a_0 = \frac{1}{4}$, $b_0 = -\frac{1}{2}$. Thus finally we obtain the solution

$$u = \frac{1}{4} - \frac{1}{2r} + \sum_{k=0}^{\infty} P_{2k+1}(\cos\theta) \left[\frac{2^{k+1}(-1)^k(2k-1)!!(4k+3)}{[(k+1)2^{4k+4} + (2k+1)]\,(k+1)!} \right] \cdot \left[\left(2k+2-\frac{1}{2^{2k+3}} \right) r^{2k+1} + \left(2k+1+2^{2k} \right) r^{-(2k+2)} \right].$$

2. Solve the following boundary-value problem on the region $\{(r,\theta,\phi)|r<2\}$:

$$\nabla^2 u = 0, \qquad u|_{r=2} = x(1+y).$$

(Here $x = r\sin\theta\cos\phi$ and $y = r\sin\theta\sin\phi$ are the standard Cartesian coordinates corresponding to the given spherical coordinate system.)

This problem is much easier than problem 1. First of all, there is a straightforward way and a tricky way. We show the trick first. Since

$$\nabla^2 x(1+y) = 0$$

for all $x, y, z \in \mathbf{R}^3$, we see that $u = x(1+y)$ satisfies Laplace's equation on the given region; it also agrees with the boundary data, and thus it must be the solution we seek.

The straightforward way is rather longer (though also very instructive in our general technique) and goes as follows. Since we are solving on a region containing the origin, the general solution can be written as

$$u = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} P_{\ell,m}(\cos\theta) r^\ell \left(c_{\ell,m}\cos m\phi + d_{\ell,m}\sin m\phi \right).$$

Thus on the boundary $r=2$ we have

$$\begin{aligned} u|_{r=2} &= \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} P_{\ell,m}(\cos\theta) 2^\ell \left(c_{\ell,m}\cos m\phi + d_{\ell,m}\sin m\phi \right) \\ &= x(1+y)|_{r=2} = 2\sin\theta\cos\phi + 4\sin^2\theta\cos\phi\sin\phi; \end{aligned}$$

since (see the lecture notes)

$$P_{1,1}(\cos\theta) = \sin\theta, \qquad P_{2,2}(\cos\theta) = 3\sin^2\theta,$$

we see that this last expression may be written as

$$2P_{1,1}(\cos\theta)\cos\phi + \frac{2}{3}P_{2,2}(\cos\theta)\sin 2\phi.$$

Orthogonality of the set $\{P_\ell\} \cup \{P_{\ell,m}\cos m\phi, P_{\ell,m}\sin m\phi\}$ then allows us to obtain

$$2^1 c_{1,1} = 2, \qquad 2^2 d_{2,2} = \frac{2}{3},$$

with all other $c_{\ell,m}$ and $d_{\ell,m}$ vanishing. This gives $c_{1,1} = 1$, $d_{1,1} = \frac{1}{6}$, and finally

$$\begin{aligned} u &= P_{1,1}(\cos\theta) r\cos\phi + \frac{1}{6}P_{2,2} r^2 \sin 2\phi \\ &= r\sin\theta\cos\phi + r^2\sin^2\theta\sin\phi\cos\phi = x + xy, \end{aligned}$$

the same as we obtained by the previous method.

Summary:
- By separating variables in Laplace's equation in cylindrical coordinates, we derive Bessel's equation, and use it to derive the Taylor series expansion for Bessel functions on nonnegative integer order.
- We then discuss the orthogonality and completeness properties of these functions.
- Finally, we then use these Taylor series expansions to deduce differentiation and recursion relations for the Bessel functions of nonnegative integer order, and say a few words about modified Bessel functions.

SEPARATION OF VARIABLES IN CYLINDRICAL COORDINATES. Recall (see the lecture notes for the week of May 23) that the Laplacian in cylindrical coordinates (ρ, ϕ, z) (which is related to Cartesian coordinates (x, y, z) by $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$) is given by

$$\nabla^2 f(\rho, \phi, z) = \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}.$$

We are interested (as usual) in solving the equation $\nabla^2 u = 0$, on a region possessing cylindrical symmetry, subject to certain conditions imposed on the boundary of that region. As before, we shall proceed by looking first for separated solutions $u = P(\rho)\Phi(\phi)Z(z)$,[1] and then investigating whether the full solution can be expressed as a series of solutions of this type.

Substituting this *ansatz* into Laplace's equation and dividing by u as usual, we obtain the equation

$$\frac{1}{P}\frac{d^2P}{d\rho^2} + \frac{1}{\rho}\frac{1}{P}\frac{dP}{d\rho} + \frac{1}{\rho^2}\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0. \tag{1}$$

We note that the last term depends only on Z, and is moreover the only term on the left-hand side dependant on Z, and must therefore be constant. Similarly, the term $\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}$ is the only term dependant on ϕ and must therefore also be constant. To proceed further, we must (as for the case of spherical coordinates) know something more about the region over which we wish to solve Laplace's equation. Let us suppose that we are interested in solving over a region which involves a full range of the angular variable ϕ (for example, a cylinder $\{(\rho, \phi, z) | \rho < a, b \le z \le c, 0 \le \phi \le 2\pi\}$). Then, just as for spherical coordinates, u and therefore Φ must be periodic in ϕ with period 2π. Now Φ satisfies the equation $\frac{d^2\Phi}{d\phi^2} = \mu\Phi$ for some constant μ; requiring Φ to be periodic forces $\mu \le 0$, say $\mu = -m^2$, giving $\Phi = a\cos m\phi + b\sin m\phi$ for some a and b; further requiring the period to be 2π gives $m \in \mathbf{Z}$. We may take $m \ge 0$ without loss of generality.[2]

The treatment of the constant corresponding to Z is more involved. To provide some context, we first recall our treatment of Laplace's equation on a square. Recall that in that case separated solutions of the form $u = X(x)Y(y)$ satisfied the equation

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0,$$

from which it is easy to see that both terms must be constant, meaning that we must have $X'' = \mu X$, $Y'' = -\mu Y$. The question then arises as to whether μ should be positive or negative (or 0, but we shall not consider that case here). Clearly, $\mu > 0$, say $\mu = m^2$, implies that we have the general solutions $Y = a\sin my + b\cos my$, $X = c\sinh mx + b\cosh mx$, i.e., Y will be oscillatory while X will be exponential, while $\mu < 0$, say $\mu = -m^2$, implies the exact opposite: $X = c\sin mx + b\cos mx$, $Y = a\sinh my + b\cosh my$, i.e., X will be oscillatory while Y will be exponential. For the boundary-value problems which we have considered so far, we had conditions like $X(0) = X(1) = 0$, which forced us to choose X to be oscillatory

[1]Here P is the capital form of the Greek letter ρ, *not* the capital form of the English letter p.

[2]Note that if $m = 0$, the general solution for Φ is not $a\cos\phi + b\sin m\phi = a$ but rather $a + b\phi$; since ϕ is not periodic as a function of ϕ, we must have $b = 0$, meaning that the solution is in fact just $\Phi = a$ for some constant a. For notational simplicity we shall write $\Phi = a\cos m\phi + b\sin m\phi$ as the general solution for all m, even $m = 0$, with the implicit understanding that when $m = 0$ we shall always (for definiteness) take $b = 0$ (otherwise b would be undefined in this case). This device can be avoided by considering the complex basis $e^{im\phi}$ instead, but we shall not do that here.

and hence Y to be exponential. Had we had instead conditions like $Y(0) = Y(1) = 0$, we would have been forced to take instead Y to be oscillatory and hence X to be exponential.

It turns out that the same duality holds in the present case.[3] Thus, depending on the given boundary conditions, we may be forced to take Z to be oscillatory, in the which case P will be non-oscillatory; or we may be forced to take P to be oscillatory, in the which case Z will be non-oscillatory. (In general, we will have a sum of solutions, one of each type.) Without prejudicing the final result, then, let us write for the moment

$$\frac{d^2Z}{dz^2} = \epsilon\lambda^2 Z,$$

where $\lambda \in \mathbf{R}$, $\lambda \geq 0$, and $\epsilon = \pm 1$.[4] Substituting this and the equation for Φ into equation (1) above, we obtain for P the equation

$$\frac{d^2P}{d\rho^2} + \frac{1}{\rho}\frac{dP}{d\rho} + \left(\epsilon\lambda^2 - \frac{m^2}{\rho^2}\right) P = 0. \tag{2}$$

If $\lambda = 0$ this equation has the general solution (much as for the r-dependent factor in separated solutions in spherical coordinates) $P = ar^{\alpha_+} + br^{\alpha_-}$, where $\alpha_\pm$ are the solutions to $\alpha(\alpha + 2) = m^2$. In this case, we have also $Z = c + dz$, whence we obtain the general separated solution

$$u = \left(ar^{\alpha_+} + br^{\alpha_-}\right)(c + dz)\left(e\cos m\phi + f\sin m\phi\right).$$

Suppose now that $\lambda > 0$, and define a new function $Q : [0, \infty) \to \mathbf{R}$ by $Q(x) = P\left(\frac{x}{\lambda}\right)$; equivalently, by $P(\rho) = Q(\lambda\rho)$. Substituting this into equation (2) above for P gives

$$\begin{aligned} 0 &= \lambda^2 Q''(\lambda\rho) + \frac{\lambda}{\rho}Q'(\lambda\rho) + \left(\epsilon\lambda^2 - \frac{m^2}{\rho^2}\right) Q(\lambda\rho) \\ &= \lambda^2\left[Q''(\lambda\rho) + \frac{1}{\lambda\rho}Q'(\lambda\rho) + \left(\epsilon - \frac{m^2}{\lambda^2\rho^2}\right) Q(\lambda\rho)\right], \end{aligned}$$

whence we obtain, writing $x = \lambda\rho$,

$$\frac{d^2Q}{dx^2} + \frac{1}{x}\frac{dQ}{dx} + \left(\epsilon - \frac{m^2}{x^2}\right) Q = 0.$$

When $\epsilon = 1$ this is called (see [1], p. 38) *Bessel's equation for functions of order m*. We now restrict to this case for the moment; thus we consider the equation

$$\frac{d^2Q}{dx^2} + \frac{1}{x}\frac{dQ}{dx} + \left(1 - \frac{m^2}{x^2}\right) Q = 0. \tag{3}$$

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[3] You may wonder why this did not play so central a part in our treatment of Laplace's equation in spherical coordinates. In spherical coordinates, assuming we solve on a region which covers a full range of ϕ and θ, we have natural boundary conditions on the corresponding factors of the separated solution Φ (identical to that here) and Θ (that it be finite at both $\theta = 0$ and $\theta = \pi$, i.e., at both poles, or equivalently, on the z-axis) which turned out to force both of them to be oscillatory. Thus the only remaining factor, $R(r)$, was forced to be the non-oscillatory (though not, we might note, in this case, exponential). This would not have happened had we solved Laplace's equation on a wedge, say for $\theta \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$ – in that case we would have to consider (in general) both oscillatory and non-oscillatory solutions in the θ direction, which could lead to oscillatory solutions in the r direction. [Note. You may recall that when we discussed the equation for R we had a restriction on the separation constant (namely – see the lecture notes for the week of May 23, p. 8 – $\alpha > \frac{1}{4}$); were this condition not satisfied, the solutions in the r direction could become oscillatory. We shall not pursue this further here.]

[4] The observant reader may note that we could drop ϵ by letting λ be a complex number, with say $\Re\lambda \geq 0$ for definiteness. It turns out that the so-called *modified Bessel functions*, which are the non-oscillatory counterparts of the oscillatory Bessel functions to be derived presently, are obtained from the latter by just this kind of transformation. We shall have more to say about all this below.

We wish to derive a power-series representation for the solutions to this equation. To do this, it is convenient to make another change of variables by setting $Q(x) = x^m q(x)$ for some function q; this gives

$$Q' = mx^{m-1}q + x^m q', \qquad Q'' = m(m-1)x^{m-2}q + 2mx^{m-1}q' + x^m q'',$$

whence, upon substituting into equation (3), we obtain

$$\begin{aligned} 0 &= x^m q'' + 2mx^{m-1}q' + m(m-1)x^{m-2}q + x^{m-1}q' + mx^{m-2}q + x^m q - m^2 x^{m-2} q \\ &= x^m q'' + (2m+1)x^{m-1}q' + \left(m(m-1) + m - m^2\right) x^{m-2} q + x^m q \\ &= x^m \left(q'' + \frac{2m+1}{x} q' + q \right), \end{aligned}$$

whence finally

$$q'' + \frac{2m+1}{x} q' + q = 0. \tag{4}$$

Now suppose that q can be expanded in a Taylor series about $x = 0$ as

$$q = \sum_{n=0}^{\infty} a_n x^n;$$

substituting into equation (4) then gives

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + (2m+1)na_n x^{n-2} + a_n x^n \\ &= (2m+1)a_1 x^{-1} + \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + (2m+1)na_n x^{n-2} + a_{n-2}x^{n-2} \\ &= (2m+1)a_1 x^{-1} + \sum_{n=0}^{\infty} x^n \left((n+1)(n+2)a_{n+2} + (2m+1)(n+2)a_{n+2} + a_n \right), \end{aligned}$$

since the first two terms in the series on the first line vanish for $n = 0$ and $n = 1$ except for the $(2m+1)a_1 x^{-1}$ term. Since the final series above contains no terms with negative powers of x, the term $(2m+1)a_1 x^{-1}$ must vanish, meaning that (since here m is a nonnegative integer) we must have $a_1 = 0$. The series itself must then vanish, which gives the recurrence relation

$$\begin{aligned} (n+2)(2m+n+2)a_{n+2} + a_n &= 0, \\ a_{n+2} &= -\frac{1}{(n+2)(2m+n+2)} a_n, \\ a_n &= -\frac{1}{n(2m+n)} a_{n-2}, \end{aligned}$$

where in the last line we have simply replace $n + 2$ by n everywhere. Since we have $a_1 = 0$ by the foregoing, this recurrence relation implies that $a_n = 0$ for all odd n, so that the power series for q only has even-order terms. Moreover, inspection of the recurrence relation above shows that we have the general formula

$$\begin{aligned} a_{2k} &= \frac{(-1)^k (2m)!!}{(2k)!!(2m+2k)!!} a_0 \\ &= \frac{(-1)^k 2^m m!}{2^k k! 2^{m+k}(m+k)!} a_0 = \frac{(-1)^k m!}{4^k k!(m+k)!} a_0, \end{aligned}$$

where as for odd numbers we define $(2k)!! = (2k)(2k-2)\cdots 4\cdot 2 = 2^k k!$. (This formula may be proved by mathematical induction as follows: when $k=0$ the coefficient above is simply $\frac{(-1)^0 m!}{4^0 0!(m+0)!} = 1$, proving the base case; supposing it holds for $2k-2$, we have

$$\begin{aligned} a_{2k} &= -\frac{1}{(2k)(2m+2k)}\frac{(-1)^{k-1}m!}{4^{k-1}(k-1)!(m+k-1)!}a_0 \\ &= \frac{(-1)^k m!}{4^k k(k-1)!(m+k)(m+k-1)!}a_0 \\ &= \frac{(-1)^k m!}{4^k k!(m+k)!}c_0, \end{aligned}$$

proving that it holds for $2k$ as well, and hence for all indices.) As with our definition of the Legendre polynomials, we are free to define a_0; we set $a_0 = \frac{1}{2^m m!}$, so that

$$\begin{aligned} a_{2k} &= \frac{(-1)^k}{2^{2k+m}k!(m+k)!}, \\ q &= \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+m}k!(m+k)!}x^{2k}, \\ Q = x^m q &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!}\left(\frac{x}{2}\right)^{2k+m}. \end{aligned}$$

This function represented by this series is called the *Bessel function of order* m and denoted $J_m(x)$.

Pulling everything back together, then, we see that the solution of equation (2) with $\epsilon = 1$ which has a power series expansion about $x=0$ is given by $J_m(\lambda\rho)$. Now when $\epsilon = 1$ we have for Z the equation

$$\frac{d^2Z}{dz^2} = \lambda^2 Z,$$

which has the general solution $Z = c\cosh\lambda z + d\sinh\lambda z$. Thus the general separated solution to Laplace's equation in this case is

$$u = J_m(\lambda\rho)\left(a\cos m\phi + b\sin m\phi\right)\left(c\cosh\lambda z + d\sinh\lambda z\right).$$

We have already restricted m to be a nonnegative integer, but note that there is as yet no restriction on λ. This is analogous to the situation we were in when solving Laplace's equation in a square in rectangular coordinates: the general solution was in terms of functions $\sin mx$, $\cos mx$, etc., where m was fixed only by the boundary conditions in the x direction. Thus we expect λ to be fixed by the boundary conditions obtaining in ρ. By requiring our solution to be regular at $x=0$, we have already given one boundary condition. Now consider the boundary condition $u|_{\rho=a} = 0$; this gives for λ the equation

$$J_m(a\lambda) = 0.$$

It can be shewn that this equation has an infinite number of solutions. In the case $a=1$, we label them $\lambda_{m,i}$, $i = 1, 2, \ldots$; in the case of general a, then, the correct values of λ are $\frac{1}{a}\lambda_{m,i}$, $i = 1, 2, \ldots$. Unfortunately, unlike for sine and cosine, there is no explicit formula for the $\lambda_{m,i}$, so we shall have to be content with just this notation. (It can be shewn – though we shall not do so here – that the zeroes are discrete (meaning that they do not 'pile up', i.e., have no accumulation point), and that as $i\to\infty$ for fixed m, the spacing becomes constant (see [1], p. 506).)

ORTHOGONALITY AND COMPLETENESS. We would now like to know something about the orthogonality and completeness properties of these Bessel functions. We first note one possible point of confusion which has not arisen in any of our previous studies. Legendre polynomials are complete in the sense that any suitably well-behaved function on $[-1, 1]$ can be expanded in a series

$$\sum_{\ell=0}^{\infty} a_\ell P_\ell(x);$$

similarly, associated Legendre functions $P_{\ell m}$ for fixed m are complete in the sense that any suitably well-behaved function on $[-1, 1]$ can be expanded in the analogous series

$$\sum_{\ell=m}^{\infty} a_\ell P_{\ell m}(x).$$

This might lead us to expect that the completeness result for Bessel functions would state that any suitably well-behaved function on some appropriate interval (perhaps their domain of definition, $[0, \infty)$) can be expanded in a series of the form

$$\sum_{m=0}^{\infty} a_m J_m(x)$$

(this is termed a *Neumann series*). It turns out that various results of this form are true (see [1], Chapter XVI). However, some reflection shows that they are not actually relevant for our current setting.[5] Roughly, this is because the index m is already 'used' in some sense by the orthogonal basis $\{\cos m\phi, \sin m\phi\}$. More precisely, we expect that a general solution to the boundary-value problem we are looking at can be expressed in the form

$$u = \sum_i \sum_{m=0}^{\infty} J_m\left(\frac{1}{a}\lambda_{m,i}\rho\right)(a_{m,i}\cos m\phi + b_{m,i}\sin m\phi)\left(c_{m,i}\cosh\frac{1}{a}\lambda_{m,i}z + d_{m,i}\sinh\frac{1}{a}\lambda_{m,i}z\right).$$

Now since this series by construction satisfies the boundary condition $u|_{\rho=a} = 0$, the only boundary conditions we might have to fit are on surfaces of constant z, say $z = L$. Suppose for the sake of definiteness that we were given the condition $u|_{z=L} = 1$. Then we would need to find an expansion (on the interval $[0, a]$, we should note)

$$1 = \sum_i \sum_{m=0}^{\infty} J_m\left(\frac{1}{a}\lambda_{m,i}\rho\right)(a'_{m,i}\cos m\phi + b'_{m,i}\sin m\phi)$$

(the constants $a'_{m,i}$ and $b'_{m,i}$ will be related but not identical to the constants $a_{m,i}$ and $b_{m,i}$ in the full expansion). As before, we may think of fixing ρ and using orthogonality of the basis $\{\cos m\phi, \sin m\phi\}$ to determine which m-valeus are present; clearly, we have only $m = 0$. Thus we are left with the expansion problem

$$1 = \sum_i a''_{0,i} J_m\left(\frac{1}{a}\lambda_{m,i}\rho\right);$$

in other words, working the expansion out in the ϕ direction gets rid of the index m. (This is analogous to what we did when considering expansions of functions on the sphere in terms of the basis $\{P_{\ell m}(\cos\theta)\cos m\phi, P_{\ell m}(\cos\theta)\sin m\phi\}$, whereby we fixed θ and expanded in $\{\cos m\phi, \sin m\phi\}$ to obtain functions $c_m(\theta)$, $d_m(\theta)$, which were then expanded in a series of $P_{\ell m}(\cos\theta)$ with m fixed.) This result also points us in the direction of the correct completeness result for Bessel functions in our current situation; namely, we expect that for any nonnegative integer m, a suitably well-behaved function on the interval $[0, a]$ will have an expansion of the form

$$\sum_{i=1}^{\infty} a_i J_m\left(\frac{1}{a}\lambda_{m,i}\rho\right), \tag{5}$$

[5] This is not to say that they are not useful for solving boundary-value problems – just that they are not needed for the type of boundary-value problem we are investigating at the moment.

where $\{\frac{1}{a}\lambda_{m,i}\}_{i=1}^{\infty}$ is the set of zeroes of $J_m(ax) = 0$. It turns out that this result is correct (though we shall not prove it here); the expansion above is called a *Fourier-Bessel series.* (See [1], Chapter XVIII, especially 18.24.)[6]

Given the correctness of the above series expansion, we can determine the coefficients a_i if we have an appropriate orthogonality result for the functions $J_m\left(\frac{1}{a}\lambda_{m,i}\rho\right)$ on the interval $[0, a]$. We shall prove an appropriate orthogonality result in the same way we proved orthogonality for the associated Legendre functions; see the lecture notes for the week of June 4, pp. 7–8. We first rewrite Bessel's equation as (here $P = J_m(\lambda\rho)$)

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) - \frac{m^2}{\rho^2}P = -\lambda^2 P.$$

Let us denote the operator on the left-hand side by L, meaning that we denote the entire left-hand side by LP; thus Bessel's equation becomes simply the eigenvalue equation for L, $LP = -\lambda^2 P$. We will now show that L is self-adjoint with respect to an appropriate inner product on $[0, a]$. For integrable functions f and g on $[0, a]$, let

$$(f, g) = \int_0^a \rho f(\rho)\overline{g(\rho)}\, d\rho.$$

Now suppose that f and g satisfy the boundary condition $f(a) = 0$, $g(a) = 0$. Then[7]

$$\begin{aligned}(Lf, g) &= \int_0^a \rho\left(\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{df}{d\rho}\right) - \frac{m^2}{\rho^2}f\right)\overline{g}\, d\rho \\ &= \int_0^a \frac{d}{d\rho}\left(\rho\frac{df}{d\rho}\right) - \frac{m^2}{\rho}f(\rho)\overline{g(\rho)}\, d\rho \\ &= -\int_0^a \rho\frac{df}{d\rho}\overline{\frac{dg}{d\rho}} + \frac{m^2}{\rho}f(\rho)\overline{g(\rho)}\, d\rho\end{aligned}$$

,

where we have performed an integration by parts and used the fact that $g(a) = 0$. Since this expression (up to conjugation, which doesn't matter when f and g are real as they are for us at this point) is clearly symmetric in f and g, we conclude that

$$(Lf, g) = (f, Lg)$$

(alternatively, this can be shewn by performing another integration by parts, as was done when dealing with associated Legendre functions). Now suppose that $f(\rho) = J_m(\frac{1}{a}\lambda_{m,i}\rho)$, $g(\rho) = J_m(\frac{1}{a}\lambda_{m,i'}\rho)$, $i \neq i'$; then the above equation gives (since f and g here clearly satisfy the boundary condition $f(a) = g(a) = 0$)

$$\begin{aligned}(Lf, g) &= -\frac{1}{a^2}\lambda_{m,i}^2(f, g) = (f, Lg) \\ &= -\frac{1}{a^2}\lambda_{m,i'}^2(f, g),\end{aligned}$$

[6]Note that Neumann series and Fourier-Bessel series do not exhaust the possibilities for series expansions in terms of Bessel functions; there are also, for example, *Kapteyn series* and *Schlömilch series* (see [1], Chapters XVII and XIX), but we shall not discuss them here.

[7]We are eliding one subtle point, namely whether the function $\frac{1}{\rho}f(\rho)\overline{g(\rho)}$ is integrable on $[0, a]$. Since we are interested in cases where $f(\rho) = J_m(\lambda\rho)$, $g(\rho) = J_m(\lambda'\rho)$ for some λ, λ', and since J_m has a zero of order m at $\rho = 0$ (i.e., $J_m(\rho) = \rho^m q(\rho)$, where $q(\rho)$ is finite at $\rho = 0$), for us these functions will be integrable when f is as long as $m \neq 0$. But when $m = 0$ this term is not present in L. Thus the calculations below are valid for the cases in which we are interested. [It would be good to see a fuller treatment of this point, but that would be (a) most importantly, outside the expertise of the current author, and (b) probably beyond the scope of the course.]

whence (since $\lambda_{m,i} \neq \lambda_{m,i'}$ as $i \neq i'$) we must have $(f, g) = 0$, showing orthogonality with respect to the given inner product. To calculate the expansion coefficients we need only the normalisation. This is found to be (see [1], 18.24)

$$\int_0^a \rho J_m^2 \left(\frac{1}{a}\lambda_{m,i}\rho\right) d\rho = \frac{1}{2}a^2 J_{m+1}^2 \left(\lambda_{m,i}\right).$$

Thus we may finally write, in expansion (5) above,

$$a_i = \frac{\left(f(\rho), J_m\left(\frac{1}{a}\lambda_{m,i}\rho\right)\right)}{\left(J_m\left(\frac{1}{a}\lambda_{m,i}\rho\right), J_m\left(\frac{1}{a}\lambda_{m,i}\rho\right)\right)} = \frac{2}{a^2 J_{m+1}^2\left(\lambda_{m,i}\right)} \int_0^a \rho f(\rho) J_m\left(\frac{1}{a}\lambda_{m,i}\rho\right) d\rho.$$

We now indicate in general how all of this may be used to solve boundary-value problems. Suppose that we are to solve Laplace's equation on the cylinder $\{(\rho, \phi, z) | \rho < a, \quad 0 \leq z \leq b\}$, with the boundary conditions

$$u|_{\rho=a} = 0, \qquad u|_{z=0} = 0, \qquad u|_{z=b} = f(\rho, \phi).$$

The first condition allows us to conclude that the series will be of the form

$$u = \sum_{m=0}^{\infty}\sum_{i=1}^{\infty} J_m\left(\frac{1}{a}\lambda_{m,i}\rho\right)\left(a_{m,i}\cos m\phi + b_{m,i}\sin m\phi\right)\left(c_{m,i}\cosh\frac{1}{a}\lambda_{m,i}z + d_{m,i}\sinh\frac{1}{a}\lambda_{m,i}z\right),$$

while the second condition then allows us to conclude (since $\cosh 0 = 1$) that $c_{m,i} = 0$ for all m, i; absorbing $d_{m,i}$ into $a_{m,i}$ and $b_{m,i}$, we are left with the expansion

$$u = \sum_{m=0}^{\infty}\sum_{i=1}^{\infty} J_m\left(\frac{1}{a}\lambda_{m,i}\rho\right)\sinh\frac{1}{a}\lambda_{m,i}z\left(a_{m,i}\cos m\phi + b_{m,i}\sin m\phi\right).$$

We may now handle this expansion and the final boundary condition in an analogous way to how we handled the expansion and condition

$$u = \sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell} P_{\ell,m}(\cos\theta) r^{\ell}\left(a_{\ell,m}\cos m\phi + b_{\ell,m}\sin m\phi\right), \qquad u|_{r=a} = f(\theta, \phi).$$

More specifically, we need to expand $f(\rho, \phi)$ in the basis $\{J_m\left(\frac{1}{a}\lambda_{m,i}\rho\right)\cos m\phi, J_m\left(\frac{1}{a}\lambda_{m,i}\rho\right)\sin m\phi\}$; we may do this by first fixing some ρ, and then expanding along ϕ to obtain ρ-dependent coefficients

$$a_m(\rho) = \frac{1}{\pi}\int_0^{2\pi} f(\rho, \phi)\cos m\phi \, d\phi, \qquad b_m(\rho) = \frac{1}{\pi}\int_0^{2\pi} f(\rho, \phi)\sin m\phi \, d\phi$$

for $m > 0$, while for $m = 0$ we have $b_0 = 0$ by convention and

$$a_0(\rho) = \frac{1}{2\pi}\int_0^{2\pi} f(\rho, \phi) \, d\phi.$$

(This separate formula for a_0 was what the factor $\frac{1}{2}$ on the constant term in the Fourier expansions we saw earlier on in class was meant to solve, but we have not adopted that convention here.) This allows us to write

$$f = \sum_{m=0}^{\infty} a_m(\rho)\cos m\phi + b_m(\rho)\sin m\phi.$$

In order to write this as a series along the lines of that for u above, we must now expand $a_m(\rho)$ and $b_m(\rho)$ in series of $\{J_m\left(\frac{1}{a}\lambda_{m,i}\rho\right)\}$, where now m is fixed and only i varies; this is exactly analogous to how we had to expand the coefficient functions $a_m(\theta)$ and $b_m(\theta)$ in the basis $\{P_{\ell,m}(\cos\theta)\}_{\ell=m}^{\infty}$, with fixed m. This will give expansions

$$a_m(\rho) = \sum_{i=1}^{\infty} a_{mi} J_m\left(\frac{1}{a}\lambda_{m,i}\rho\right),$$

and similarly for $b_m(\rho)$. Equating coefficients then allows us to determine u, as usual.
DIFFERENTIATION FORMULAS AND RECURRENCE RELATIONS. In order to calculate the integrals needed to find the coefficients in expansions such as those above, we need results on Bessel functions similar to those we derived for the Legendre polynomials previously. We now take up this question.

PROPOSITION. The Bessel functions satisfy the following four identities:

1. $J_{m-1}(x) - J_{m+1}(x) = 2J_m'(x)$, $m > 0$; $J_0'(x) = -J_1(x)$,

while for $m > 0$ we have

2. $J_{m-1}(x) + J_{m+1}(x) = \frac{2m}{x} J_m(x)$;
3. $J_{m-1}(x) = J_m'(x) + \frac{m}{x} J_m(x)$;
4. $J_{m+1}(x) = -J_m'(x) + \frac{m}{x} J_m(x)$;
5. $\frac{d}{dx}\left(x^m J_m(x)\right) = x^m J_{m-1}(x)$;
6. $\frac{d}{dx}\left(x^{-m} J_m(x)\right) = -x^{-m} J_{m+1}(x)$.

Proof. Recall the series expansion

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!}\left(\frac{x}{2}\right)^{2k+m}.$$

Differentiating this expression term-by-term, we obtain

$$J_m'(x) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!}(k+m+k)\left(\frac{x}{2}\right)^{2k+m-1},$$

where we have written $2k + m = k + m + k$. We expand out these two series separately since they will be useful in proving the second identity also. We see that

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!} k \left(\frac{x}{2}\right)^{2k+m-1} &= -\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k-1)!(m+1+(k-1))!}\left(\frac{x}{2}\right)^{2(k-1)+m+1} \\ &= -J_{m+1}(x),\end{aligned}$$

where in the second sum we may start at $k = 1$ since the $k = 0$ term in the first sum clearly vanishes. Similarly, we see that, for $m > 0$,

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!}(m+k)\left(\frac{x}{2}\right)^{2k+m-1} &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m-1+k)!}\left(\frac{x}{2}\right)^{2k+(m-1)} \\ &= J_{m-1}(x),\end{aligned}$$

while if $m = 0$ then we have as before

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!}(m+k)\left(\frac{x}{2}\right)^{2k+m-1} = -J_1(x).$$

Thus we have, in particular, for $m > 0$,

$$J_m'(x) = \frac{1}{2}\left(J_{m-1}(x) - J_{m+1}(x)\right),$$

while for $m = 0$

$$J_0'(x) = -J_1(x).$$

This proves the first identity. For the second identity, note that, by the foregoing,

$$J_{m-1}(x) + J_{m+1}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!}(m+k-k)\left(\frac{x}{2}\right)^{2k+m-1}$$
$$= \frac{2m}{x}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!}\left(\frac{x}{2}\right)^{2k+m} = \frac{2m}{x}J_m(x).$$

The next two follow by adding and subtracting the first two; specifically,

$$J_{m-1}(x) = \frac{1}{2}\left(2J'_m(x) + \frac{2m}{x}J_m(x)\right) = J'_m(x) + \frac{m}{x}J_m(x),$$

while

$$J_{m+1}(x) = \frac{1}{2}\left(\frac{2m}{x}J_m(x) - 2J'_m(x)\right) = -J'_m(x) + \frac{m}{x}J_m(x).$$

Finally, identites 3 and 4 give

$$\frac{d}{dx}\left(x^m J_m(x)\right) = mx^{m-1}J_m(x) + x^m J'_m(x) = x^m\left(\frac{m}{x}J_m(x) + J'_m(x)\right) = x^m J_{m-1}(x),$$
$$\frac{d}{dx}\left(x^{-m} J_m(x)\right) = -mx^{-m-1}J_m(x) + x^{-m} J'_m(x) = x^{-m}\left(-\frac{m}{x}J_m(x) + J'_m(x)\right) = -x^{-m} J_{m+1}(x).$$

This completes the proof. QED.

Identity 5 above gives rise, for example, to the following integral formula:

$$\int x^m J_{m-1}(x)\,dx = x^m J_m(x) + C,$$

which may be used to calculate the coefficients in the expansion of x^m in a Fourier-Bessel series in Bessel functions of order m. This type of expansion is needed on Homework 6.

Finally, we say a few words about the case $\epsilon = -1$, corresponding to oscillatory behaviour in the z direction; explicitly, Z obeys the equation $Z'' = -\lambda^2 Z$, with general solution $Z = a\cos\lambda z + b\sin\lambda z$, while P satisfies the equation

$$\frac{d^2P}{d\rho^2} + \frac{1}{\rho}\frac{dP}{d\rho} + \left(-\lambda^2 - \frac{m^2}{\rho^2}\right)P = 0. \tag{6}$$

We recall that with $\epsilon = 1$ the solution to this equation is given by

$$J_m(\lambda\rho) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!}\left(\frac{\lambda\rho}{2}\right)^{2k+m}.$$

Now it seems reasonable that replacing λ by $i\lambda$ (where $i = \sqrt{-1}$) should give a solution to equation (6); substituting in to the expression above, and dividing by i^m, we obtain the function

$$I_m(\lambda\rho) = \sum_{k=0}^{\infty} \frac{1}{k!(m+k)!}\left(\frac{\lambda\rho}{2}\right)^{2k+m}.$$

This function, known as a *modified Bessel function of order* m, is in fact a solution to equation (6) which is moreover well-behaved (i.e., finite) at $x = 0$. The other linearly independent solution to equation (6) is denoted $K_m(x)$ and will not be discussed here. Assuming that only the $I_m(\lambda\rho)$ factors occur in our separated solutions, a general separated solution to Laplace's equation in this case is of the form

$$I_m(\lambda\rho)\left(a\cos m\phi + b\sin m\phi\right)\left(c\cos\lambda z + d\sin\lambda z\right).$$

Inspecting and comparing the series expansions of $J_m(x)$ and $I_m(x)$, we note the similarity between them and the series for $\sin x$ and $\sinh x$:

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!} \left(\frac{x}{2}\right)^{2k+m}, \qquad I_m(x) = \sum_{k=0}^{\infty} \frac{1}{k!(m+k)!} \left(\frac{x}{2}\right)^{2k+m},$$
$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}, \qquad \sinh x = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} x^{2k+1}.$$

Thus we see that $J_m(x)$ is the parallel for cylindrical coordinates of the oscillatory solution $\sin x$ in rectangular coordinates, while $I_m(x)$ is the parallel for the non-oscillatory (in fact, exponential) solution $\sinh x$. The parallels between these pairs go even deeper, as can be seen from the derivative and recurrence identities satisfied by $I_m(x)$ (see [1], 3.7); but we shall not go any deeper into these here.

The third practice problem for week 6 makes use of the functions $I_m(x)$.

REFERENCES

1. Watson, G. N. A Treatise on the Theory of Bessel Functions, 2nd ed. Cambridge: Cambridge University Press, 1952.

APM 346 (Summer 2019), Homework 6 solutions.

APM 346, Homework 6. Due Wednesday, June 19, at 6.00 AM EDT. To be marked completed/not completed.

1. Solve the following boundary-value problem on the region $\{(\rho, \phi, z) | \rho < 1, 0 < z < 1\}$ in cylindrical coordinates:

$$\nabla^2 u = 0, \qquad u|_{\rho=1} = 0, \qquad u|_{z=0} = 0, \qquad u|_{z=1} = 1.$$

We know from class that the general solution to Laplace's equation on the given region which satisfies the first boundary condition $u|_{\rho=1} = 0$ is of the form

$$u(\rho, \phi, z) = \sum_{m=0}^{\infty} \sum_{i=1}^{\infty} J_m \left(\lambda_{m,i} \rho\right) \left(a_{mi} \cos m\phi + b_{mi} \sin m\phi\right) \left(c_{mi} \cosh \lambda_{m,i} z + d_{mi} \sinh \lambda_{m,i} z\right),$$

where $\{\lambda_{m,i}\}_{i=1}^{\infty}$ is the set of all positive zeroes of $J_m(x)$. It is now just a matter of determining the coefficients in the above expansion which will make it satisfy the remaining boundary conditions. At $z = 0$, we have

$$u_{z=0} = \sum_{m=0}^{\infty} \sum_{i=1}^{\infty} J_m \left(\lambda_{m,i} \rho\right) \left(a_{mi} \cos m\phi + b_{mi} \sin m\phi\right) c_{mi} = 0;$$

thus (since $\{J_m \left(\lambda_{m,i} \rho\right) \cos m\phi, J_m \left(\lambda_{m,i} \rho\right) \sin m\phi\}$ is a complete orthogonal set on $[0, 1] \times [0, 2\pi]$) we must have $c_{mi} = 0$ for all m and all i. Then we may absorb the coefficients d_{mi} into a_{mi} and b_{mi} and write

$$u(\rho, \phi, z) = \sum_{m=0}^{\infty} \sum_{i=1}^{\infty} J_m \left(\lambda_{m,i} \rho\right) \left(a_{mi} \cos m\phi + b_{mi} \sin m\phi\right) \sinh \lambda_{m,i} z.$$

At $z = 1$, then, we have

$$u|_{z=1} = \sum_{m=0}^{\infty} \sum_{i=1}^{\infty} J_m \left(\lambda_{m,i} \rho\right) \left(a_{mi} \cos m\phi + b_{mi} \sin m\phi\right) \sinh \lambda_{m,i} = 1.$$

Taking the inner product of this with functions $\cos m\phi$, $\sin m\phi$, $m > 0$, we have

$$0 = (1, \cos m\phi) = \sum_{i=1}^{\infty} J_m \left(\lambda_{m,i} \rho\right) \left(a_{mi} \pi\right) \sinh \lambda_{m,i},$$
$$0 = (1, \sin m\phi) = \sum_{i=1}^{\infty} J_m \left(\lambda_{m,i} \rho\right) \left(b_{mi} \pi\right) \sinh \lambda_{m,i},$$

which gives (since $\{J_m \left(\lambda_{m,i} \rho\right)\}_{i=1}^{\infty}$ is a complete orthogonal set on $[0, 1]$) that $a_{mi} = 0$ and $b_{mi} = 0$ for all $m > 0$ and all i. Now $b_{0i} = 0$ for all i by definition, so we are left simply with the condition

$$\sum_{i=1}^{\infty} a_{0i} J_0 \left(\lambda_{0,i} \rho\right) \sinh \lambda_{0,i} = 1.$$

Using the orthogonality properties of the $J_0 \left(\lambda_{0,i} \rho\right)$, we conclude that

$$a_{0i} \sinh \lambda_{0,i} = \frac{\left(1, J_0 \left(\lambda_{0,i} \rho\right)\right)}{\left(J_0 \left(\lambda_{0,i} \rho\right), J_0 \left(\lambda_{0,i} \rho\right)\right)} = \frac{2}{J_1^2 \left(\lambda_{0,i}\right)} \int_0^1 \rho J_0 \left(\lambda_{0,i} \rho\right) \, d\rho.$$

Now

$$\int x J_0(x) \, dx = x J_1(x) + C,$$

so

$$\int_0^1 \rho J_0(\lambda_{0,i}\rho)\, d\rho = \frac{1}{\lambda_{0,i}^2}\int_0^{\lambda_{0,i}} xJ_0(x)\,dx = \frac{1}{\lambda_{0,i}^2}\lambda_{0,i}J_1(\lambda_{0,i}) = \frac{1}{\lambda_{0,i}}J_1(\lambda_{0,i}),$$

and

$$a_{0i} = \frac{2}{\lambda_{0,i}J_1(\lambda_{0,i})\sinh\lambda_{0,i}},$$

so finally

$$u = \sum_{i=1}^{\infty}\frac{2}{\lambda_{0,i}J_1(\lambda_{0,i})\sinh\lambda_{0,i}}J_0(\lambda_{0,i}\rho)\sinh\lambda_{0,i}z.$$

2. The same as 1, except with the condition $u|_{z=1} = 1$ replaced by $u|_{z=1} = \rho\cos\phi$.

The first few steps are of course the same as problem 1; thus we may start from the series expansion

$$u(\rho,\phi,z) = \sum_{m=0}^{\infty}\sum_{i=1}^{\infty} J_m(\lambda_{m,i}\rho)(a_{mi}\cos m\phi + b_{mi}\sin m\phi)\sinh\lambda_{m,i}z.$$

At $z = 1$ we now have

$$u|_{z=1} = \sum_{m=0}^{\infty}\sum_{i=1}^{\infty} J_m(\lambda_{m,i}\rho)(a_{mi}\cos m\phi + b_{mi}\sin m\phi)\sinh\lambda_{m,i} = \rho\cos\phi,$$

whence as before we may conclude that $a_{mi} = 0$, $b_{mi} = 0$ for $m \neq 1$, all i, and also that $b_{1i} = 0$, while

$$\sum_{i=1}^{\infty} a_{1i}J_1(\lambda_{1,i}\rho)\sinh\lambda_{1,i} = \rho.$$

Thus we have as in 1

$$a_{1i}\sinh\lambda_{1,i} = \frac{(\rho, J_1(\lambda_{1,i}\rho))}{(J_1(\lambda_{1,i}\rho), J_1(\lambda_{1,i}\rho))} = \frac{2}{J_2^2(\lambda_{1,i})}\int_0^1 \rho^2 J_1(\lambda_{1,i}\rho)\, d\rho.$$

Now since

$$\int x^2 J_1(x)\,dx = x^2 J_2(x) + C,$$

we have

$$\int_0^1 \rho^2 J_1(\lambda_{1,i}\rho)\, d\rho = \frac{1}{\lambda_{1,i}^3}\lambda_{1,i}^2 J_2(\lambda_{1,i}) = \frac{1}{\lambda_{1,i}}J_2(\lambda_{1,i}),$$

so

$$a_{1i} = \frac{2}{\lambda_{1,i}J_2(\lambda_{1,i})\sinh\lambda_{1,i}}$$

and

$$u = \sum_{i=1}^{\infty}\frac{2}{\lambda_{1,i}J_2(\lambda_{1,i})\sinh\lambda_{1,i}}J_1(\lambda_{1,i}\rho)\cos\phi\sinh\lambda_{1,i}z.$$

Generalities; Laplace's equation

If $\{e_\alpha\}$ is a complete, orthogonal set with respect to an inner product $(\cdot,\cdot)$, then any f can be written $f = \sum_\alpha a_\alpha e_\alpha$, where $a_\alpha = \frac{(f,e_\alpha)}{(e_\alpha,e_\alpha)}$.

Laplace's equation $\nabla^2 u = 0$ has the following general series expansions as its solutions when solved in the indicated regions and with the indicated boundary conditions:

Region and boundary conditions	Series expansion, related complete orthogonal set, and inner product
$\{(x,y)\|0 \le x \le 1,\ \ 0 \le y \le 1\}$ $u\|_{x=0} = u\|_{x=1} = 0$	$u = \sum_{n=0}^{\infty} \sin n\pi x\,(a_n \sinh n\pi y + b_n \cosh n\pi y)$ $\{\sin n\pi x\}_{n=1}^{\infty}, (f(x),g(x)) = \int_0^1 f(x)\overline{g(x)}\,dx$
$\{(r,\theta,\phi)\|r \le a\}$ azimuthally symmetric u finite and single-valued	$u = \sum_{\ell=0}^{\infty} a_\ell P_\ell(\cos\theta) r^\ell$ $\{P_\ell(\cos\theta)\}_{\ell=0}^{\infty}, (f(x),g(x)) = \int_{-1}^1 f(x)\overline{g(x)}\,dx, (f(\theta),g(\theta)) = \int_0^\pi f(\theta)\overline{g(\theta)}\sin\theta\,d\theta$
$\{(r,\theta,\phi)\|a \le r \le b\}$ azimuthally symmetric u finite and single-valued	$u = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta)\left(a_\ell r^\ell + b_\ell r^{-(\ell+1)}\right)$ $\{P_\ell(\cos\theta)\}_{\ell=0}^{\infty}, (f(x),g(x)) = \int_{-1}^1 f(x)\overline{g(x)}\,dx, (f(\theta),g(\theta)) = \int_0^\pi f(\theta)\overline{g(\theta)}\sin\theta\,d\theta$
$\{(\rho,\phi,z)\|\rho \le a,\ \ 0 \le z \le z_0\}$ azimuthally symmetric $u\|_{\rho=a} = 0$, u finite	$u = \sum_{i=1}^{\infty} J_0\left(\frac{\lambda_{0i}}{a}\rho\right)\left(a_i \cosh\frac{\lambda_{0i}}{a}z + b_i \sinh\frac{\lambda_{0i}}{a}z\right)$ $\{J_0\left(\frac{\lambda_{0i}}{a}\rho\right)\}_{i=1}^{\infty}$, $(f(\rho),g(\rho)) = \int_0^a f(\rho)\overline{g(\rho)}\rho\,d\rho$ λ_{mi}, $m \in \mathbf{Z}$, $m \ge 0$, $i \in \mathbf{Z}$, $i \ge 1$ denotes the ith *positive* zero of $J_m(x)$
$\{(r,\theta,\phi)\|a \le r \le b\}$ u finite and single-valued	$u = \sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell} P_{\ell m}(\cos\theta)\,(a_{\ell m}\cos m\phi + b_{\ell m}\sin m\phi)\left(c_{\ell m} r^\ell + d_{\ell m} r^{-(\ell+1)}\right)$ $\{P_{\ell m}(\cos\theta)\cos m\phi, P_{\ell m}(\cos\theta)\sin m\phi \| \ell \in \mathbf{Z}, \ell \ge 0, m \in \mathbf{Z}, 0 \le m \le \ell\}$ $(f(\theta,\phi),g(\theta,\phi)) = \int_0^\pi \int_0^{2\pi} f(\theta,\phi)\overline{g(\theta,\phi)}\sin\theta\,d\phi\,d\theta$
$\{(\rho,\phi,z)\|\rho \le a,\ \ 0 \le z \le z_0\}$ $u\|_{\rho=a} = 0$, u finite	$u = \sum_{m=0}^{\infty}\sum_{i=1}^{\infty} J_m\left(\frac{\lambda_{mi}}{a}\rho\right)(a_{mi}\cos m\phi + b_{mi}\sin m\phi)\left(c_{mi}\cosh\frac{\lambda_{mi}}{a}z + d_{mi}\sinh\frac{\lambda_{mi}}{a}z\right)$ $\{J_m\left(\frac{\lambda_{mi}}{a}\rho\right)\cos m\phi, J_m\left(\frac{\lambda_{mi}}{a}\rho\right)\sin m\phi \| m \in \mathbf{Z}, m \ge 0, i \in \mathbf{Z}, i \ge 1\}$ $(f(\rho,\phi),g(\rho,\phi)) = \int_0^a \int_0^{2\pi} f(\rho,\phi)\overline{g(\rho,\phi)}\rho\,d\phi\,d\rho$ λ_{mi}, $m \in \mathbf{Z}$, $m \ge 0$, $i \in \mathbf{Z}$, $i \ge 1$ denotes the ith *positive* zero of $J_m(x)$

In all cases, solving Laplace's equation proceeds as follows:

1. Determine the correct coordinate system and boundary conditions (including azimuthal symmetry or lack thereof).
2. Assuming this corresponds to an entry in the above table, write down the corresponding general series expansion.
3. Apply the remaining boundary conditions to this series and equate the result to the given boundary data to determine the expansion coefficients.

For the first four examples above, the boundary data is essentially one-dimensional, so that only one set of integrals occurs in step 3. In the last two examples, the expansion part of step 3 can be split into two steps, as follows:

3.1. Expand in ϕ for fixed θ (resp. ρ) to obtain θ- (resp. ρ-) dependent coefficients a_m, b_m.

3.2. Expand a_m and b_m in the basis $\{P_{\ell m}(\cos\theta)\}_{\ell=m}^{\infty}$ (resp. $\{J_m\left(\lambda_{mi}\frac{\rho}{a}\right)\}_{i=1}^{\infty}$; both of these are complete orthogonal sets) to obtain the final expansion coefficients $a_{\ell m}$, $b_{\ell m}$ (resp. a_{mi}, b_{mi}).

Special functions: equations and properties

Associated Legendre functions. These are solutions $P_{\ell m}(x)$, $\ell \in \mathbf{Z}$, $\ell \geq 0$, $m \in \mathbf{Z}$, $0 \leq m \leq \ell$ to the equation

$$\left(1-x^2\right)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + \left(\ell(\ell+1) - \frac{m^2}{1-x^2}\right)P = 0$$

which are finite at $x = 0$. For fixed m, the set $\{P_{\ell m}(x)\}_{\ell=m}^{\infty}$ is complete and orthogonal on the interval $[-1, 1]$ with respect to the inner product $(f(x), g(x)) = \int_{-1}^{1} f(x)\overline{g(x)}\,dx$; equivalently, $\{P_{\ell m}(\cos\theta)\}_{\ell=m}^{\infty}$ is complete and orthogonal (in θ) on the interval $[0, \pi]$ with respect to the inner product $(f(\theta), g(\theta)) = \int_0^{\pi} f(\theta)\overline{g(\theta)}\sin\theta\,d\theta$. They have normalisation

$$\int_{-1}^{1} P_{\ell m}^2(x)\,dx = \int_0^{\pi} P_{\ell m}^2(\cos\theta)\sin\theta\,d\theta = \frac{(\ell+m)!}{(\ell-m)!}\frac{2}{2\ell+1}.$$

The first few for $m > 0$ are as follows. (For $m = 0$, see the Legendre polynomials below.)

$$P_{1,1}(\cos\theta) = \sin\theta, \quad P_{2,1}(\cos\theta) = 3\sin\theta\cos\theta, \quad P_{2,2}(\cos\theta) = 3\sin^2\theta.$$

The associated Legendre functions satisfy the following relation:

$$P_{\ell m}(x) = \left(1-x^2\right)^{\frac{m}{2}}\frac{d^m}{dx^m}P_{\ell,0}(x).$$

Legendre polynomials. When $m = 0$, the associated Legendre functions $P_{\ell,0}(x)$ are polynomials and denoted by $P_\ell(x)$. By the foregoing, they satisfy the equation

$$\left(1-x^2\right)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + \ell(\ell+1)P = 0$$

and form a complete orthogonal set on $[-1, 1]$ with respect to the above-given inner product, with normalisation

$$\int_{-1}^{1} P_\ell^2(x)\,dx = \int_0^{\pi} P_\ell^2(\cos\theta)\sin\theta\,d\theta = \frac{2}{2\ell+1}.$$

The first few are as follows:

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}.$$

They satisfy the following recursion and differentiation relations:

$$(n+1)P_{n+1} - (2n+1)xP_n + nP_{n-1} = 0, \quad P'_{n+1} - 2xP'_n + P'_{n-1} = P_n, \quad xP'_n - P'_{n-1} = nP_n,$$
$$P'_{n+1} - P'_{n-1} = (2n+1)P_n, \quad (1-x^2)P'_n = nP_{n-1} - nxP_n.$$

$P_\ell(x)$ is an odd or even function as ℓ is odd or even. Thus $P_\ell(0) = 0$ if ℓ is odd.

Bessel functions. These are solutions $J_m(x)$, $m \in \mathbf{Z}$, $m \geq 0$ to the equation

$$\frac{d^2J}{dx^2} + \frac{1}{x}\frac{dJ}{dx} + \left(1 - \frac{m^2}{x^2}\right)J = 0$$

which are finite at $x = 0$. It can be shewn that each $J_m(x)$ has infinitely many zeroes, and we denote the ith *positive* zero of J_m by λ_{mi}, $i = 1, 2, \ldots$. It can be shewn that the spacing between zeroes approaches a constant value when $i \to +\infty$, but there is no closed-form formula for them. $J_m(x)$ has the Taylor series expansion

$$J_m(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!(m+k)!}\left(\frac{x}{2}\right)^{2k+m}.$$

For any positive number a and any $m \geq 0$, the set $\{J_m\left(\lambda_{mi}\frac{\rho}{a}\right)\}_{i=1}^{\infty}$ is complete orthogonal on the interval $[0, a]$ with respect to the inner product $(f(\rho), g(\rho)) = \int_0^a f(\rho)\overline{g(\rho)}\rho\,d\rho$. They have normalisation

$$\int_0^a J_m^2\left(\lambda_{mi}\frac{\rho}{a}\right)\rho\,d\rho = \frac{1}{2}a^2J_{m+1}^2\left(\lambda_{mi}\right).$$

The Bessel functions cannot be expressed in any simple way in terms of elementary functions. They satisfy the relations ($m > 0$)

$$J'_0(x) = -J_1(x),$$
$$J_{m-1}(x) - J_{m+1}(x) = 2J'_m(x), \quad J_{m-1}(x) + J_{m+1}(x) = \frac{2m}{x}J_m(x), \quad J_{m-1}(x) = J'_m(x) + \frac{m}{x}J_m(x),$$
$$J_{m+1}(x) = -J'_m(x) + \frac{m}{x}J_m(x), \quad \frac{d}{dx}\left(x^mJ_m(x)\right) = x^mJ_{m-1}(x), \quad \frac{d}{dx}\left(x^{-m}J_m(x)\right) = -x^{-m}J_{m+1}(x).$$

APM346 (Summer 2019), Term Test.

Instructor: Nathan Carruth

Name: Nathan Carruth

– Dude, who asked you to write your own test?
– me, of course! Someone as told me to mark it afterwards! ☺

Student Number: ______

Signature: ______

This test will run for 120 minutes, beginning at 7.00 PM EDT.

No aids of any form are allowed. Do not open the test until instructed to do so.

There are six questions on this test, for a total of 70 marks. The weighting is indicated on each question. You must show all of your work for credit.

You may use the back sides of the pages, as well as pages 15 and 16, to continue your solutions, as long as this is clearly indicated.

1. [10 marks] Solve on $\{(x,y)|0 \le x \le 1,\ 0 \le y \le 1\}$:

$$\nabla^2 u = 0, \quad u|_{x=0} = u|_{x=1} = 0, \ u|_{y=0} = 0, \ u|_{y=1} = \begin{cases} x, & 0 \le x \le \frac{1}{2} \\ 1-x, & \frac{1}{2} \le x \le 1 \end{cases}.$$

From the first two boundary conditions, we see that the solution may be expressed as a series [this is one of the standard series from the review sheet]

1 $$u = \sum_{n=1}^{\infty} \sin n\pi x \left(a_n \cosh n\pi y + b_n \sinh n\pi y\right).$$

The third boundary condition then gives

$$0 = u|_{y=0} = \sum_{n=1}^{\infty} a_n \sin n\pi x,$$

1 which means that $a_n = 0$ for all n. Thus

$$u = \sum_{n=1}^{\infty} b_n \sin n\pi x \sinh n\pi y.$$

The final condition then gives

1 $$u|_{y=1} = \sum_{n=1}^{\infty} b_n \sinh n\pi \sin n\pi x = \begin{cases} x, & 0 \le x \le 1/2 \\ 1-x, & 1/2 \le x \le 1. \end{cases}$$

This gives (denoting the function on the right by $h(x)$)

1 0.5 1

$$\sinh n\pi\, b_n = \frac{(h(x), \sin n\pi x)}{(\sin n\pi x, \sin n\pi x)} = 2\left(\int_0^{1/2} x \sin n\pi x\, dx + \int_{1/2}^{1} (1-x) \sin n\pi x\, dx\right).$$

– since $\sin[n\pi(1-x)] = \sin(n\pi - n\pi x) = (-1)^{n+1} \sin n\pi x$, you might be able to use symmetry to simplify things here. At any rate this could explain why so many terms end up cancelling in your final answer below.

Now, using integration by parts,

1.5 $$\int x \sin n\pi x\, dx = \frac{-1}{n\pi} x \cos n\pi x + \frac{1}{n\pi}\int \cos n\pi x\, dx = -\frac{1}{n\pi} x \cos n\pi x + \frac{1}{n^2\pi^2} \sin n\pi x + C$$

so

$$\sinh n\pi\, b_n = 2\left(-\frac{1}{n\pi} x \cos n\pi x + \frac{1}{n^2\pi^2} \sin n\pi x \Big|_0^{1/2} - \left[-\frac{1}{n\pi} x \cos n\pi x + \frac{1}{n^2\pi^2} \sin n\pi x\right]\Big|_{1/2}^{1} - \frac{1}{n\pi} \cos n\pi x \Big|_{1/2}^{1}\right)$$

2 $$= 2\left(-\frac{1}{2n\pi} \cos\frac{n\pi}{2} + \frac{1}{n^2\pi^2} \sin\frac{n\pi}{2} - \left(-\frac{1}{n\pi}(-1)^n - \left[-\frac{1}{2n\pi} \cos\frac{n\pi}{2} + \frac{1}{n^2\pi^2} \sin\frac{n\pi}{2}\right]\right) - \frac{1}{n\pi}\left((-1)^n - \cos\frac{n\pi}{2}\right)\right)$$

$$= -\frac{2}{n\pi} \cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2} \sin\frac{n\pi}{2} + \frac{2}{n\pi} \cos\frac{n\pi}{2} = \frac{4}{n^2\pi^2} \sin\frac{n\pi}{2}.$$ 0.5

Now $\sin\frac{n\pi}{2} = 0$ when n is even, while $\sin\frac{(2k+1)\pi}{2} = (-1)^k$; thus we have for u the expansion

0.5 $$u = 4\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2\pi^2} \sin[(2k+1)\pi x] \sinh[(2k+1)\pi y] / \sinh[(2k+1)\pi].$$

2. [10 marks] Solve on $\{(x,y)|0 \le x \le 1,\ 0 \le y \le 1\}$:

$$\nabla^2 u = 0, \quad u|_{x=0} = 0, \ u|_{x=1} = -\left.\frac{\partial u}{\partial x}\right|_{x=1}, \quad u|_{y=0} = 0, \ u|_{y=1} = x.$$

We start by looking for separated solutions; ~~satisfying the first two boundary conditions;~~ i.e., for solutions $u(x,y) = X(x)Y(y)$. This gives as usual

This is a good way of looking at things, though we haven't necessarily talked about it in class much

.5 $\frac{X''}{X} + \frac{Y''}{Y} = 0, \quad X'' = -\lambda^2 X, \quad Y'' = \lambda^2 Y,$

OK

where we take $X''/X < 0$ .5 so as to be able to fit the first two boundary conditions. Now applying these to the separated solution gives

$$X'' = -\lambda^2 X, \quad X(0) = 0, \quad X(1) = -X'(1),$$

so $X = a\sin\lambda x + b\cos\lambda x$; .5 $X(0) = 0$ means $b = 0$, .5 while $X(1) = -X'(1)$ means

If $\lambda = 0$, the sol'n for X would be $a + bx$; $X(0) = 0$ gives $a = 0$, while $X(1) = -X'(1)$ gives $b = -b$, so $b = 0$. Thus we can ignore the $\lambda = 0$ solution and only take $\lambda > 0$. ($\lambda < 0$ would give the same equation and hence would just duplicate the functions we have here.)

$$a\sin\lambda = -a\lambda\cos\lambda, \quad \lambda = -\tan\lambda. \quad .5$$

This equation has infinitely many positive solutions, which we denote $\lambda_1, \lambda_2, \ldots$ in increasing order. Then the general solution to Laplace's equation satisfying the first two boundary conditions will be

1 $$u = \sum_{n=1}^{\infty} \sin\lambda_n x\,(a_n \sinh\lambda_n y + b_n \cosh\lambda_n y).$$

Note the similarity to the series sol'n in problem 1.

.5 $u|_{y=0} = 0$ gives $b_n = 0$ (as in problem 1), while $u|_{y=1} = x$ gives

.5 $$\sum_{n=1}^{\infty} a_n \sinh\lambda_n \sin\lambda_n x = x,$$

whence (we know from the homework that $\{\sin\lambda_n x\}$ is an orthogonal set)

on $[0,1]$!

Perhaps I should have put a ~~reminder about this on the test paper.~~ On the other hand I'm pretty sure this came up on the practice problems.

1 $$a_n \sinh\lambda_n = \frac{(x, \sin\lambda_n x)}{(\sin\lambda_n x, \sin\lambda_n x)}.$$

(~~from~~ see problem 1) OK

1 Now $\int x\sin\lambda_n x\,dx = -\frac{1}{\lambda_n} x\cos\lambda_n x + \frac{1}{\lambda_n^2}\sin\lambda_n x + C$, so

1 $$(x, \sin\lambda_n x) = \int_0^1 x\sin\lambda_n x\,dx = -\frac{1}{\lambda_n} x\cos\lambda_n x + \frac{1}{\lambda_n^2}\sin\lambda_n x\Big|_0^1 = -\frac{1}{\lambda_n}\cos\lambda_n + \frac{1}{\lambda_n^2}\sin\lambda_n$$

$$= -\frac{1}{\lambda_n}\cos\lambda_n\left(1 - \frac{1}{\lambda_n}\frac{\sin\lambda_n}{\cos\lambda_n}\right) = -\frac{2}{\lambda_n}\cos\lambda_n,$$

good [the previous expression above would be sufficient though]

while

$$(\sin\lambda_n x, \sin\lambda_n x) = \int_0^1 \sin^2\lambda_n x\,dx = \int_0^1 \frac{1}{2}(1 - \cos 2\lambda_n x)\,dx = \frac{1}{2} - \frac{1}{4\lambda_n}\sin 2\lambda_n$$

1 $$= \frac{1}{2} - \frac{1}{2\lambda_n}\sin\lambda_n\cos\lambda_n = \frac{1}{2}(1 + \cos^2\lambda_n) = \cancel{\frac{1}{2}\sin^2\lambda_n},$$

You forgot the − sign in $\frac{1}{\lambda_n}\sin\lambda_n = -\cos\lambda_n \ldots$

So (see p. 6)

$$\overset{a_n}{\cancel{b_n}} \sinh \lambda_n = \frac{-2/\lambda_n \cos\lambda_n}{\frac{1}{2}\cancel{\sin^2\lambda_n}\,(1+\cos^2\lambda_n)} = -\cancel{\frac{4\cos\lambda_n}{\lambda_n \sin^2\lambda_n}} = -\cancel{\frac{4}{\lambda_n^2 \sin\lambda_n}}, \quad -\frac{4\cos\lambda_n}{\lambda_n(1+\cos^2\lambda_n)}$$

and the solution is

1 $$u = \sum_{n=1}^{\infty} -\frac{4\cos\lambda_n}{\lambda_n^2 \cancel{\sin\lambda_n}\,(1+\cos^2\lambda_n)} \sin \lambda_n x \, \frac{\sinh \lambda_n y}{\sinh \lambda_n}.$$

3. [5 marks] Solve on $\{(r,\theta,\phi) | 1 < r < 2\}$:

$$\nabla^2 u = 0, \quad u|_{r=1} = \cos\theta, \quad u|_{r=2} = 0.$$

Since we are solving on a spherical shell, and the boundary data are azimuthally symmetric, we have that u can be written as

.5 $$u = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta)\left(a_\ell r^\ell + b_\ell r^{-(\ell+1)}\right).$$

The boundary conditions then give

.5 $$u|_{r=1} = \cos\theta = P_1(\cos\theta) = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta)\,(a_\ell + b_\ell)$$

.5 $$u|_{r=2} = 0 = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta)\left(a_\ell 2^\ell + b_\ell 2^{-(\ell+1)}\right);$$ .5

by inspection, then, we have $a_\ell 2^\ell + b_\ell 2^{-(\ell+1)} = 0$ for all ℓ, while $a_\ell + b_\ell = 0$ if $\ell \neq 1$ and $a_1 + b_1 = 1$. .5 Thus if $\ell = 1$ we have

1 $$\begin{aligned} a_1 + b_1 &= 1 \\ 2a_1 + \tfrac{1}{\cancel{2}\,4} b_1 &= 0 \end{aligned} \qquad a_1 = -\tfrac{1}{\cancel{4}\,8} b_1 \qquad \tfrac{7}{8}\,\cancel{\tfrac{3}{4}}\, b_1 = 1 \qquad b_1 = \cancel{\tfrac{4}{3}}\ \tfrac{8}{7}, \quad a_1 = \cancel{-\tfrac{1}{3}}\ -\tfrac{1}{7}$$

while if $\ell \neq 1$

1 $$\begin{aligned} a_\ell + b_\ell &= 0 \\ 2^\ell a_\ell + 2^{-(\ell+1)} b_\ell &= 0 \end{aligned} \qquad a_\ell = -2^{-(2\ell+1)} b_\ell \qquad b_\ell\left(1 - \frac{1}{2^{2\ell+1}}\right) = 0 \qquad b_\ell = 0,\ a_\ell = 0.$$

Thus finally

.5 $$u = P_1(\cos\theta)\left(\cancel{-\tfrac{1}{3}}\,{-\tfrac{1}{7}}\, r + \cancel{\tfrac{4}{3}}\,\tfrac{8}{7}\, r^{-2}\right).$$

4. [10 marks] Solve on $\{(r,\theta,\phi)|1<r<2\}$:

$$\nabla^2 u = 0, \quad u|_{r=1} = \left.\frac{\partial u}{\partial r}\right|_{r=1}, \quad u|_{r=2} = \begin{cases} -1, & 0 \le \theta < \frac{\pi}{2} \\ 1, & \frac{\pi}{2} < \theta \le \pi \end{cases}.$$

The following may be useful: $(2\ell+1)P_\ell(x) = P'_{\ell+1}(x) - P'_{\ell-1}(x)$. [Hint: do not confuse this problem with problem 2!]

As in problem 3, we have the series expansion

.5 $$u = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta)\left(a_\ell r^\ell + b_\ell r^{-(\ell+1)}\right);$$

the boundary conditions give (letting $h(x) = \begin{cases} 1, & -1 \le x < 0 \\ -1, & 0 < x \le 1 \end{cases}$, so that $u|_{r=2} = h(\cos\theta)$) ✓

$$u|_{r=1} = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta)(a_\ell + b_\ell) = u_r|_{r=1} = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta)(\ell a_\ell - (\ell+1)b_\ell);$$ .5

so $(1-\ell)a_\ell + (\ell+2)b_\ell = 0$ for all ℓ, .5

.5 $$u|_{r=2} = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta)\left(a_\ell 2^\ell + b_\ell 2^{-(\ell+1)}\right) = h(\cos\theta);$$

this gives 1

$$a_\ell 2^\ell + b_\ell 2^{-(\ell+1)} = \frac{(h(\cos\theta), P_\ell(\cos\theta))}{(P_\ell(\cos\theta), P_\ell(\cos\theta))} = \frac{2\ell+1}{2}(h(\cos\theta), P_\ell(\cos\theta)).$$ ✓ .5

Now (let us denote this quantity by c_ℓ)

$$(h(\cos\theta), P_\ell(\cos\theta)) = \int_0^\pi h(\cos\theta)P_\ell(\cos\theta)\sin\theta\, d\theta = \int_{-1}^{1} h(x)P_\ell(x)\,dx$$ .5 .5

$$= \int_{-1}^{0} P_\ell(x)\,dx - \int_0^1 P_\ell(x)\,dx = -\int_0^1 (P_\ell(x) - P_\ell(-x))\,dx,$$ ✓ .5

so $c_\ell = 0$ if ℓ is even, .5 since then $P_\ell(x) = P_\ell(-x)$, while if $\ell = 2k+1$ we have

$$c_{2k+1} = -2\int_0^1 P_{2k+1}(x)\,dx = -\frac{2}{4k+3}\int_0^1 P'_{2k+2}(x) - P'_{2k}(x)\,dx = -\frac{2}{4k+3}\left[P_{2k+2}(1) - P_{2k}(1) - (P_{2k+2}(0) - P_{2k}(0))\right]$$ .5

$$= + \frac{2}{4k+3}(P_{2k+2}(0) - P_{2k}(0)),$$ .5

So if ℓ is even $a_\ell 2^\ell + b_\ell 2^{-(\ell+1)} = 0$, while if ℓ is odd

$$a_\ell 2^\ell + b_\ell 2^{-(\ell+1)} = +(P_{\ell+1}(0) - P_{\ell-1}(0)).$$

(See p. 10.)

Thus for ℓ even we have

1 $\quad \begin{cases} (1-\ell)a_\ell + (\ell+2)b_\ell = 0 \\ a_\ell 2^\ell + b_\ell 2^{-(\ell+1)} = 0 \end{cases} \quad a_\ell = -2^{-(2\ell+1)} b_\ell \quad b_\ell\left[(\ell+2) - (\ell-1)2^{-(2\ell+1)}\right] = 0$

$b_\ell = 0,\ a_\ell = 0,$

while if ℓ is odd we have

$(1-\ell)a_\ell + (\ell+2)b_\ell = 0 \qquad b_\ell = \frac{\ell-1}{\ell+2} a_\ell$, .5

$a_\ell 2^\ell + b_\ell 2^{-(\ell+1)} = +(P_{\ell+1}(0) - P_{\ell-1}(0))$

$a_\ell\left[2^\ell + 2^{-(\ell+1)}\frac{\ell-1}{\ell+2}\right] = -(P_{\ell-1}(0) - P_{\ell+1}(0))$, .5

2 ✓ It looks rather like you made a copy error which cancelled the above sign error...

$a_\ell = \dfrac{P_{\ell+1}(0) - P_{\ell-1}(0)}{2^\ell + 2^{-(\ell+1)}\frac{\ell-1}{\ell+2}}$, .5 $\quad b_\ell = \dfrac{\ell-1}{\ell+2} \cdot \dfrac{P_{\ell+1}(0) - P_{\ell-1}(0)}{2^\ell + 2^{-(\ell+1)}\cdot\frac{\ell-1}{\ell+2}}$ . .5

This gives Finally for u

.5 $\quad u = \sum_{k=0}^{\infty} P_{2k+1}(\cos\theta)\left(\dfrac{P_{2k+2}(0) - P_{2k}(0)}{2^{2k+1} + 2^{-(2k+2)}\frac{2k}{2k+3}}\right)\left(r^{2k+1} + \dfrac{2k}{2k+3} r^{-(2k+\cancel{2})}\right).$ ok

5. [20 marks] Solve on $\{(r,\theta,\phi) | r < 1\}$:

$$\nabla^2 u = 0, \quad u|_{r=1} = \begin{cases} \sin\theta\cos\phi, & 0 \le \theta < \frac{\pi}{2} \\ -\sin\theta\cos\phi, & \frac{\pi}{2} < \theta \le \pi \end{cases}.$$

The following identities may be useful: $(1-x^2)P_\ell' = \ell P_{\ell-1} - \ell x P_\ell$, $xP_\ell = \frac{\ell+1}{2\ell+1}P_{\ell+1} + \frac{\ell}{2\ell+1}P_{\ell-1}$, $\int_0^1 P_\ell(x)\,dx = \frac{1}{2\ell+1}(P_{\ell-1}(0) - P_{\ell+1}(0))$. You may leave your final answer in terms of $P_\ell(0)$ if it is nonzero. [Hint: what is the relationship between $P_{\ell m}$ and P_ℓ?] $\int_{-1}^1 P_{\ell m}^2(x)\,dx = \frac{(\ell+m)!}{(\ell-m)!}\frac{2}{2\ell+1}$

Since we no longer have azimuthal symmetry, but are solving on the whole sphere, we have the series expansion

$$u = \sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell} P_{\ell m}(\cos\theta)(a_{\ell m}\cos m\phi + b_{\ell m}\sin m\phi) r^\ell.$$

The boundary condition gives (letting h be the function from 3)

$$u|_{r=1} = \sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell} P_{\ell m}(\cos\theta)(a_{\ell m}\cos m\phi + b_{\ell m}\sin m\phi) = -\sin\theta\, h(\cos\theta)\cos\phi,$$

whence we see that $b_{\ell m}=0$ for all ℓ, m, while $a_{\ell m}=0$ unless $m=1$ and

$$\sum_{\ell=1}^{\infty} a_{\ell 1} P_{\ell 1}(\cos\theta) = -\sin\theta\, h(\cos\theta).$$

By the orthogonality of $\{P_{\ell 1}(x)\}$ on $[-1,1]$, this gives

$$a_{\ell 1} = \frac{(-\sin\theta\, h(\cos\theta), P_{\ell 1}(\cos\theta))}{(P_{\ell 1}(\cos\theta), P_{\ell 1}(\cos\theta))} = \frac{(\ell-1)!}{(\ell+1)!}\frac{2\ell+1}{2}\int_0^\pi -\sin^2\theta\, h(\cos\theta) P_{\ell 1}(\cos\theta)\,d\theta.$$

Performing the usual change of variables $x = \cos\theta$ on this integral gives

$$\int_{-1}^1 -(1-x^2)^{1/2} h(x) P_{\ell 1}(x)\,dx = -\int_{-1}^0 (1-x^2)^{1/2} P_{\ell 1}(x)\,dx + \int_0^1 (1-x^2)^{1/2} P_{\ell 1}(x)\,dx.$$

Now

$$P_{\ell m}(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_\ell(x), \qquad P_{\ell 1}(x) = (1-x^2)^{1/2}P_\ell'(x),$$

so

$$\int (1-x^2)^{1/2} P_{\ell 1}(x)\,dx = \int (1-x^2)P_\ell'(x)\,dx = \int \ell P_{\ell-1}(x) - \ell x P_\ell(x)\,dx$$

$$= \int \ell P_{\ell-1}(x) - \left[\frac{\ell^2+\ell}{2\ell+1} P_{\ell+1}(x) + \frac{\ell^2}{2\ell+1}P_{\ell-1}(x)\right] dx$$

(see p. 12)

$$\int (1-x^2)\cancel{^{1/2}} P_\ell'(x)\,dx = \frac{\ell^2+\ell}{2\ell+1}\int P_{\ell-1}(x) - P_{\ell+1}(x)\,dx,$$

so (note that we have $\ell \geq 1$, so $\ell - 1 \geq 0$)

$$\int_0^1 (1-x^2)^{1/2} P_{\ell 1}(x)\,dx = \int_0^1 (1-x^2) P_\ell'(x)\,dx = \frac{\ell^2+\ell}{2\ell+1}\int_0^1 P_{\ell-1}(x) - P_{\ell+1}(x)\,dx$$

1 $$= \frac{\ell^2+\ell}{2\ell+1}\cdot\left[\frac{1}{2\ell-1}\left(P_{\ell-2}(0) - P_\ell(0)\right) - \frac{1}{2\ell+3}\left(P_\ell(0) - P_{\ell+2}(0)\right)\right]$$

while

1 $$\int_{-1}^0 (1-x^2)^{1/2} P_{\ell 1}(x)\,dx = \int_{-1}^0 (1-x^2) P_\ell'(x)\,dx = \int_0^1 (1-x^2) P_\ell'(-x)\,dx;$$

now P_ℓ' is odd when ℓ is even, and vice versa, so that when ℓ is even we have

$$a_{\ell 1} = \frac{1}{\ell(\ell+1)}\cdot\frac{2\ell+1}{2}\left[2\int_0^1 (1-x^2)^{1/2} P_{\ell 1}(x)\,dx\right] = \frac{2\ell+1}{\ell(\ell+1)}\cdot\frac{\ell^2+\ell}{2\ell+1}\left[\frac{1}{2\ell-1}\left(P_{\ell-2}(0) - P_\ell(0)\right) - \frac{1}{2\ell+3}\left(P_\ell(0) - P_{\ell+2}(0)\right)\right]$$

2 $$= \frac{1}{2\ell-1}\left(P_{\ell-2}(0) - P_\ell(0)\right) - \frac{1}{2\ell+3}\left(P_\ell(0) - P_{\ell+2}(0)\right),$$

while if ℓ is odd we have $\int_0^\pi -\sin\theta\, h(\cos\theta) P_{\ell 1}(\cos\theta)\,d\theta = 0$, so $a_\ell = 0$. Thus we have for u (remember that $\ell \geq 1$!)

2 $$u = \sum_{k=1}^\infty P_{2k}(\cos\theta)\, r^{2k}\cdot\left[\frac{1}{4k-1}\left(P_{2k-2}(0) - P_{2k}(0)\right) - \frac{1}{4k+3}\left(P_{2k}(0) - P_{2k+2}(0)\right)\right].$$

cos φ

6. [15 marks] Solve on $\{(\rho,\phi,z)|\rho<1,\ 0\le z\le 1\}$:

$$\nabla^2 u = 0,\quad u|_{\rho=1}=0,\ u|_{z=0}=0,\ u|_{z=1}=\begin{cases}-\rho^3\cos 3\phi, & 0\le\rho<\frac12\\ \rho^3\cos 3\phi, & \frac12<\rho<1\end{cases}.$$

The following identity may be useful: $\frac{d}{dx}(x^m J_m(x)) = x^m J_{m-1}(x)$. You may leave your final answer in terms of quantities of the form $J_{m+1}(\frac12\lambda_{mi})$ and $J_{m+1}(\lambda_{mi})$ (as long as you say what the λ_{mi} are!).

We have the series expansion (since $u|_{\rho=1}=0$; here λ_{mi} is the ith positive zero of J_m) .5

1 $$u=\sum_{m=0}^{\infty}\sum_{i=1}^{\infty} J_m(\lambda_{mi}\rho)(a_{mi}\cos m\phi + b_{mi}\sin m\phi)(c_{mi}\cosh\lambda_{mi}z + d_{mi}\sinh\lambda_{mi}z)$$

The condition $u|_{z=0}=0$ gives

$$\sum_{m=0}^{\infty}\sum_{i=1}^{\infty} J_m(\lambda_{mi}\rho)(a_{mi}\cos m\phi+b_{mi}\sin m\phi)c_{mi}=0,$$ .5

whence we take $c_{mi}=0$ .5 for all m and all i. Redefining a_{mi} and b_{mi} by absorbing d_{mi}, we may write the third boundary condition as

$$\sum_{m=0}^{\infty}\sum_{i=1}^{\infty} J_m(\lambda_{mi}\rho)(a_{mi}\cos m\phi+b_{mi}\sin m\phi)\sinh\lambda_{mi}=g(\rho)\cos 3\phi,$$ .5

where $g(\rho)=\begin{cases}-\rho^3, & 0\le\rho<\frac12\\ \rho^3, & \frac12<\rho<1\end{cases}$. 1 As in problem 5, this allows us to conclude that $b_{mi}=0$ for all m and all i, while $a_{mi}=0$ for $m\ne 3$ and all i 1 and finally

$$\sum_{i=1}^{\infty} J_3(\lambda_{3i}\rho)a_{3i}\sinh\lambda_{3i}=g(\rho).$$ .5

By the orthogonality of $\{J_3(\lambda_{3i}\rho)\}$ on $[0,1]$, we have

$$a_{3i}\sinh\lambda_{3i}=\frac{(g(\rho),J_3(\lambda_{3i}\rho))}{(J_3(\lambda_{3i}\rho),J_3(\lambda_{3i}\rho))}=\frac{2}{J_4^2(\lambda_{3i})}\int_0^1 g(\rho)J_3(\lambda_{3i}\rho)\rho\,d\rho$$ 1 1 1

$$=\frac{2}{J_4^2(\lambda_{3i})}\left(-\int_0^{1/2}\rho^4 J_3(\lambda_{3i}\rho)\,d\rho+\int_{1/2}^1\rho^4 J_3(\lambda_{3i}\rho)\,d\rho\right).$$ .5

Now from the given identity,

$$\int x^4 J_3(x)\,dx = x^4 J_4(x)+C,\quad \int\rho^4 J_3(\lambda_{3i}\rho)\,d\rho=\frac{1}{\lambda_{3i}^5}\left[\lambda_{3i}^4\rho^4 J_4(\lambda_{3i}\rho)+C\right]$$ 0.5 1

$$=\frac{1}{\lambda_{3i}}\rho^4 J_4(\lambda_{3i}\rho)+C,$$ .5

(see p. 14)

whence

$$a_{3i} = \frac{2}{J_4^2(\lambda_{3i})\sinh\lambda_{3i}}\left(-\frac{1}{\lambda_{3i}}\rho^4 J_4(\lambda_{3i}\rho)\Big|_0^{1/2} + \frac{1}{\lambda_{3i}}\rho^4 J_4(\lambda_{3i}\rho)\Big|_{1/2}^{1}\right) \quad .5$$

$$= \frac{2}{J_4^2(\lambda_{3i})\sinh\lambda_{3i}}\left(-\frac{1}{16\lambda_{3i}}J_4(\tfrac{1}{2}\lambda_{3i}) + \frac{1}{\lambda_{3i}}J_4(\lambda_{3i}) - \frac{1}{16\lambda_{3i}}J_4(\tfrac{1}{2}\lambda_{3i})\right) \quad 1$$

$$= \frac{2}{\lambda_{3i}J_4(\lambda_{3i})\sinh\lambda_{3i}}\left(1 - \frac{1}{8}\frac{J_4(\frac{1}{2}\lambda_{3i})}{J_4(\lambda_{3i})}\right). \quad .5$$

and we have finally for u

2 $$u = \sum_{i=1}^{\infty} J_3(\lambda_{3i}\rho)\cos 3\varphi \sinh\lambda_{3i}z \cdot \frac{2}{\lambda_{3i}J_4(\lambda_{3i})\sinh\lambda_{3i}}\left(1 - \frac{1}{8}\frac{J_4(\frac{1}{2}\lambda_{3i})}{J_4(\lambda_{3i})}\right).$$

Scratch work

Summary:

- We introduce the modified Bessel functions I_m in greater detail, and show how they can be used to solve certain boundary-value problems for Laplace's equation on a cylinder.
- We then show how to use J_m and I_m together to solve the most general kind of boundary-value problem for Laplace's equation on a cylinder.
- We show how to solve Laplace's equation on a rectangular prism using rectangular coordinates in three dimensions, and point out that the most general problem requires using three separate series.
- We then give a brief introduction to the eigenvalue problem for the Laplacian, including why it is useful.

MODIFIED BESSEL FUNCTIONS. Recall that Laplace's equation in cylindrical coordinates is given by

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = 0,$$

while substituting in the separated $u = P(\rho)\Phi(\phi)Z(z)$ and dividing by u gives the equation

$$\frac{P''}{P} + \frac{P'}{\rho P} + \frac{1}{\rho^2}\frac{\Phi''}{\Phi} + \frac{Z''}{Z} = 0,$$

from which we see that we must have both $\frac{\Phi''}{\Phi}$ and $\frac{Z''}{Z}$ constant. If we are considering problems on the whole range $[0, 2\pi]$ of ϕ, then Φ must be periodic with period 2π, and this means that $\frac{\Phi''}{\Phi} = -m^2$ for some $m \in \mathbf{Z}$, $m \geq 0$. This leaves the question as to what $\frac{Z''}{Z}$ is. Previously we considered the case where $\frac{Z''}{Z} > 0$ and then showed that this together with the boundary condition $u|_{\rho=1} = 0$ gave rise to solutions for P of the form $J_m(\lambda_{mi}\rho)$, where λ_{mi} is the ith zero of J_m. At the end of the last set of lecture notes (June 11 – 13), we gave a brief discussion of the case where $\frac{Z''}{Z} < 0$. We would now like to treat this in greater detail.

Thus suppose that $\frac{Z''}{Z} = -\mu^2$, where we may assume $\mu \geq 0$. This means that $Z(z) = c\cos\mu z + d\sin\mu z$ for some constants c and d, and that P satisfies the equation

$$P'' + \frac{1}{\rho}P' - \left(\mu^2 + \frac{m^2}{\rho^2}\right)P = 0.$$

We see that this is formally the same as the equation satisfied by $J_m(\lambda\rho)$, but with $\lambda = i\mu$. This suggests that a solution to this equation which is well-behaved at 0 is

$$P(\rho) = J_m(i\mu\rho).$$

However, we have so far only defined J_m for real values of the independent variable, so it is not clear a priori what this expression should mean. Recall though that we defined J_m via the power series

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!}\left(\frac{x}{2}\right)^{2k+m},$$

which converges for all real x. It can be shown that this power series also converges for all complex x also, and thus we define $J_m(x)$ for any complex number x to be equal to the sum of the above power series. (This is analogous to how we used the power series expansion $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ to define e^x when x is a complex number; in the case $x = i\theta$, that gives rise to the formula $e^{i\theta} = \cos\theta + i\sin\theta$, cf. the review sheet on complex numbers.) Thus the solution above is

$$\begin{aligned} P(\rho) = J_m(i\mu\rho) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!}\left(\frac{i\mu\rho}{2}\right)^{2k+m} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!} i^{2k+m}\left(\frac{\mu\rho}{2}\right)^{2k+m} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!}(-1)^k i^m\left(\frac{\mu\rho}{2}\right)^{2k+m} \\ &= i^m \sum_{k=0}^{\infty} \frac{1}{k!(m+k)!}\left(\frac{\mu\rho}{2}\right)^{2k+m}. \end{aligned}$$

Since it is convenient to have functions of a real variable take real values, we drop the factor of i^m and define the *modified Bessel function of degree* m to be

$$I_m(x) = i^{-m} J_m(ix) = \sum_{k=0}^{\infty} \frac{1}{k!(m+k)!} \left(\frac{x}{2}\right)^{2k+m}.$$

It is useful to note the similarity between the pair $J_m(x)$, $I_m(x)$ and the pair $\sin x$, $\sinh x$; see the notes for June 11 – 13, p. 10 for discussion.

Given the foregoing, then, we see that the general separated solution to Laplace's equation on a cylinder (well-behaved at $\rho = 0$) in the case where $\frac{Z''}{Z} = -\mu^2$ is given by[1]

$$I_m(\mu\rho)(\alpha \cos m\phi \cos \mu z + \beta \cos m\phi \sin \mu z + \gamma \sin m\phi \cos \mu z + \delta \sin m\phi \sin \mu z). \tag{1}$$

We now face the problem of determining which values for μ are appropriate. Recall that when dealing with the case $\frac{Z''}{Z} = \lambda^2 > 0$, the values for λ were determined by the boundary condition $u|_{\rho=a} = 0$, which forced $J_m(\lambda a) = 0$, which meant that $\lambda a = \lambda_{mi}$ for some i (where λ_{mi}, again, is the ith zero of J_m), or $\lambda = \frac{\lambda_{mi}}{a}$. This suggests that in the present case μ should be determined by a boundary condition in z.[2] We now give an example to show which kinds of boundary-value problems make use of separated solutions of the foregoing type.

EXAMPLE. Solve on $\{(\rho, \phi, z) | \rho < 1,\ \ 0 < z < 1\}$:

$$\nabla^2 u = 0, \quad u|_{z=0} = u|_{z=1} = 0, \quad u_{\rho=1} = 1.$$

Since we have the conditions $u|_{z=0} = u|_{z=1} = 0$, we see that the solution must be oscillatory in the z-direction, so that we must use the above form of separated solution, i.e., our general solution will be a series in solutions of the type in equation (1). Applying the z boundary conditions $u|_{z=0} = u|_{z=1} = 0$ gives $c = 0$, $\sin \mu = 0$, so $\mu = n\pi$, where $n \in \mathbf{Z}$ and we may take $n > 0$ (this is exactly the same as what we did to implement the boundary conditions $u|_{x=0} = u|_{x=1} = 0$ when we solved Laplace's equation in rectangular coordinates earlier on in the course). Thus the general solution to Laplace's equation on the above region which satisfies the first two boundary conditions above will be (absorbing d into a and b)

$$u = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} I_m(n\pi\rho)(a_{nm} \cos m\phi + b_{nm} \sin m\phi) \sin n\pi z.$$

We note that $\{\cos m\phi \sin n\pi z, \sin m\phi \sin n\pi z | n, m \in \mathbf{Z}, m \geq 0, n > 0\}$ is a complete orthogonal set on $\{(\phi, z) | \phi \in [0, 2\pi], z \in [0, 1]\}$ with respect to the inner product

$$(f(\phi, z), g(\phi, z)) = \int_0^{2\pi} \int_0^1 f(\phi, z) \overline{g(\phi, z)} \, dz \, d\phi;$$

this can be shewn exactly as was done for the set $\{P_{\ell m} \cos m\phi, P_{\ell m} \sin m\phi | m, \ell \in \mathbf{Z}, m \geq 0, \ell \geq m\}$ previously (by first expanding in ϕ, obtaining z-dependent coefficients, and then expanding each of these coefficients in a series in $\sin n\pi z$, for example). The relevant normalisation integrals are

$$(\cos m\phi \sin n\pi z, \cos m\phi \sin n\pi z) = \int_0^{2\pi} \cos^2 m\phi \, d\phi \int_0^1 \sin^2 n\pi z \, dz = \frac{\pi}{2},$$
$$(\sin m\phi \sin n\pi z, \sin m\phi \sin n\pi z) = \int_0^{2\pi} \sin^2 m\phi \, d\phi \int_0^1 \sin^2 n\pi z \, dz = \frac{\pi}{2}.$$

[1]See the example below for a discussion of this form, which is more general than the form we have been using for separated solutions.

[2]Note that this is in accordance with how we have determined separation constants so far: they are determined by boundary conditions in the *oscillatory* directions, not in the nonoscillatory ones.

We now need only to determine a_{nm} and b_{nm} by implementing the final boundary condition $u|_{\rho=1} = 1$. This gives

$$\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} I_m(n\pi)(a_{nm}\cos m\phi + b_{nm}\sin m\phi)\sin n\pi z = 1;$$

by our general results on expansions in complete orthogonal sets, we may write

$$\begin{aligned}
a_{nm}I_m(n\pi) &= \frac{(1, \cos m\phi \sin n\pi z)}{(\cos m\phi \sin n\pi z, \cos m\phi \sin n\pi z)} = \frac{2}{\pi}\int_0^{2\pi}\int_0^1 \cos m\phi \sin n\pi z \, dz \, d\phi \\
&= \frac{2}{\pi}\int_0^{2\pi} \cos m\phi \, d\phi \int_0^1 \sin n\pi z \, dz = \begin{cases} \frac{2}{n\pi}(1-(-1)^n), & m = 0 \\ 0, & m \neq 0 \end{cases}, \\
b_{nm}I_m(n\pi) &= \frac{(1, \sin m\phi \sin n\pi z)}{(\sin m\phi \sin n\pi z, \sin m\phi \sin n\pi z)} = \frac{2}{\pi}\int_0^{2\pi}\int_0^1 \sin m\phi \sin n\pi z \, dz \, d\phi = 0,
\end{aligned}$$

where we have used orthogonality of the set $\{\cos m\phi, \sin m\phi | m \in \mathbf{Z}, m \geq 0\}$ together with the fact that $\cos 0 \cdot \phi = \cos 0 = 1$ and the integral $\int_0^1 \sin n\pi z \, dz = \frac{1}{n\pi}(1-(-1)^n)$. Thus our final solution is given by (noting that $1-(-1)^n = 0$, n even, 2, n odd)

$$u = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\frac{I_0((2k+1)\pi\rho)}{I_0((2k+1)\pi)}\sin{(2k+1)\pi z}.$$

The above method can clearly be used with any problem of the form

$$\nabla^2 u = 0, \quad u|_{z=0} = u|_{z=1} = 0, \quad u|_{\rho=1} = f(\phi, z),$$

for suitably well-behaved functions $f(\phi, z)$. Should we be working on a cylinder like $\{(\rho, \phi, z) | \rho < a, 0 < z < b\}$, the only difference would be that we would take $\mu = \frac{n\pi}{b}$ instead of $\mu = n\pi$. The a factor would only show up in the coefficients, not in the separation constants (just as, when we solved problems with $u|_{\rho=1} = 0$, the length of the cylinder did not show up in the separation constants, only the radius). We now consider how to treat still more general problems.

GENERAL BOUNDARY VALUE PROBLEMS ON A CYLINDER. We shall proceed by means of an example.

EXAMPLE. Solve on $\{(\rho, \phi, z) | \rho < 2, 0 < z < 3\}$:

$$\nabla^2 u = 0, \quad u|_{z=0} = 0, \quad u|_{z=3} = \rho^2\cos 2\phi, \quad u|_{\rho=2} = z\phi.$$

This problem does not look quite exactly like anything we have encountered before. By this point we have had a great deal of experience solving problems of the form

$$\nabla^2 u = 0, \quad u|_{z=0} = 0, \quad u|_{z=3} = \rho^2\cos 2\phi, \quad u|_{\rho=2} = 0, \tag{2}$$

and in the previous example we saw how to solve problems like

$$\nabla^2 u = 0, \quad u|_{z=0} = u|_{z=3} = 0, \quad u|_{\rho=2} = z\phi, \tag{3}$$

but the current problem is not of either of these forms: actually it looks rather like a mix of the two! It turns out that this is exactly the key to solving it, too: since the equation $\nabla^2 u = 0$ is linear and homogeneous, the sum of any two solutions is still a solution; thus if we let u_1 denote the solution to problem (2) and u_2 the solution to problem (3), then $u = u_1 + u_2$ will still solve $\nabla^2 u = 0$, and a moment's thought shows that it satisfies all of the boundary conditions of the original problem.

[We pause to note that this is a very general technique. As we have had occasion to note multiple times, when solving Laplace's equation we must have at least one direction which is not oscillatory. But nonoscillatory functions do not form complete orthogonal sets, so this means that there will be at least one part of the boundary on which we cannot specify arbitrary boundary data (and must in general have homogeneous boundary data). We can solve general problems with nonhomogeneous boundary data on all boundaries using the above method: split the problem up into multiple (in three dimensions we never need more than 3) subproblems, each of which has nonhomogeneous data on at most one set of boundaries; if this is done correctly, so that the nonhomogeneous data do not add on top of each other when the solutions are added, the sum of the solution to each subproblem will be the solution to the original problem, just as here.]

Let us consider first problem (2):

$$\nabla^2 u_1 = 0, \quad u_1|_{z=0} = 0, \quad u_1|_{z=3} = \rho^2 \cos 2\phi, \quad u|_{\rho=2} = 0.$$

We see that the general solution satisfying the third boundary condition will be of the form

$$u_1 = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m\left(\frac{\lambda_{mn}}{2}\rho\right)(a_{mn}\cos m\phi + b_{mn}\sin m\phi)\left(c_{mn}\cosh\frac{\lambda_{mn}}{2}z + d_{mn}\sinh\frac{\lambda_{mn}}{2}z\right).$$

Before proceeding we pause to indicate another way of writing out this sum which is more convenient in cases where we have inhomogeneous data on both ends of the cylinder (here, where we have $u_1|_{z=0} = 0$, it does not make that much difference). This comes from noting that sometimes it can be hard or even impossible to determine the individual quantities a_{mn}, b_{mn}, etc.: what we obtain naturally are various *products* of these quantities, e.g., $a_{mn}c_{mn}$, etc.. (This impossibility of determining the individual factors in these products is the reason why we constantly speak of 'absorbing' (e.g.) d_{mn} into a_{mn} and b_{mn}, etc..) However, a moment's thought shows that we actually *don't care* about the individual quantities either: the only things that matter for the solution are exactly the products $a_{mn}c_{mn}$, etc., which we are able to calculate. Thus it makes sense to get rid of unknowable and irrelevant quantities and write out the sum only in terms of knowable and relevant ones. Further, since we typically think of expanding in ϕ first, it makes sense to write the series in such a way that the $\cos\phi$ terms and $\sin\phi$ terms are clearly separated. Thus instead of the above form, we consider the alternate form

$$\begin{aligned} u_1 = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m\left(\frac{\lambda_{mn}}{2}\rho\right)\Bigg[&\left(\alpha_{mn}\cosh\frac{\lambda_{mn}}{2}z + \beta_{mn}\sinh\frac{\lambda_{mn}}{2}z\right)\cos m\phi \\ &+ \left(\gamma_{mn}\cosh\frac{\lambda_{mn}}{2}z + \delta_{mn}\sinh\frac{\lambda_{mn}}{2}z\right)\sin m\phi\Bigg]. \end{aligned}$$

This is exactly equivalent to the above form, with the definitions

$$\alpha_{mn} = a_{mn}c_{mn}, \quad \beta_{mn} = a_{mn}d_{mn}, \quad \gamma_{mn} = b_{mn}c_{mn}, \quad \delta_{mn} = b_{mn}d_{mn},$$

and moreover it is exactly these four quantities which can be determined uniquely in terms of the boundary data.

With this expression in hand, we may now determine the coefficients from the boundary data, as follows (recall that the normalisation for $J_m\left(\frac{\lambda_{mn}}{2}\rho\right)$ is $\left(J_m\left(\frac{\lambda_{mn}}{2}\rho\right), J_m\left(\frac{\lambda_{mn}}{2}\rho\right)\right) = \frac{1}{2}2^2 J_{m+1}^2(\lambda_{mn}) = 2J_{m+1}^2(\lambda_{mn})$):

$$\begin{aligned} 0 = u_1|_{z=0} &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m\left(\frac{\lambda_{mn}}{2}\rho\right)[\alpha_{mn}\cos m\phi + \gamma_{mn}\sin m\phi], \\ \alpha_{mn} &= \frac{1}{2\pi J_{m+1}^2(\lambda_{mn})}(0, J_m\left(\frac{\lambda_{mn}}{2}\rho\right)\cos m\phi) = 0, \\ \gamma_{mn} &= \frac{1}{2\pi J_{m+1}^2(\lambda_{mn})}(0, J_m\left(\frac{\lambda_{mn}}{2}\rho\right)\sin m\phi) = 0, \end{aligned}$$

a result we could also have obtained by inspection (though it is important to remember the logic that goes behind it). The other boundary condition then gives

$$0 = u_1|_{z=1} = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m\left(\frac{\lambda_{mn}}{2}\rho\right)\left[\beta_{mn}\sinh\frac{\lambda_{mn}}{2}\cos m\phi + \delta_{mn}\sinh\frac{\lambda_{mn}}{2}\sin m\phi\right],$$

$$\beta_{mn}\sinh\frac{\lambda_{mn}}{2} = \frac{1}{2\pi J_{m+1}^2(\lambda_{mn})}\left(\rho^2\cos 2\phi, J_m\left(\frac{\lambda_{mn}}{2}\rho\right)\cos m\phi\right) = \begin{cases}\frac{\left(\rho^2, J_m\left(\frac{\lambda_{mn}}{2}\rho\right)\right)}{2J_{m+1}^2(\lambda_{mn})}, & m = 2,\\ 0, & m \neq 2\end{cases},$$

$$\delta_{mn}\sinh\frac{\lambda_{mn}}{2} = \frac{1}{2\pi J_{m+1}^2(\lambda_{mn})}\left(\rho^2\cos 2\phi, J_m\left(\frac{\lambda_{mn}}{2}\rho\right)\sin m\phi\right) = 0,$$

where we have used orthogonality of the set $\{\cos m\phi, \sin m\phi\}$. Now we may calculate further (making the change of variables $x = \frac{\lambda_{2n}}{2}\rho$)

$$\left(\rho^2, J_2\left(\frac{\lambda_{2n}}{2}\rho\right)\right) = \int_0^2 \rho^2 J_2\left(\frac{\lambda_{2n}}{2}\rho\right)\rho\,d\rho = \frac{16}{\lambda_{2n}^4}\int_0^{\lambda_{2n}} x^3 J_2(x)\,dx = \frac{16}{\lambda_{2n}^4}x^3 J_3(x)\Big|_0^{\lambda_{2n}}$$
$$= \frac{16 J_3(\lambda_{2n})}{\lambda_{2n}},$$

whence we have

$$\beta_{2n}\sinh\frac{\lambda_{2n}}{2} = \frac{8}{\lambda_{2n}J_3(\lambda_{2n})},$$
$$\beta_{2n} = \frac{8}{\lambda_{2n}\sinh\frac{\lambda_{2n}}{2}J_3(\lambda_{2n})},$$

and finally

$$u_1 = \sum_{n=1}^{\infty}\frac{8}{\lambda_{2n}\sinh\frac{\lambda_{2n}}{2}J_3(\lambda_{2n})}J_2\left(\frac{\lambda_{2n}}{2}\rho\right)\sinh\frac{\lambda_{2n}}{2}z\cos 2\phi.$$

We now turn to problem (3). In this case, as shewn in the previous example, the general solution satisfying the first two boundary conditions will be of the form

$$u_2 = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} I_m(n\pi\rho)\left(a_{mn}\cos m\phi + b_{mn}\sin m\phi\right)\sin\frac{n\pi}{3}z.$$

The final boundary condition gives

$$z\phi = u_2|_{\rho=2} = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} I_m(2n\pi)\left(a_{mn}\cos m\phi + b_{mn}\sin m\phi\right)\sin\frac{n\pi}{3}z.$$

As before, we may calculate the coefficients a_{mn} and b_{mn} using our general formula for coefficients in orthogonal expansions, viz. (assuming for the moment that $m > 0$) –

$$\begin{aligned}
a_{mn}I_m(2n\pi) &= \frac{2}{3\pi}\left(z\phi, \cos m\phi\sin\frac{n\pi}{3}z\right) = \frac{2}{3\pi}\int_0^{2\pi}\int_0^3 z\phi\cos m\phi\sin\frac{n\pi}{3}z\,dz\,d\phi\\
&= \frac{2}{3\pi}\int_0^{2\pi}\phi\cos m\phi\,d\phi\int_0^3 z\sin\frac{n\pi}{3}z\,dz\\
&= \frac{2}{3\pi}\left(\frac{1}{m}\phi\sin m\phi + \frac{1}{m^2}\cos m\phi\right)\Big|_0^{2\pi}\left(-\frac{3}{n\pi}z\cos\frac{n\pi}{3}z + \frac{9}{n^2\pi^2}\sin\frac{n\pi}{3}z\right)\Big|_0^3 = 0,\\
b_{mn}I_m(2n\pi) &= \frac{2}{3\pi}\left(z\phi, \sin m\phi\sin\frac{n\pi}{3}z\right) = \frac{2}{3\pi}\int_0^{2\pi}\int_0^3 z\phi\sin m\phi\sin\frac{n\pi}{3}z\,dz\,d\phi\\
&= \frac{2}{3\pi}\int_0^{2\pi}\phi\sin m\phi\,d\phi\int_0^3 z\sin\frac{n\pi}{3}z\,dz\\
&= \frac{2}{3\pi}\left(-\frac{1}{m}\phi\cos m\phi + \frac{1}{m^2}\sin m\phi\right)\Big|_0^{2\pi}\left(-\frac{3}{n\pi}z\cos\frac{n\pi}{3}z + \frac{9}{n^2\pi^2}\sin\frac{n\pi}{3}z\right)\Big|_0^3 = (-1)^n\frac{36}{\pi mn},
\end{aligned}$$

while for $m = 0$ we have $b_{0n} = 0$ by definition and

$$a_{0n} I_0(2n\pi) = \frac{1}{3\pi}\left(z\phi, \sin\frac{n\pi}{3}z\right) = \frac{1}{3\pi}\,\frac{1}{2}\phi^2\Big|_0^{2\pi}\left(-\frac{3}{n\pi}z\cos\frac{n\pi}{3}z + \frac{9}{n^2\pi^2}\sin\frac{n\pi}{3}z\right)\Big|_0^3$$
$$= \frac{4\pi}{3}(-1)^{n+1}\frac{9}{n\pi} = (-1)^{n+1}\frac{12}{n}.$$

This gives finally

$$a_{0n} = (-1)^{n+1}\frac{12}{nI_0(2n\pi)}, \quad a_{mn} = 0, \ m \neq 0,$$
$$b_{0n} = 0, \quad b_{mn} = (-1)^n\frac{36}{\pi mnI_m(2n\pi)}, \ m \neq 0,$$

and hence the solution

$$u_2 = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{12}{nI_0(2n\pi)}I_0(n\pi\rho)\sin\frac{n\pi}{3}z + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^n\frac{36}{\pi mnI_m(2n\pi)}I_m(n\pi\rho)\sin m\phi\sin\frac{n\pi}{3}z.$$

Thus we obtain as the final solution to our original problem

$$u = \sum_{n=1}^{\infty}\frac{8}{\lambda_{2n}\sinh\frac{\lambda_{2n}}{2}J_3(\lambda_{2n})}J_2\left(\frac{\lambda_{2n}}{2}\rho\right)\sinh\frac{\lambda_{2n}}{2}z\cos 2\phi + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{12}{nI_0(2n\pi)}I_0(n\pi\rho)\sin\frac{n\pi}{3}z$$
$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^n\frac{36}{\pi mnI_m(2n\pi)}I_m(n\pi\rho)\sin m\phi\sin\frac{n\pi}{3}z.$$

LAPLACE'S EQUATION IN THREE-DIMENSIONAL RECTANGULAR COORDINATES. In three-dimensional rectangular coordinates, Laplace's equation has the form

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

We attempt to solve this by the method of separation of variables. Thus we look for solutions of the form $u = X(x)Y(y)Z(z)$; substituting in and dividing by u, we obtain

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0. \tag{4}$$

By standard arguments ($\frac{X''}{X}$ depends only on x, and nothing else on the left-hand side depends on x, and analogously for the remaining terms) we have that there must be constants μ_1, μ_2, μ_3 such that

$$X'' = \mu_1 X, \quad Y'' = \mu_2 Y, \quad Z'' = \mu_3 Z.$$

Note that we have not yet attempted to determine the signs of these constants. Substituting in to equation (4), we have

$$\mu_1 + \mu_2 + \mu_3 = 0.$$

Thus we see that at least one of μ_1, μ_2, μ_3 must be positive and at least one must be negative. (We ignore for the moment the case where all of them are zero.) Which are positive and which are negative depends on the type of problem we wish to solve. We shall indicate the general method for determining this by means of a specific example.

EXAMPLE. Solve on $\{(x, y, z) | x, y, z \in [0, 1]\}$:

$$\nabla^2 u = 0, \quad u|_{x=0} = u|_{x=1} = u|_{z=0} = u|_{z=1} = 0, \quad u|_{y=0} = 0, \quad u|_{y=1} = 1.$$

We begin by looking for separated solutions of $\nabla^2 u = 0$ which satisfy the homogeneous boundary conditions; thus we look for solutions $X(x)Y(y)Z(z)$ which satisfy $X(0) = X(1) = Z(0) = Z(1) = 0$. Now it can be shewn that any linear combination of sinh and cosh can vanish at at most one point (I should have given the proof a long time ago; it is very simple: if $a\cosh x + b\sinh x = 0$, then letting $\alpha = \frac{1}{2}(a+b)$ and $\beta = \frac{1}{2}(a-b)$, we have

$$\begin{aligned} \alpha e^x + \beta e^{-x} &= 0 \\ \alpha e^{2x} + \beta &= 0 \\ e^{2x} &= -\frac{\beta}{\alpha}, \end{aligned}$$

which has at most one real solution x (and none if $\frac{\beta}{\alpha} > 0$)). Similarly, any linear function can vanish at at most one point. This implies that neither X nor Z can be a linear combination of sinh and cosh, nor can they be linear; since X and Z are either linear combinations of sinh and cosh (when $\mu_i > 0$), or are linear (when $\mu_i = 0$), or are linear combinations of sin and cos (when $\mu_i < 0$), the latter case must obtain. This implies that $\frac{X''}{X}$ and $\frac{Z''}{Z}$ must both be negative, i.e., that $\mu_1 = -\lambda_1^2$, $\mu_3 = -\lambda_3^2$ for some $\lambda_1, \lambda_3 > 0$. Hence we must have $\mu_2 > 0$, say $\mu_2 = \lambda_2^2$, $\lambda_2 > 0$. The equation $\mu_1 + \mu_2 + \mu_3 = 0$ then gives

$$\lambda_2^2 = \lambda_1^2 + \lambda_3^2.$$

(This illustrates the general procedure for determining when we take $\mu_i > 0$ and when we take $\mu_i < 0$: the μ_i corresponding to coordinates which have homogeneous boundary data at both ends will be negative, while the remaining one will be positive. If we have inhomogeneous data along more than one coordinate direction, we should split the problem up into multiple subproblems as we did in the previous example.)

The general separated solution is thus

$$\begin{aligned} &\cos\lambda_1 x(\alpha\cos\lambda_3 z\cosh\lambda_2 y + \beta\cos\lambda_3 z\sinh\lambda_2 y + \gamma\sin\lambda_3 z\cosh\lambda_2 y + \delta\sin\lambda_3 z\sinh\lambda_2 y) \\ &\qquad + \sin\lambda_1 x(\alpha'\cos\lambda_3 z\cosh\lambda_2 y + \beta'\cos\lambda_3 z\sinh\lambda_2 y + \gamma'\sin\lambda_3 z\cosh\lambda_2 y + \delta'\sin\lambda_3 z\sinh\lambda_2 y). \end{aligned}$$

Now $X(0) = X(1) = 0$ implies that $\alpha = \beta = \gamma = \delta = 0$, $\lambda_1 = n\pi$, exactly as we found when we solved Laplace's equation on a rectangle; similarly, now, $Z(0) = Z(1) = 0$ implies that $\alpha' = \beta' = 0$, $\lambda_3 = m\pi$. Thus the general separated solution satisfying the first four boundary conditions is of the form

$$\sin n\pi x\sin m\pi z\left(a\cosh y\pi\sqrt{n^2+m^2} + b\sinh y\pi\sqrt{n^2+m^2}\right),$$

and the general solution will be a series in these solutions, i.e.,

$$u = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin n\pi x\sin m\pi z\left(a_{nm}\cosh y\pi\sqrt{n^2+m^2} + b_{nm}\sinh y\pi\sqrt{n^2+m^2}\right).$$

The boundary conditions in y now give

$$0 = u|_{y=0} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin n\pi x\sin m\pi z(a_{nm}),$$

whence we see that (since, similarly to what we mentioned in the first example above, $\{\sin n\pi x\sin m\pi z | n, m \in \mathbf{Z}, n, m > 0\}$ is a complete orthogonal set on $[0,1]\times[0,1]$ with respect to the standard inner product, with normalisation constant $(\sin n\pi x\sin m\pi z, \sin n\pi x\sin m\pi z) = \frac{1}{4}$)

$$a_{nm} = 4(0, \sin n\pi x\sin m\pi z) = 0.$$

(We could have implemented this condition at the level of the separated solutions, and written our original series for u without the cosh term; we have proceeded this way in order to emphasise that when the boundary data on one side of the cube are inhomogeneous, the direction perpendicular to that side (here, y) should be treated differently than the other sides. In particular, the full procedure as illustrated here would allow us to also treat the case where the boundary data at $y = 0$ were not homogeneous, and this could not in general be implemented at the level of the separated solution.) Similarly, the other boundary condition gives

$$1 = u|_{y=1} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm} \sin n\pi x \sin m\pi z \sinh \pi\sqrt{n^2+m^2},$$

whence we obtain

$$\begin{aligned} b_{nm} \sinh \pi\sqrt{n^2+m^2} &= 4(1, \sin n\pi x \sin m\pi z) = 4\int_0^1\int_0^1 \sin n\pi x \sin m\pi z\, dx\, dz \\ &= 4\int_0^1 \sin n\pi x\, dx \int_0^1 \sin m\pi z\, dz = \frac{4}{nm}(1-(-1)^n)(1-(-1)^m), \end{aligned}$$

which is 0 if either of n or m is even and $\frac{16}{nm}$ when both are odd. Thus we have

$$b_{2k+1,2\ell+1} = \frac{16}{(2k+1)(2\ell+1)\sinh \pi\sqrt{(2k+1)^2+(2\ell+1)^2}}$$

and finally the solution

$$u = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{16}{(2k+1)(2\ell+1)\sinh \pi\sqrt{(2k+1)^2+(2\ell+1)^2}} \sin{(2k+1)\pi x} \sin{(2\ell+1)\pi z} \sinh y\pi\sqrt{(2k+1)^2+(2\ell+1)^2}.$$

(The example I did in class was actually much simpler than this, involving just a single separated solution as the final answer. I didn't realise I was doing a different problem until I was almost finished typing it up though – and anyway it doesn't hurt to see another (and more complicated!) example.)

EIGENFUNCTIONS OF THE LAPLACIAN. The next topics which we wish to treat are Green's functions, the heat equation, and the wave equation (though we may take some time off to talk about Fourier transforms at some point). The study of all of these, especially of the first two, benefit from a knowledge of the eigenfunctions of the Laplacian, so we now turn to that topic. First we give an example from linear algebra as motivation. (See also the examples we gave related to diagonalisation in the first week or two of the course.)

EXAMPLE. Let A be an $n \times n$ matrix, and x and y be column vectors of length n. Consider the equation $Ax = y$. If we know the inverse matrix A^{-1}, then we can solve this by writing $x = A^{-1}y$. In general, though, finding the inverse of a matrix is hard. If A were diagonal, though, it would be easy, since the inverse of a diagonal matrix

$$D = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$$

is

$$D = \begin{bmatrix} d_1^{-1} & 0 & 0 & \cdots & 0 \\ 0 & d_2^{-1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n^{-1} \end{bmatrix}.$$

More abstractly, suppose that $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ were a basis of eigenvectors for the matrix A; suppose also that A is symmetric, so that this set can be taken to be orthogonal.

[This can be shewn in an analogous fashion to how we showed that the Legendre polynomials and Bessel functions formed orthogonal sets. For simplicity we work with the standard real inner product. Symmetry of A means that for any vectors v and w, we have

$$(v, Aw) = \sum_{i=1}^{n} v_i (Aw)_i = \sum_{i=1}^{n}\sum_{j=1}^{n} v_i A_{ij} w_j = \sum_{i=1}^{n}\sum_{j=1}^{n} v_i A_{ji} w_j = (Av, w),$$

so if $\mathbf{e}_i$ and $\mathbf{e}_j$ are eigenvectors corresponding to distinct eigenvalues, say λ_i and λ_j, then we may write

$$(\mathbf{e}_i, A\mathbf{e}_j) = \lambda_j(\mathbf{e}_i, \mathbf{e}_j) = (A\mathbf{e}_i, \mathbf{e}_j) = \lambda_i(\mathbf{e}_i, \mathbf{e}_j),$$

so $(\lambda_j - \lambda_i)(\mathbf{e}_i, \mathbf{e}_j) = 0$ and $(\mathbf{e}_i, \mathbf{e}_j) = 0$ since $\lambda_i \neq \lambda_j$. (In the event that $\mathbf{e}_i$ and $\mathbf{e}_j$ belong to the same eigenvalue, they can be taken orthogonal by applying the Graham-Schmidt process if needed.)]

Then we can write

$$y = \sum_{i=1}^{n} \frac{(y, \mathbf{e}_i)}{(\mathbf{e}_i, \mathbf{e}_i)}\mathbf{e}_i,$$
$$x = \sum_{i=1}^{n} x_i \mathbf{e}_i,$$

whence the system $Ax = y$ becomes

$$\sum_{i=1}^{n} \lambda_i x_i \mathbf{e}_i = \sum_{i=1}^{n} \frac{(y, \mathbf{e}_i)}{(\mathbf{e}_i, \mathbf{e}_i)}\mathbf{e}_i.$$

Since $\{\mathbf{e}_i\}$ is a basis, this implies that $\lambda_i x_i = \frac{(y,\mathbf{e}_i)}{(\mathbf{e}_i,\mathbf{e}_i)}$, so

$$x_i = \frac{1}{\lambda_i}\frac{(y, \mathbf{e}_i)}{(\mathbf{e}_i, \mathbf{e}_i)},$$
$$x = \sum_{i=1}^{n} \frac{1}{\lambda_i}\frac{(y, \mathbf{e}_i)}{(\mathbf{e}_i, \mathbf{e}_i)}.$$

Note that this procedure did not require us to invert any matrix; in fact, the computations involved were nothing more than the taking of inner products and multiplication and division. (Finding the eigenvectors of A, of course, is highly nontrivial, so this method is not necessarily any faster overall at solving a single system.)

The idea behind this example may be applied to, among other things, the study of a generalisation of Laplace's equation called *Poisson's equation*. So far we have only studied the homogeneous equation $\nabla^2 u = 0$; however, there are many cases (such as, for example, when one has a source of heat inside a region and wishes to find the equilibrium temperature distribution, or when one has a nonzero charge density inside a region and wishes to find the electrostatic potential) when one wishes to solve an equation of the form $\nabla^2 u = f$ for some function f. Generally one still has boundary conditions which u is also required to satisfy. Suppose now that there were a complete orthogonal set of (nonzero) functions $\{e_i\}$, where i is an abstract index, such that $\nabla^2 e_i = \Lambda_i e_i$, and such that each e_i satisfied the relevant boundary conditions. Then we would be able to expand the function f as

$$f = \sum_i \frac{(f, e_i)}{(e_i, e_i)} e_i,$$

and also any potential solution u as

$$u = \sum_i u_i e_i.$$

Substituting both of these into the equation $\nabla^2 u = f$, we obtain

$$\sum_i \Lambda_i u_i e_i = \sum_i \frac{(f, e_i)}{(e_i, e_i)} e_i;$$

since the set $\{e_i\}$ is orthogonal and does not contain 0, this implies that for each i

$$\Lambda_i u_i = \frac{(f, e_i)}{(e_i, e_i)}. \tag{5}$$

If $\Lambda_i \neq 0$ for all i, then we may solve this for u_i and then substitute in to the expansion $u = \sum_i u_i e_i$ to obtain a series expansion for the solution u to Poisson's equation in the functions e_i, much as we have been doing for solutions to Laplace's equation (though the eigenfunctions e_i may well be different from the orthogonal bases we have used so far). If $\Lambda_i = 0$ for some i then things are more complicated. From equation (5) it is evident that in this case there can be no solution (at least, not one expressible as a series in the $\{e_i\}$) if $(f, e_i) \neq 0$. If, however, we happen to have $(f, e_i) = 0$ whenever $\Lambda_i = 0$, then clearly there will still be a solution; though it is not necessarily unique, since the u_i will not be determined by equation (5). We may obtain a unique solution by requiring $u_i = 0$ for such i. Thus we see that the equation $\nabla^2 u = f$ will have a unique solution if we restrict both f and u to lie in the space of functions which are orthogonal to all eigenfunctions of ∇^2 with zero eigenvalues. We shall probably have more to say on this point later.

Let us assume for the moment, for simplicity, that none of the eigenvalues are zero (or that we have restricted f and u as just indicated, and then restricted i to run over the eigenfunctions corresponding to nonzero eigenvalues). Then we may write the solution u as

$$u = \sum_i \frac{1}{\Lambda_i} \frac{(f, e_i)}{(e_i, e_i)} e_i;$$

now if our inner product (f, e_i) were given by an integral, say (writing things schematically for generality) $(f, e_i) = \int_D f \overline{e_i}\, dx'$, then we may express this equation as follows (formally interchanging summation and integration):

$$u(x) = \sum_i e_i(x) \frac{1}{\Lambda_i (e_i, e_i)} \int_D f(x') \overline{e_i(x')}\, dx' = \int_D \left(\sum_i \frac{e_i(x) \overline{e_i(x')}}{(e_i, e_i)} \frac{1}{\Lambda_i} \right) f(x')\, dx'.$$

A function $G(x, x')$ such as that in the parentheses above is called a *Green's function* for the given boundary-value problem. We shall study such functions systematically starting next week. The above expression gives (at least formally) the Green's function in terms of the eigenfunctions and eigenvalues of the Laplacian for the given boundary conditions.

[The formula above has a formal analogue in linear algebra as well. We may write the formula as

$$u(x) = \int_D G(x, x') f(x')\, dx'.$$

Now the solution to an equation $Ax = y$ can be written as

$$x_i = \sum_j A^{-1}_{ij} y_j;$$

if we think of i as corresponding to x, j as corresponding to x', and $\sum$ as corresponding to $\int$, then we see that in some sense G corresponds to $\left(\nabla^2\right)^{-1}$; i.e., the integral operator given above involving G is an 'inverse' to the Laplacian.]

Another place where the eigenfunctions of ∇^2 are useful is in studying the heat equation $\frac{\partial u}{\partial t} = \nabla^2 u$. Suppose that we are interested in studying this equation subject to certain boundary conditions on u (which are constant in time), and suppose that we have a complete orthogonal set of eigenfunctions $\{e_i\}$ for the Laplacian ∇^2 subject to these boundary conditions. Then we could write for each time t, as before,

$$u(t, \mathbf{x}) = \sum_i u_i(t) e_i(\mathbf{x}),$$

and substituting this into the heat equation gives

$$\sum_i u_i'(t) e_i = \sum_i \Lambda_i u_i(t) e_i,$$

whence we have $u_i'(t) = \Lambda_i u_i(t)$, i.e., the system completely decouples, exactly as we discussed in the first couple weeks of class. This last equation has solution $u_i(t) = u_{i,0} e^{\Lambda_i t}$, and thus our solution u is

$$u = \sum_i u_{i,0} e^{\Lambda_i t} e_i,$$

where the constants $u_{i,0}$ are to be determined from the initial condition $u|_{t=0}$, exactly as we determine coefficients in orthogonal expansions for Laplace's equation using boundary conditions. We shall go over all of this in more detail later on in the course.

APM 346 (Summer 2019), Homework 7 solutions.

APM 346, Homework 7. Due Monday, July 8, at 6.00 AM EDT. To be marked completed/not completed.

1. Solve on $\{(\rho, \phi, z) | \rho < 2, \ \ 0 \le z \le 3\}$:

$$\nabla^2 u = 0, \ \ u|_{\rho=2} = 0, \ \ u|_{z=0} = \rho \cos\phi, \ \ u|_{z=3} = \rho \sin\phi.$$

We have the general series expansion (as introduced in class on Thursday; this form for the expansion turns out to be much more convenient for this particular problem than the one we have been using)

$$u = \sum_{m=0}^{\infty} \sum_{i=1}^{\infty} J_m \left(\frac{1}{2}\lambda_{mi}\rho\right) \left(\alpha_{mi} \cos m\phi \cosh \frac{1}{2}\lambda_{mi} z + \beta_{mi} \cos m\phi \sinh \frac{1}{2}\lambda_{mi} z \right.$$
$$\left. + \gamma_{mi} \sin m\phi \cosh \frac{1}{2}\lambda_{mi} z + \delta_{mi} \sin m\phi \sinh \frac{1}{2}\lambda_{mi} z \right),$$

where as usual λ_{mi} denotes the ith positive zero of J_m. At $z = 0$, this gives

$$\sum_{m=0}^{\infty} \sum_{i=1}^{\infty} J_m \left(\frac{1}{2}\lambda_{mi}\rho\right) (\alpha_{mi} \cos m\phi + \gamma_{mi} \sin m\phi) = \rho \cos\phi,$$

whence we see that $\gamma_{mi} = 0$ for all m, i, while $\alpha_{mi} = 0$ unless $m = 1$, and in this case

$$\sum_{i=1}^{\infty} \alpha_{1i} J_1 \left(\frac{1}{2}\lambda_{1i}\rho\right) = \rho,$$

whence

$$\alpha_{1i} = \frac{2}{2^2 J_2^2(\lambda_{1i})} \int_0^2 \rho^2 J_1 \left(\frac{1}{2}\lambda_{1i}\rho\right) d\rho = \frac{1}{2J_2^2(\lambda_{1i})} \left(\frac{2}{\lambda_{1i}}\right)^3 \int_0^{\lambda_{1i}} x^2 J_1(x)\, dx$$
$$= \frac{4}{\lambda_{1i}^3 J_2^2(\lambda_{1i})} \lambda_{1i}^2 J_2(\lambda_{1i}) = \frac{4}{\lambda_{1i} J_2(\lambda_{1i})}.$$

At $z = 3$ we have

$$\sum_{m=0}^{\infty} \sum_{i=1}^{\infty} J_m \left(\frac{1}{2}\lambda_{mi}\rho\right) \left(\alpha_{mi} \cos m\phi \cosh \frac{3}{2}\lambda_{mi} + \beta_{mi} \cos m\phi \sinh \frac{3}{2}\lambda_{mi} + \delta_{mi} \sin m\phi \sinh \frac{3}{2}\lambda_{mi}\right) = \rho^2 \sin\phi,$$

whence we see that $\alpha_{mi} \cosh \frac{3}{2}\lambda_{mi} + \beta_{mi} \sinh \frac{3}{2}\lambda_{mi} = 0$ for all m, i, which gives $\beta_{mi} = 0$ for $m \ne 1$ and $\beta_{1i} = -\coth \frac{3}{2}\lambda_{1i} \frac{4}{\lambda_{1i} J_2^2(\lambda_{1i})}$; also $\delta_{mi} = 0$ for $m \ne 1$, while

$$\sum_{i=1}^{\infty} \delta_{1i} \sinh \frac{3}{2}\lambda_{1i} J_1 \left(\frac{1}{2}\lambda_{1i}\rho\right) = \rho,$$

whence we see from the above calculation for α_{1i} that

$$\delta_{1i} = \frac{4}{\lambda_{1i} J_2(\lambda_{1i}) \sinh \frac{3}{2}\lambda_{1i}}.$$

Thus finally

$$u = \sum_{i=1}^{\infty} J_1 \left(\frac{1}{2}\lambda_{1i}\rho\right) \left[\left(\cosh \frac{1}{2}\lambda_{1i} z - \coth \frac{3}{2}\lambda_{1i} \sinh \frac{1}{2}\lambda_{1i} z\right) \cos\phi + \frac{\sinh \frac{1}{2}\lambda_{1i} z}{\sinh \frac{3}{2}\lambda_{1i}} \sin\phi\right] \frac{4}{\lambda_{1i} J_2(\lambda_{1i})}.$$

[It is worth noting how the quantity in parentheses interpolates between $\cos\phi$ at $z=0$ and $\sin\phi$ at $z=3$: the coefficients of $\cos\phi$ and $\sin\phi$ are exactly those linear combinations of $\cosh\frac{1}{2}\lambda_{1i}z$ and $\sinh\frac{1}{2}\lambda_{1i}z$ which are 1 at $z=0$ and $z=3$ and 0 at $z=3$ and $z=0$, respectively. In both cases, the remaining coefficient is exactly that needed for the ρ part to come out to ρ.]

2. Solve on $\{(\rho,\phi,z)|\rho<1,\ 0\le z\le 1\}$:

$$\nabla^2 u = 0,\ \ u|_{z=0} = u|_{z=1} = 0,\ \ u|_{\rho=1} = \begin{cases} -\phi, & 0<\phi<\pi, \\ \phi, & \pi<\phi<2\pi \end{cases}.$$

In this case we have the general series expansion

$$u = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} I_m(n\pi\rho)\left(a_{mn}\cos m\phi + b_{mn}\sin m\rho\right)\sin n\pi z.$$

Thus

$$\begin{aligned} a_{0n}I_0(n\pi) &= \frac{1}{\pi}\int_0^1 \sin n\pi z\left[\int_0^\pi -\phi\,d\phi + \int_\pi^{2\pi}\phi\,d\phi\right]dz = \frac{1}{\pi}\left(-\frac{1}{n\pi}\cos n\pi z\Big|_0^1\right)\left(-\frac{pi^2}{2}+\frac{1}{2}\left(4\pi^2-\pi^2\right)\right) \\ &= \frac{1}{n}\left(1-(-1)^n\right), \end{aligned}$$

while $b_{0i}=0$ by definition and for $m>0$

$$\begin{aligned} a_{mn}I_m(n\pi) &= \frac{2}{\pi}\int_0^1\left[\int_0^\pi -\phi\cos m\phi\sin n\pi z\,d\phi + \int_\pi^{2\pi}\phi\cos m\phi\sin n\pi z\,d\phi\right]dz \\ &= \frac{2}{\pi}\int_0^1 \sin n\pi z\left[-\frac{\phi}{m}\sin m\phi - \frac{1}{m^2}\cos m\phi\Big|_0^\pi + \frac{\phi}{m}\sin m\phi + \frac{1}{m^2}\cos m\phi\Big|_\pi^{2\pi}\right]dz \\ &= \frac{2}{\pi}\int_0^1 \sin n\pi z\left[\frac{1}{m^2}\left(1-(-1)^m\right) + \frac{1}{m^2}\left(1-(-1)^m\right)\right]dz = \frac{4}{n\pi^2 m^2}\left(1-(-1)^m\right)\left(1-(-1)^n\right), \end{aligned}$$

and

$$\begin{aligned} b_{mn}I_m(n\pi) &= \frac{2}{\pi}\int_0^1\left[\int_0^\pi -\phi\sin m\phi\sin n\pi z\,d\phi + \int_0^{2\pi}\phi\cos m\phi\sin n\pi z\,d\phi\right]dz \\ &= \frac{2}{\pi}\int_0^1 \sin n\pi z\left[\frac{\phi}{m}\cos m\phi - \frac{1}{m^2}\sin m\phi\Big|_0^\pi - \frac{\phi}{m}\cos m\phi + \frac{1}{m^2\sin m\phi}\Big|_\pi^{2\pi}\right]dz \\ &= \frac{2}{n\pi^2}\left(1-(-1)^n\right)\left[\frac{\pi}{m}(-1)^m - \left(\frac{\pi}{m}\left(2-(-1)^m\right)\right)\right] = -\frac{4}{mn\pi}\left(1-(-1)^n\right)\left(1-1(-1)^m\right), \end{aligned}$$

so finally

$$\begin{aligned} u &= \sum_{n=1}^{\infty}\frac{1}{n}\left(1-(-1)^n\right)\frac{I_0(n\pi\rho)}{I_0(n\pi)}\sin n\pi z \\ &\quad + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{I_m(n\pi\rho)}{I_m(n\pi)}\left(\frac{4}{nm\pi}\left(1-(-1)^n\right)\left(1-(-1)^m\right)\right)\left[\frac{2}{m\pi}\cos m\phi - \sin m\phi\right]\sin n\pi z. \end{aligned}$$

3. Solve on $\{(\rho,\phi,z)|\rho<1,\ 0\le z\le 1\}$:

$$\nabla^2 u = 0,\ \ u|_{z=0} = \rho^2\cos 2\phi,\ \ u|_{z=1} = \rho^2\sin 2\phi,\ \ u|_{\rho=1} = \begin{cases} -\phi, & 0<\phi<\pi, \\ \phi, & \pi<\phi<2\pi \end{cases}.$$

[Hint: This is basically just problems 1 and 2 combined.]

We decompose this problem as $u = u_1 + u_2$, where u_1 is the solution to problem 2 and u_2 satisfies on $\{(\rho, \phi, z) | \rho < 1, 0 \le z \le 1\}$

$$\nabla^2 u_2 = 0, \quad u_2|_{\rho=1} = 0, \quad u_2|_{z=0} = \rho^2 \cos 2\phi, \quad u_2|_{\mathbf{z}=1} = \rho^2 \sin 2\phi.$$

This has solution

$$u_2 = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(\lambda_{mi}\rho) \Big(\alpha_{mi} \cos m\phi \cosh \lambda_{mi} z + \beta_{mi} \cos m\phi \sinh \lambda_{mi} z$$
$$+ \gamma_{mi} \sin m\phi \cosh \lambda_{mi} z + \delta_{mi} \sin m\phi \sinh \lambda_{mi} z \Big).$$

Now at $z = 0$

$$u_2 = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(\lambda_{mi}\rho) (\alpha_{mi} \cos m\phi + \gamma_{mi} \sin m\phi) = \rho^2 \cos 2\phi,$$

so just as in problem 1 we see that $\gamma_{mi} = 0$ for all m and all i, while $\alpha_{mi} = 0$ for $m \neq 2$ and we have after a standard calculation [we have done this calculation a number of times by this point, but on a test I would expect you to write it out anyway!]

$$\alpha_{2i} = \frac{2}{J_3^2(\lambda_{2i})} \int_0^1 \rho^3 J_2(\lambda_{2i})\, d\rho = \frac{2}{\lambda_{2i} J_3(\lambda_{2i})}.$$

Similarly, at $z = 1$

$$u_2 = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(\lambda_{mi}\rho) (\alpha_{mi} \cos m\phi \cosh \lambda_{mi} + \beta_{mi} \cos m\phi \sinh \lambda_{mi} + \delta_{mi} \sin m\phi \sinh \lambda_{mi}) = \rho^2 \sin 2\phi,$$

so $\alpha_{mi} \cosh \lambda_{mi} + \beta_{mi} \sinh \lambda_{mi} = 0$ for all m and all i, meaning that $\beta_{mi} = 0$ for $m \neq 2$ while

$$\beta_{2i} = -\coth \lambda_{mi} \frac{2}{\lambda_{2i} J_3(\lambda_{2i})},$$

and $\delta_{mi} = 0$ for $m \neq 2$ while

$$\delta_{2i} \sinh \lambda_{2i} = \frac{2}{J_3^2(\lambda_{2i})} \int_0^1 \rho^3 J_2(\lambda_{2i})\, d\rho = \frac{2}{\lambda_{2i} J_3(\lambda_{2i})},$$

giving

$$\delta_{2i} = \frac{2}{\lambda_{2i} J_3(\lambda_{2i}) \sinh \lambda_{2i}},$$

so

$$u_2 = \sum_{i=1}^{\infty} J_2(\lambda_{2i}\rho) \left(\frac{2}{\lambda_{2i} J_3(\lambda_{2i})} \right) \left[(\cosh \lambda_{2i} z - \coth \lambda_{2i} \sinh \lambda_{2i} z) \cos 2\phi + \frac{\sinh \lambda_{2i} z}{\sinh \lambda_{2i}} \sin 2\phi \right]$$

[note again how the coefficients on $\cos 2\phi$ and $\sin 2\phi$ interpolate between 0 and 1, exactly as with the solution in problem 1!] and finally

$$u = u_1 + u_2 = \sum_{i=1}^{\infty} J_2(\lambda_{2i}\rho) \left(\frac{2}{\lambda_{2i} J_3(\lambda_{2i})} \right) \left[(\cosh \lambda_{2i} z - \coth \lambda_{2i} \sinh \lambda_{2i} z) \cos 2\phi + \frac{\sinh \lambda_{2i} z}{\sinh \lambda_{2i}} \sin 2\phi \right]$$
$$+ \sum_{n=1}^{\infty} \frac{1}{n} (1 - (-1)^n) \frac{I_0(n\pi\rho)}{I_0(n\pi)} \sin n\pi z$$
$$+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{I_m(n\pi\rho)}{I_m(n\pi)} \left(\frac{4}{nm\pi} (1 - (-1)^n)(1 - (-1)^m) \right) \left[\frac{2}{m\pi} \cos m\phi - \sin m\phi \right] \sin n\pi z.$$

4. [Optional. This problem requires knowledge of basic complex function theory. I am only putting it here because I think it is exceptionally cool and can't resist.] Solve on $\{(\rho, \phi, z) | \rho < 1,\ \ 0 \le z \le 1\}$:

$$\nabla^2 u = 0,\ \ u|_{\rho=1} = 0,\ \ u|_{z=0} = 0,\ \ u|_{z=1} = \cos(\rho\cos\phi)\cosh\,(\rho\sin\phi)\,.$$

[Hint: can you recognise the boundary datum at $z = 1$ as the real part of an analytic function of $x + iy$? Try writing out the power series of that function and solving the above problem term-by-term in that power series, noting that $x + iy = \rho e^{i\phi}$.]

[Sketch.] We note that $x + iy = \rho\cos\phi + i\rho\sin\phi = \rho e^{i\phi}$, so

$$\begin{aligned}
\cos(x+iy) &= \cos x\cos iy - \sin x\sin iy = \cos x\cosh y + i\sin x\sinh y = \cos(\rho\cos\phi)\cosh\,(\rho\cos\phi) + \cdots \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(x+iy)^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\rho e^{i\phi}\right)^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\rho^{2n}e^{2in\phi} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\rho^{2n}\left(\cos 2n\phi + i\sin 2n\phi\right),
\end{aligned}$$

so

$$\cos(\rho\cos\phi)\cosh\,(\rho\sin\phi) = \Re\cos(x+iy) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\rho^{2n}\cos(2n+1)\phi.$$

Now the general solution to $\nabla^2 u = 0$ on the given region satisfying the first two boundary conditions is

$$u = \sum_{m=0}^{\infty}\sum_{i=1}^{\infty} J_m(\lambda_{mi}\rho)\left(a_{mi}\cos m\phi + b_{mi}\sin m\phi\right)\sinh\lambda_{mi}z;$$

applying the boundary condition at $z = 1$ then gives

$$\sum_{m=0}^{\infty}\sum_{i=1}^{\infty} J_m(\lambda_{mi}\rho)\left(a_{mi}\cos m\phi + b_{mi}\sin m\phi\right)\sinh\lambda_{mi} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\rho^{2n}\cos 2n\phi.$$

From this we easily see that $b_{mi} = 0$ for all m, i, while $a_{mi} = 0$ when m is odd. If $m = 2n$ for some $n \in \mathbf{Z}$, $n \ge 0$, then we obtain

$$\begin{aligned}
a_{2n,i}\sinh\lambda_{2n,i} &= \frac{(-1)^n}{(2n)!}\frac{2}{J_{2n+1}^2(\lambda_{2n,i})}\int_0^1 \rho^{2n}J_{2n}(\lambda_{2n,i}\rho)\rho\,d\rho = \frac{(-1)^n 2}{(2n)!\lambda_{2n,i}J_{2n+1}(\lambda_{2n,i})}, \\
a_{2n,i} &= \frac{(-1)^n 2}{(2n)!\lambda_{2n,i}J_{2n+1}(\lambda_{2n,i})\sinh\lambda_{2n,i}},
\end{aligned}$$

and

$$u = \sum_{n=0}^{\infty}\sum_{i=1}^{\infty}\frac{(-1)^n 2}{(2n)!\lambda_{2n,i}J_{2n+1}(\lambda_{2n,i})\sinh\lambda_{2n,i}}J_{2n}(\lambda_{2n,i}\rho)\cos 2n\phi\sinh\lambda_{2n,i}z.$$

Summary:
- We calculate the eigenfunctions and eigenvalues of the Laplacian on the unit cube.
- We then give examples of how to use these eigenfunctions and eigenvalues to solve Possion's equation and the heat equation on this cube.

[NOTE. These lecture notes do not cover all of the material discussed in lecture but are rather of a summary form, intended to indicate the main points related to problems in rectangular coordinates and also to give the two examples. They will be supplemented later by additional information, particularly concerning the eigenfunctions of the Laplacian in cylindrical coordinates.]

EIGENFUNCTIONS AND EIGENVALUES OF THE LAPLACIAN ON THE UNIT CUBE. Throughout the rest of these lecture notes we shall denote the unit cube in $\mathbf{R}^3$ by

$$Q = \{(x, y, z) | 0 \leq x, y, z \leq 1\}.$$

Now consider the following problem: determine all functions $u : Q \to \mathbf{R}$ not identically zero and all real numbers λ such that

$$\nabla^2 u = \lambda u, \quad u|_{\partial Q} = 0.$$

We attempt to solve this by applying separation of variables. Thus let $u = X(x)Y(y)Z(z)$; substituting this into the equation $\nabla^2 u = \lambda u$ and dividing through by u, we obtain

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = \lambda.$$

We see from this that all three of the quantities $\frac{X''}{X}$, $\frac{Y''}{Y}$ and $\frac{Z''}{Z}$ must be constants. The next step is clearly to try to determine whether they are positive or negative. This is determined by the boundary conditions. In our case, the boundary conditions give

$$u|_{x=0,1} = u|_{y=0,1} = u_{z=0,1} = 0;$$

in terms of X, Y, and Z, these become

$$X(0) = X(1) = 0, \quad Y(0) = Y(1) = 0, \quad Z(0) = Z(1) = 0.$$

Thus we see that each of the functions X, Y and Z must be oscillatory, meaning that each of $\frac{X''}{X}$, $\frac{Y''}{Y}$ and $\frac{Z''}{Z}$ must be negative. Let us work with X first. We may write $\frac{X''}{X} = -\mu^2$, where at this point all we know is that $\mu \in \mathbf{R}$ (and we may take $\mu > 0$: $\mu \neq 0$ since the only solution if $\mu = 0$ that satisfies the boundary conditions would be $X = 0$ which would give $u = 0$). Thus

$$X = a \cos \mu x + b \sin \mu x;$$

$X(0) = 0$ gives $a = 0$, while $X(1) = 0$ then gives (since $b \neq 0$ as $b = 0$ implies that $X = 0$, hence $u = 0$) $\mu = \ell\pi$, $\ell \in \mathbf{Z}$, $\ell > 0$. Similarly, we find that

$$Y = \sin m\pi y, \quad Z = \sin n\pi z,$$

$m, n \in \mathbf{Z}$, $m, n > 0$. Thus we have for u

$$u = \sin \ell\pi x \sin m\pi y \sin n\pi z,$$

$\ell, m, n \in \mathbf{Z}$, $\ell, m, n > 0$. The corresponding eigenvalue is clearly

$$\lambda = \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = -\ell^2\pi^2 - m^2\pi^2 - n^2\pi^2 = -\pi^2\left(\ell^2 + m^2 + n^2\right).$$

We shall denote the above function by $\mathbf{e}_{\ell mn}(x,y,z)$ and the above eigenvalue (if needed) by $-\lambda_{\ell mn}^2$ (changing the notation slightly); thus we have

$$\mathbf{e}_{\ell mn}(x,y,z) = \sin \ell\pi x \sin m\pi y \sin n\pi z, \quad -\lambda_{\ell mn}^2 = -\pi^2\left(\ell^2+m^2+n^2\right).$$

Here we have $\ell, m, n \in \mathbf{Z}, \ell, m, n > 0$.

Let us pause to consider the properties of the set of eigenfunctions here. We note that the set $\{\sin \ell\pi x | \ell \in \mathbf{Z}, \ell > 0\}$ is complete on $[0,1]$, and similarly for the sets $\{\sin m\pi y | m \in \mathbf{Z}, m > 0\}$ and $\{\sin n\pi z | n \in \mathbf{Z}, n > 0\}$. Now let $f : Q \to \mathbf{R}$ be any suitably well-behaved (for example, piecewise continuous) function on Q. Then we may expand successively as follows:

$$\begin{aligned}
f(x,y,z) &= \sum_{\ell=1}^{\infty} f_\ell(y,z)\sin \ell\pi x \\
&= \sum_{\ell=1}^{\infty}\left(\sum_{m=1}^{\infty} f_{\ell m}(z)\sin m\pi y\right)\sin \ell\pi x \\
&= \sum_{\ell=1}^{\infty}\left(\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty} f_{\ell mn}\sin n\pi z\right]\sin m\pi y\right)\sin \ell\pi x \\
&= \sum_{\ell,m,n=1}^{\infty} f_{\ell mn}\sin \ell\pi x \sin m\pi y \sin n\pi z,
\end{aligned}$$

where

$$\begin{aligned}
f_\ell(y,z) &= 2\int_0^1 f(x,y,z)\sin \ell\pi x\, dx \\
f_{\ell m}(z) &= 2\int_0^1 f_\ell(y,z)\sin m\pi y\, dy = 4\int_0^1\int_0^1 f(x,y,z)\sin \ell\pi x \sin m\pi y\, dx\, dy \\
f_{\ell mn} &= 2\int_0^1 f_{\ell m}(z)\sin n\pi z\, dz = 8\int_0^1\int_0^1\int_0^1 f(x,y,z)\sin \ell\pi x \sin m\pi y \sin n\pi z\, dx\, dy\, dz,
\end{aligned}$$

or in other words

$$f_{\ell mn} = \frac{(f, \mathbf{e}_{\ell mn})}{(\mathbf{e}_{\ell mn}, \mathbf{e}_{\ell mn})},$$

where we use the inner product

$$(f,g) = \iiint_Q f(x,y,z)\overline{g(x,y,z)}\, dV.$$

The foregoing is exactly what we would expect were the set $\{\mathbf{e}_{\ell mn}\}$ a complete orthogonal set on Q, and it turns out that this is the case. The foregoing is the closest we shall probably get to showing that the set is complete; orthogonality can be shewn as follows:

$$\begin{aligned}
(\mathbf{e}_{\ell mn}, \mathbf{e}_{\ell' m' n'}) &= \int_0^1\int_0^1\int_0^1 \mathbf{e}_{\ell mn}(x,y,z)\overline{\mathbf{e}_{\ell' m' n'}(x,y,z)}\, dV \\
&= \int_0^1\int_0^1\int_0^1 \sin \ell\pi x \sin m\pi y \sin n\pi z \sin \ell'\pi x \sin m'\pi y \sin n'\pi z\, dx\, dy\, dz \\
&= \int_0^1 \sin \ell\pi x \sin \ell'\pi x\, dx \int_0^1 \sin m\pi y \sin m'\pi y\, dy \int_0^1 \sin n\pi z \sin n'\pi z\, dz,
\end{aligned}$$

which is easily seen to be $\frac{1}{8}$ in case $\ell = \ell'$, $m = m'$, and $n = n'$, and to be zero if any of these equalities fails to hold. This means that the set $\{\mathbf{e}_{\ell mn}\}$ is orthogonal on Q, as claimed, with normalisation constant $\frac{1}{8}$ (in accord with the expansion formula above).

SOLVING POISSION'S EQUATION ON THE UNIT CUBE. The general idea behind this procedure has been explained already (see the notes for July 2 – 4). We shall illustrate it in this specific case with the following example.

Example. Solve the following problem on Q:

$$\nabla^2 u = (1-x^2)(1-y^2)(1-z^2), \quad u|_{\partial Q} = 0.$$

By our general method, the first step is to expand the right-hand side of the equation above in the above complete orthogonal set. To do this, we first calculate the following integral ($\ell \in \mathbf{Z}$, $\ell > 0$):

$$\begin{aligned}\int_0^1 (1-x^2)\sin \ell\pi x \, dx &= -(1-x^2)\frac{\cos \ell\pi x}{\ell\pi}\bigg|_0^1 - \int_0^1 (2x)\frac{\cos \ell\pi x}{\ell\pi}\, dx = \frac{1}{\ell\pi} - \frac{2}{\ell\pi}\int_0^1 x\cos \ell\pi x \, dx \\ &= \frac{1}{\ell\pi} - \frac{2}{\ell\pi}\left[x\frac{\sin \ell\pi x}{\ell\pi}\bigg|_0^1 - \int_0^1 \frac{\sin \ell\pi x}{\ell\pi}\, dx\right] \\ &= \frac{1}{\ell\pi} + \frac{2}{\ell^2\pi^2}\left(-\frac{\cos \ell\pi x}{\ell\pi}\bigg|_0^1\right) = \frac{1}{\ell\pi} + \frac{2}{\ell^3\pi^3}\left(1-(-1)^\ell\right).\end{aligned}$$

Thus we see that the expansion coefficients for the function $f = (1-x^2)(1-y^2)(1-z^2)$ are

$$\begin{aligned} f_{\ell mn} &= 8\int_0^1\int_0^1\int_0^1 (1-x^2)(1-y^2)(1-z^2) u_{\ell mn}\, dx\, dy\, dz \\ &= 8\int_0^1\int_0^1\int_0^1 (1-x^2)(1-y^2)(1-z^2)\sin \ell\pi x \sin m\pi y \sin n\pi z\, dx\, dy\, dz \\ &= 8\int_0^1 (1-x^2)\sin \ell\pi x\, dx \int_0^1 (1-y^2)\sin m\pi y\, dy \int_0^1 (1-z^2)\sin n\pi z\, dz \\ &= 8\left(\frac{1}{\ell\pi} + \frac{2}{\ell^3\pi^3}\left(1-(-1)^\ell\right)\right)\left(\frac{1}{m\pi} + \frac{2}{m^3\pi^3}\left(1-(-1)^m\right)\right)\left(\frac{1}{n\pi} + \frac{2}{n^3\pi^3}\left(1-(-1)^n\right)\right).\end{aligned}$$

Now suppose that u is a solution to the given problem, and let the expansion coefficients for u be $u_{\ell mn}$, so that

$$u = \sum_{\ell,m,n=1}^\infty u_{\ell mn}\sin \ell\pi x \sin m\pi y \sin n\pi z.$$

Substituting this into the equation $\nabla^2 u = (1-x^2)(1-y^2)(1-z^2)$, and assuming that we may differentiate term-by-term, we obtain

$$\begin{aligned}\nabla^2 u &= \sum_{\ell,m,n=1}^\infty u_{\ell mn}(-\lambda_{\ell mn}^2)\sin \ell\pi x \sin m\pi y \sin n\pi z \\ &= \sum_{\ell,m,n=1}^\infty 8\left(\frac{1}{\ell\pi} + \frac{2}{\ell^3\pi^3}\left(1-(-1)^\ell\right)\right)\left(\frac{1}{m\pi} + \frac{2}{m^3\pi^3}\left(1-(-1)^m\right)\right)\left(\frac{1}{n\pi} + \frac{2}{n^3\pi^3}\left(1-(-1)^n\right)\right) \\ &\qquad \cdot \sin \ell\pi x \sin m\pi y \sin n\pi z.\end{aligned}$$

Since $\{\sin \ell\pi x \sin m\pi y \sin n\pi z\}$ is a complete orthogonal set on Q, the coefficients in these two sums must be equal; thus we obtain

$$u_{\ell mn} = -\frac{8\left(\frac{1}{\ell\pi} + \frac{2}{\ell^3\pi^3}\left(1-(-1)^\ell\right)\right)\left(\frac{1}{m\pi} + \frac{2}{m^3\pi^3}\left(1-(-1)^m\right)\right)\left(\frac{1}{n\pi} + \frac{2}{n^3\pi^3}\left(1-(-1)^n\right)\right)}{\pi^2(\ell^2+m^2+n^2)},$$

whence finally we have the solution

$$u = \sum_{\ell,m,n=1}^\infty -\frac{8\left(\frac{1}{\ell\pi} + \frac{2}{\ell^3\pi^3}\left(1-(-1)^\ell\right)\right)\left(\frac{1}{m\pi} + \frac{2}{m^3\pi^3}\left(1-(-1)^m\right)\right)\left(\frac{1}{n\pi} + \frac{2}{n^3\pi^3}\left(1-(-1)^n\right)\right)}{\pi^2(\ell^2+m^2+n^2)} \cdot \sin \ell\pi x \sin m\pi y \sin n\pi z.$$

SOLVING THE HEAT EQUATION ON THE UNIT CUBE. We now consider a different problem which can also be treated using the above eigenfunctions for the Laplacian on the unit cube. We recall that early on in the course we derived the heat equation

$$\frac{\partial u}{\partial t} = \nabla^2 u,$$

which describes the time evolution of the temperature distribution of an object, and also other physical processes. We would like to learn how to solve this equation. First of all we must consider the question of what forms of initial or boundary data are appropriate. First we briefly recall what we know about ordinary differential equations: To solve a first-order ordinary differential equation, it suffices to know one piece of information, such as the value of the unknown function at some point (typically the initial point); to solve a second-order ordinary differential equation, we need to know two different pieces of information, such as the value of the function and its derivative at the initial point, or the value of the function at the end points. We have seen this latter situation play out in our study of Laplace's equation: in order to find a solution to Laplace's equation, we need to know something about the function on the boundary of the region we are considering; for example, its value over the whole boundary. (If we think back to the case of Laplace's equation on a cube, we see that this corresponds to the case in ordinary differential equations of giving the value of the function at the endpoints.) Now the heat equation is second-order in its spatial derivatives (just like Laplace's equation), but it is first-order in time. Thus we anticipate that we shall need to be given boundary data at each time of a sort similar to that we are given for Laplace's equation, while we also need to be given some kind of initial data. Since we are basically evolving the value of the function at each point in space, it seems reasonable to suspect that we may need to give as initial data the value of the function u at each point of Q, at some initial time (typically taken to be $t = 0$). We shall now give an example to indicate how this is done.

First a word about notation. A solution to the heat equation on Q is a function of *four* variables, three spatial ones and one temporal one, which we denote as x, y, z and t, respectively, so that a solution is written $u = u(t, x, y, z)$. We assume further that we are interested in finding the function u for all positive times, given its value at $t = 0$; thus we solve the heat equation on the region $(0, +\infty) \times Q$.

Example. Solve the following problem on $(0, +\infty) \times Q$:

$$\frac{\partial u}{\partial t} = \nabla^2 u, \quad u|_{t=0} = xyz, \quad u|_{(0,+\infty)\times\partial Q} = 0.$$

(Here $u|_{t=0} = xyz$ is the initial data discussed above, while $u|_{(0,+\infty)\times\partial Q} = 0$ is the boundary data. Note that we do not need to give any 'boundary data' for the future in t; thus, if we consider $(0, +\infty) \times Q$ as a long rectangular prism, then we are given data only on the bottom and sides, not on the top. This is because the heat equation is only first-order in time.) For each t, we may expand the function $u(t, x, y, z)$ in the basis $\{\mathbf{e}_{\ell mn}\}$ as

$$u(t, x, y, z) = \sum_{\ell,m,n=1}^{\infty} u_{\ell mn}(t)\mathbf{e}_{\ell mn} = \sum_{\ell,m,n=1}^{\infty} u_{\ell mn}(t) \sin \ell\pi x \sin m\pi y \sin n\pi z.$$

Substituting this into the above equation, and assuming that we may differentiate term-by-term (and that the expansion coefficients $u_{\ell mn}(t)$ are differentiable), we obtain

$$u'_{\ell mn} = -\pi^2(\ell^2 + m^2 + n^2)u_{\ell mn}.$$

This equation can be solved easily to obtain

$$u_{\ell mn}(t) = u_{\ell mn}(0)e^{-\pi^2(\ell^2+m^2+n^2)t}$$

(that the multiplicative constant is in fact $u_{\ell mn}(0)$ may be seen by setting $t = 0$ in both sides of the above equation). Thus, to determine $u_{\ell mn}(t)$, and hence to determine the desired series expansion for u, it suffices to determine the expansion coefficients $u_{\ell mn}(0)$ for the initial data xyz. Now

$$\int_0^1 x \sin \ell\pi x \, dx = -x\frac{\cos \ell\pi x}{\ell\pi}\bigg|_0^1 + \frac{1}{\ell\pi}\int_0^1 \cos \ell\pi x \, dx = \frac{(-1)^{\ell+1}}{\ell\pi},$$

whence we see that the expansion coefficients $u_{\ell mn}(0)$ are

$$u_{\ell mn}(0) = 8\frac{(-1)^{\ell+1}}{\ell\pi}\frac{(-1)^{m+1}}{m\pi}\frac{(-1)^{n+1}}{n\pi} = -8\frac{(-1)^{\ell+m+n}}{\pi^3\ell mn},$$

so

$$u_{\ell mn}(t) = -8\frac{(-1)^{\ell+m+n}}{\pi^3\ell mn}e^{-\pi^2(\ell^2+m^2+n^2)t}$$

and the solution to our problem is

$$u(t,x,y,z) = \sum_{\ell,m,n=1}^{\infty} -8\frac{(-1)^{\ell+m+n}}{\pi^3\ell mn}e^{-\pi^2(\ell^2+m^2+n^2)t}\sin\ell\pi x\sin m\pi y\sin n\pi z.$$

A WORD ABOUT BOUNDARY CONDITIONS. So far we have considered various different kinds of boundary conditions without giving them names. Generally, the types of boundary conditions one considers for Laplace's equation (hence, for the spatial boundary conditions in the heat equation) on some region D are the following:

- Dirichlet: $u|_{\partial D} = f$.
- Neumann: $(\nabla u \cdot \mathbf{n})|_{\partial D} = g$.
- Robin: $(au + b\nabla u \cdot \mathbf{n})|_{\partial D} = h$.

Here $\mathbf{n}$ indicates the outer unit normal to the region D at its boundary.

One may also consider more general conditions (for example, Dirichlet over part of the boundary and Neumann over another part), which we may then term *mixed* boundary conditions.

APM 346, Homework 8. Due Monday, July 15, at 6.00 AM EDT. To be marked completed/not completed.

Using our derivation of the eigenfunctions and eigenvalues of the Laplacian in class, solve the following problems.

1. Write out a series expansion for the solution to the following problem on $Q = \{(x, y, z)|0 \leq x, y, z \leq 1\}$:

$$\nabla^2 u = \chi(x)\chi(y)\chi(z), \quad u|_{\partial Q} = 0,$$

where ∂Q is the boundary of the cube Q and

$$\chi(x) = \begin{cases} 0, & 0 \leq x < \frac{1}{2} \\ 1, & \frac{1}{2} < x \leq 1 \end{cases}.$$

We note the following integral:

$$\begin{aligned} \int_0^1 \chi(x) \sin \ell\pi x \, dx &= \int_{\frac{1}{2}}^1 \sin \ell\pi x \, dx = -\frac{\cos \ell\pi x}{\ell\pi}\bigg|_{\frac{1}{2}}^1 \\ &= \frac{(-1)^{\ell+1}}{\ell\pi} + \frac{\cos \frac{1}{2}\ell\pi}{\ell\pi} \\ &= \begin{cases} \frac{1}{\ell\pi}, & \ell \text{ odd} \\ \frac{1}{\ell\pi}\left((-1)^{\frac{\ell}{2}} - 1\right), & \ell \text{ even} \end{cases}. \end{aligned}$$

There is no convenient way to simplify a triple sum over a product of three versions of this last quantity, so we shall use the second-to-last line instead in the formulæ below. Thus we write

$$\begin{aligned} &\int_0^1 \int_0^1 \int_0^1 \chi(x)\chi(y)\chi(z) \sin \ell\pi x \sin m\pi x \sin n\pi x \, dx \, dy \, dz \\ &\qquad = \frac{1}{\pi^3 \ell mn}\left((-1)^{\ell+1} + \cos\frac{1}{2}\ell\pi\right)\left((-1)^{m+1} + \cos\frac{1}{2}m\pi\right)\left((-1)^{n+1} + \cos\frac{1}{2}n\pi\right), \end{aligned}$$

whence we may write

$$\begin{aligned} \chi(x)\chi(y)\chi(z) = \sum_{\ell,m,n=1}^{\infty} \frac{8}{\pi^3 \ell mn}&\left((-1)^{\ell+1} + \cos\frac{1}{2}\ell\pi\right)\left((-1)^{m+1} + \cos\frac{1}{2}m\pi\right)\left((-1)^{n+1} + \cos\frac{1}{2}n\pi\right) \\ &\cdot \sin \ell\pi x \sin m\pi y \sin n\pi z. \end{aligned}$$

From our general technique, if we denote the coefficients in the above series by $\chi_\ell \chi_m \chi_n$, then the coefficients $u_{\ell mn}$ in the series expansion for u will be given by

$$u_{\ell mn} = -\frac{1}{\pi^2(\ell^2 + m^2 + n^2)}\chi_\ell \chi_m \chi_n,$$

whence the solution for u will be

$$\begin{aligned} u = \sum_{\ell,m,n=1}^{\infty} -\frac{8}{\pi^5 \ell mn(\ell^2 + m^2 + n^2)}\Bigg[&\left((-1)^{\ell+1} + \cos\frac{1}{2}\ell\pi\right)\left((-1)^{m+1} + \cos\frac{1}{2}m\pi\right)\left((-1)^{n+1} + \cos\frac{1}{2}n\pi\right) \\ &\cdot \sin \ell\pi x \sin m\pi y \sin n\pi z\Bigg]. \end{aligned}$$

2. Write out a series expansion for the solution to the following problem on $Q \times (0, +\infty)$, where Q is as in problem 1:

$$\frac{\partial u}{\partial t} = \nabla^2 u, \quad u|_{\partial Q} = 0, \quad u|_{t=0} = \sin \pi x \sin \pi y,$$

where we denote an arbitrary point in $Q \times (0, +\infty)$ by (x, y, z, t).

We proceed similarly to question 1 and first calculate the expansion coefficients for the nonhomogeneous boundary term $\sin \pi x \sin \pi y$. Since

$$\int_0^1 \sin \ell\pi x \sin \pi x \, dx = \begin{cases} \frac{1}{2}, & \ell = 1 \\ 0, & \ell \neq 1 \end{cases}$$

and

$$\int_0^1 \sin n\pi z \, dz = -\frac{1}{n\pi} \cos n\pi z \Big|_0^1 = \frac{1}{n\pi} \left(1 - (-1)^n\right),$$

we see that

$$\int_0^1 \int_0^1 \int_0^1 \sin \pi x \sin \pi y \sin \ell\pi x \sin m\pi y \sin n\pi z \, dx \, dy \, dz = \begin{cases} \frac{1}{4n\pi} \left(1 - (-1)^n\right), & \ell = m = 1 \\ 0, & \text{otherwise} \end{cases},$$

whence

$$\sin \pi x \sin \pi y = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} \sin \pi x \sin \pi y \sin (2k+1)\pi z.$$

We note that the eigenvalue corresponding to the kth term in the above sum is $-\pi^2(2 + (2k+1)^2)$ (since the kth term corresponds to the (ℓ, m, n) term in the original sum with $\ell = m = 1$ and $n = 2k + 1$). Thus the solution to our original problem is simply

$$u = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} \sin \pi x \sin \pi y \sin (2k+1)\pi z e^{-\pi^2(2+(2k+1)^2)t}.$$

Summary:
- We use the eigenfunctions and eigenvalues of the Laplacian on a cylinder, derived last week, to solve a problem involving Poisson's equation on a cylinder.
- We then derive the eigenfunctions and eigenvalues of the Laplacian on the unit ball.
- We give a method for solving Poisson's equation and the heat equation with inhomogeneous boundary conditions, and give an example in spherical coordinates.

EXAMPLE. Solve the following problem on the cylinder $C = \{(\rho, \theta, z) | \rho < 1, 0 < z < 1\}$:

$$\nabla^2 u = z\rho^2 \cos 2\phi, \quad u|_{\partial C} = 0.$$

From last time, we know that the eigenfunctions of the Laplacian on C are

$$\mathbf{e}_{nmi} = \begin{cases} J_m(\lambda_{mi}\rho)\cos m\phi \sin n\pi z \\ J_m(\lambda_{mi}\rho)\sin m\phi \sin n\pi z \end{cases}$$

with corresponding eigenvalues

$$\lambda_{nmi} = -\lambda_{mi}^2 - n^2\pi^2.$$

Thus we must expand the function $z\rho^2 \cos 2\phi$ in this basis. To do this, we compute as follows:

$$\begin{aligned}\left(z\rho^2\cos 2\phi, J_m(\lambda_{mi}\rho)\cos m\phi \sin n\pi z\right) &= \int_0^1 \int_0^1 \int_0^{2\pi} z\rho^2 \cos 2\phi J_m(\lambda_{mi}\rho)\cos m\phi \sin n\pi z \, d\phi \, dz \, \rho d\rho \\ &= \int_0^1 z \sin n\pi z \, dz \int_0^1 \rho^3 J_m(\lambda_{mi}\rho) \, d\rho \int_0^{2\pi} \cos 2\phi \cos m\phi \, d\phi,\end{aligned}$$

which is seen to be zero when $m \neq 2$, while when $m = 2$ it becomes

$$\frac{(-1)^{n+1}}{n\pi} \frac{J_3(\lambda_{mi})}{\lambda_{mi}} \pi,$$

whence the coefficient of $J_m(\lambda_{mi}\rho)\cos m\phi \sin n\pi z$ in the expansion of $z\rho^2 \cos 2\phi$ when $m \neq 2$ is zero, while when $m = 2$ it is

$$\frac{\frac{1}{n\pi}(-1)^{n+1}\frac{J_3(\lambda_{2i})}{\lambda_{2i}}\pi}{\frac{1}{2}\cdot\frac{1}{2}J_3^2(\lambda_{2i})\cdot\pi} = \frac{4(-1)^{n+1}}{n\pi\lambda_{2i}J_3(\lambda_{2i})}.$$

A similar calculation shows immediately that the coefficient of $J_m(\lambda_{mi}\rho)\sin m\phi \sin n\pi z$ is zero, since $\cos 2\phi$ is orthogonal to $\sin m\phi$ for all m. Thus we have finally

$$z\rho^2\cos 2\phi = \sum_{n=1}^\infty \sum_{i=1}^\infty \frac{4(-1)^{n+1}}{n\pi\lambda_{2i}J_3(\lambda_{2i})} J_2(\lambda_{2i}\rho)\sin n\pi z \cos 2\phi.$$

Given this, the solution to our original problem is almost immediate: we assume as usual that we have an expansion of the form

$$u(\rho, \theta, z) = \sum_{n=1}^\infty \sum_{m=0}^\infty \sum_{i=1}^\infty J_m(\lambda_{mi}\rho)\sin n\pi z \left(c_{nmi}\cos m\phi + d_{nmi}\sin m\phi\right);$$

then, assuming that we may differentiate term-by-term, we have, since both $J_m(\lambda_{mi}\rho)\cos m\phi \sin n\pi z$ and $J_m(\lambda_{mi}\rho)\sin m\phi \sin n\pi z$ are eigenfunctions of the Laplacian with the same eigenvalue $\lambda_{nmi} = -\lambda_{mi}^2 - n^2\pi^2$,

$$\begin{aligned}\nabla^2 u &= \sum_{n=1}^\infty \sum_{m=0}^\infty \sum_{i=1}^\infty \left(-\lambda_{mi}^2 - n^2\pi^2\right) J_m(\lambda_{mi}\rho)\sin n\pi z \left(c_{nmi}\cos m\phi + d_{nmi}\sin m\phi\right) \\ &= \sum_{n=1}^\infty \sum_{i=1}^\infty \frac{4(-1)^{n+1}}{n\pi\lambda_{2i}J_3(\lambda_{2i})} J_2(\lambda_{2i}\rho)\sin n\pi z \cos 2\phi,\end{aligned}$$

whence we see that $d_{nmi} = 0$ for all n, m, i, while $c_{nmi} = 0$ for all n and i unless $m = 2$ and

$$c_{n2i} = -\frac{4(-1)^{n+1}}{n\pi\lambda_{2i}J_3\left(\lambda_{2i}\right)\left(\lambda_{2i}^2 + n^2\pi^2\right)},$$

so that finally we have the solution

$$u(\rho, \theta, z) = \sum_{n=1}^{\infty}\sum_{i=1}^{\infty} \frac{4(-1)^n}{n\pi\lambda_{2i}J_3\left(\lambda_{2i}\right)\left(\lambda_{2i}^2 + n^2\pi^2\right)} J_2\left(\lambda_{2i}\rho\right)\sin n\pi z\cos 2\phi.$$

EXAMPLE. Solve the following problem on $(0, +\infty) \times C$:

$$\frac{\partial u}{\partial t} = \nabla^2 u, \quad u|_{t=0} = z\rho^2\cos 2\phi, \quad u|_{(0,+\infty)\times\partial Q} = 0.$$

From the previous example, we have the expansion

$$z\rho^2\cos 2\phi = \sum_{k=0}^{\infty}\sum_{i=1}^{\infty} \frac{8}{(2k+1)\pi\lambda_{2i}J_3\left(\lambda_{2i}\right)} J_2\left(\lambda_{2i}\rho\right)\sin(2k+1)\pi z\cos 2\phi.$$

Expanding u as

$$u = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{i=1}^{\infty} J_m\left(\lambda_{mi}\rho\right)\sin n\pi z\left(c_{nmi}(t)\cos m\phi + d_{nmi}(t)\sin m\phi\right)$$

and substituting this into the heat equation $\frac{\partial u}{\partial t} = \nabla^2 u$ as before, we see that

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{i=1}^{\infty} & J_m\left(\lambda_{mi}\rho\right)\sin n\pi z\left(c'_{nmi}(t)\cos m\phi + d'_{nmi}(t)\sin m\phi\right) \\ &= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{i=1}^{\infty}\left(-\lambda_{mi}^2 - n^2\pi^2\right) J_m\left(\lambda_{mi}\rho\right)\sin n\pi z\left(c_{nmi}(t)\cos m\phi + d_{nmi}(t)\sin m\phi\right),\end{aligned}$$

whence equating coefficients of like terms gives the equations

$$\begin{aligned} c'_{nmi} &= -\left(\lambda_{mi}^2 + n^2\pi^2\right)c_{nmi} \\ d'_{nmi} &= -\left(\lambda_{mi}^2 + n^2\pi^2\right)d_{nmi}. \end{aligned} \tag{1}$$

Now the initial condition gives

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{i=1}^{\infty} & J_m\left(\lambda_{mi}\rho\right)\sin n\pi z\left(c_{nmi}(0)\cos m\phi + d_{nmi}(0)\sin m\phi\right) \\ &= \sum_{n=1}^{\infty}\sum_{i=1}^{\infty}\frac{4\left(1-(-1)^n\right)}{n\pi\lambda_{2i}J_3\left(\lambda_{2i}\right)} J_2\left(\lambda_{2i}\rho\right)\sin n\pi z\cos 2\phi,\end{aligned}$$

so

$$c_{nmi}(0) = \begin{cases} 0, & m \neq 2, \\ \frac{4\left(1-(-1)^n\right)}{n\pi\lambda_{2i}J_3\left(\lambda_{2i}\right)}, & m = 2 \end{cases}$$
$$d_{nmi}(0) = 0,$$

whence the system (1) gives $d_{nmi}(t) = 0$ for all n, m, i and all t, while $c_{nmi}(t) = 0$ for all t unless $m = 2$ and finally

$$c_{n2i}(t) = \frac{4\left(1-(-1)^n\right)}{n\pi\lambda_{2i}J_3\left(\lambda_{2i}\right)}e^{-\left(\lambda_{2i}^2 + n^2\pi^2\right)t},$$

so that the solution to our original problem is finally

$$\begin{aligned}
u(\rho,\theta,z) &= \sum_{n=1}^{\infty}\sum_{i=1}^{\infty}\frac{4\left(1-(-1)^n\right)}{n\pi\lambda_{2i}J_3\left(\lambda_{2i}\right)}e^{-\left(\lambda_{2i}^2+n^2\pi^2\right)t}J_2\left(\lambda_{2i}\rho\right)\sin n\pi z\cos 2\phi \\
&= \sum_{k=0}^{\infty}\sum_{i=1}^{\infty}\frac{8}{(2k+1)\pi\lambda_{2i}J_3\left(\lambda_{2i}\right)}e^{-\left(\lambda_{2i}^2+(2k+1)^2\pi^2\right)t}J_2\left(\lambda_{2i}\rho\right)\sin\left(2k+1\right)\pi z\cos 2\phi.
\end{aligned}$$

EIGENVALUES AND EIGENFUNCTIONS FOR THE LAPLACIAN ON THE UNIT BALL. We now turn our attention to the task of finding the eigenfunctions and eigenvalues of the Laplacian on the unit ball with homogeneous[1] Dirichlet boundary conditions. In other words, let $B = \{(r,\theta,\phi)|r<1\}$ denote the unit ball in spherical coordinates, and consider the problem

$$\nabla^2 u = \lambda u, \quad u|_{\partial B} = 0.$$

We approach this problem as before, by separating variables; thus we set

$$u(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi),$$

and recalling that in spherical coordinates the Laplacian is given by

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}$$

we obtain, substituting into $\nabla^2 u = \lambda u$ and dividing by u

$$\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{\cot\theta}{r^2}\frac{\Theta'}{\Theta} + \frac{1}{r^2\sin^2\theta}\frac{\Phi''}{\Phi} = \lambda. \tag{2}$$

Since only the quantity $\frac{\Phi''}{\Phi}$ depends on ϕ, this quantity must be constant. Considerations identical to those used when solving Laplace's equation in spherical and cylindrical coordinates and when finding the eigenvalues of the Laplacian in cylindrical coordinates show that we must in fact have $\frac{\Phi''}{\Phi} = -m^2$, where $m \in \mathbf{Z}$, $m \geq 0$, which has solutions $\{hi = \cos m\phi$, $\Phi = \sin m\phi$ (the latter only for $m > 0$). Substituting this back into equation (2) above, we obtain

$$\begin{aligned}
\lambda &= \frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{\cot\theta}{r^2}\frac{\Theta'}{\Theta} - \frac{m^2}{r^2\sin^2\theta} \\
&= \frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \frac{1}{r^2}\left(\frac{\Theta''}{\Theta} + \cot\theta\frac{\Theta'}{\Theta} - \frac{m^2}{\sin^2\theta}\right);
\end{aligned}$$

as when we solved Laplace's equation in spherical coordinates (see notes for May 23 – 30), this implies that the quantity in parentheses above is constant. By analogy with what we did there, we set it equal to $-\ell(\ell+1)$, where $\ell \in \mathbf{Z}$, $\ell \geq 0$. Then Θ must satisfy the equation

$$\Theta'' + \cot\theta\Theta' + \left(\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right)\Theta = 0,$$

whence we see that $\Theta(\theta) = P_{\ell m}(\cos\theta)$, as when solving Laplace's equation. We are thus left only with the following equation for R:

$$\begin{aligned}
\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} - \frac{1}{r^2}\ell(\ell+1) &= \lambda, \\
R'' + \frac{2}{r}R' + \left(-\lambda - \frac{\ell(\ell+1)}{r^2}\right)R &= 0.
\end{aligned}$$

[1]Since eigenvalue problems must of necessity be linear, it makes no sense to ask for an eigenfunction of the Laplacian satisfying inhomogeneous boundary conditions; or at any rate, while one could certainly write out the equations, it is hard to see how the resulting solutions could be of use.

As it stands, this is close to Bessel's equation (see notes for June 11 – 13, p. 2, Equation (2))

$$P'' + \frac{1}{\rho}P' + \left(\lambda^2 - \frac{m^2}{\rho^2}\right)P = 0,$$

but it is not identical. We may transform it into Bessel's equation by the following method. Let $S = r^{\frac{1}{2}}R$, so that $R = r^{-\frac{1}{2}}S$; then we have

$$R' = -\frac{1}{2}r^{-\frac{3}{2}}S + r^{-\frac{1}{2}}S',$$
$$R'' = \frac{3}{4}r^{-\frac{5}{2}}S - r^{-\frac{3}{2}}S' + r^{-\frac{1}{2}}S'',$$

whence we see that

$$\begin{aligned}
0 &= R'' + \frac{2}{r}R' + \left(-\lambda - \frac{\ell(\ell+1)}{r^2}\right)R \\
&= \left(\frac{3}{4}r^{-\frac{5}{2}}S - r^{-\frac{3}{2}}S' + r^{-\frac{1}{2}}S''\right) + \frac{2}{r}\left(-\frac{1}{2}r^{-\frac{3}{2}}S + r^{-\frac{1}{2}}S'\right) + \left(-\lambda - \frac{\ell(\ell+1)}{r^2}\right)r^{-\frac{1}{2}}S \\
&= r^{-\frac{1}{2}}\left(S'' + \left(-r^{-1} + \frac{2}{r}\right)S' + \left(\frac{3}{4}r^{-2} - \frac{1}{r^2} - \lambda - \frac{\ell(\ell+1)}{r^2}\right)S\right) \\
&= r^{-\frac{1}{2}}\left(S'' + \frac{1}{r}S' + \left(-\lambda - \frac{\ell(\ell+1)+\frac{1}{4}}{r^2}\right)S\right) = r^{-\frac{1}{2}}\left(S'' + \frac{1}{r}S' + \left(-\lambda - \frac{\left(\ell+\frac{1}{2}\right)^2}{r^2}\right)S\right),
\end{aligned}$$

so that S must satisfy the equation

$$S'' + \frac{1}{r}S' + \left(-\lambda - \frac{\left(\ell+\frac{1}{2}\right)^2}{r^2}\right)S = 0.$$

Now the boundary condition $u|_{\partial B} = 0$ means that R must satisfy $R(1) = 0$; since $S = r^{\frac{1}{2}}R$, this implies that $S(1) = 0$ also. Thus S cannot be a modified Bessel function, which implies that we must have $\lambda < 0$ and (up to a multiplicative constant) $S = J_{\ell+\frac{1}{2}}(\sqrt{\lambda}r)$. Again, $S(1) = 0$ implies that $\sqrt{\lambda} = \kappa_{\ell i}$ for some i, where $\kappa_{\ell i}$ denotes the ith positive zero of $J_{\ell+\frac{1}{2}}(x)$ (thus, if we were to extend our earlier notation and let $\lambda_{\nu i}$ denote the ith positive zero of $J_\nu(x)$ for any real $\nu \geq 0$, we have $\kappa_{\ell i} = \lambda_{\ell+\frac{1}{2},i}$; this latter expression is the notation which we used in class). We thus obtain that up to a multiplicative constant

$$R = r^{-\frac{1}{2}}J_{\ell+\frac{1}{2}}\left(\kappa_{\ell i}r\right).$$

It turns out to be convenient to take the multiplicative constant to be $\sqrt{\frac{\pi}{2}}$. The resulting functions are called *spherical Bessel functions* and are denoted by j_ℓ, $\ell \in \mathbf{Z}$, $\ell \geq 0$; explicitly,

$$j_\ell(x) = \sqrt{\frac{\pi}{2x}}J_{\ell+\frac{1}{2}}(x).$$

We thus obtain finally that the eigenfunctions for the Laplacian on the unit ball are

$$\mathbf{e}_{m\ell i} = \begin{cases} j_\ell\left(\kappa_{\ell i}r\right)P_{\ell m}(\cos\theta)\cos m\phi \\ j_\ell\left(\kappa_{\ell i}r\right)P_{\ell m}(\cos\theta)\sin m\phi \end{cases},$$

with corresponding eigenvalue

$$\lambda_{m\ell i} = -\kappa_{\ell i}^2.$$

We note that the eigenvalue does not depend on m (though it does depend on both ℓ and i).

We now derive the orthogonality properties of the j_ℓ. First, we note without proof that the Bessel functions J_ν satisfy the same orthogonality relations as the J_m for all real (not just integer) $\nu \geq 0$, namely

$$\int_0^1 x J_\nu(\lambda_{\nu i} x) J_\nu(\lambda_{\nu j} x)\, dx = \begin{cases} 0, & i \neq j \\ \frac{1}{2} J_{\nu+1}^2(\lambda_{\nu i}), & i = j \end{cases}.$$

From this we may derive the orthogonality property of the spherical Bessel functions, as follows:

$$\begin{aligned} \int_0^1 x^2 j_\ell(\kappa_{\ell i} x) j_\ell(\kappa_{\ell j} x)\, dx &= \frac{\pi}{2\sqrt{\kappa_{\ell i}\kappa_{\ell j}}} \int_0^1 x J_{\ell+\frac{1}{2}}\left(\lambda_{\ell+\frac{1}{2},i} x\right) J_{\ell+\frac{1}{2}}\left(\lambda_{\ell+\frac{1}{2},j} x\right) dx \\ &= \begin{cases} 0, & i \neq j \\ \frac{\pi}{4\lambda_{\ell+\frac{1}{2},i}} J_{\ell+\frac{1}{2}+1}^2\left(\lambda_{\ell+\frac{1}{2},i}\right), & i = j \end{cases}, \end{aligned}$$

whence we see that $\{j_\ell(\kappa_{\ell i} x)\}$ is an orthogonal set on the interval $[0,1]$ with the normalisation integral

$$\begin{aligned} \int_0^1 x^2 j_\ell^2(\kappa_{\ell i} x)\, dx &= \frac{\pi}{4\lambda_{\ell+\frac{1}{2},i}} J_{\ell+\frac{1}{2}+1}^2\left(\lambda_{\ell+\frac{1}{2},i}\right) = \frac{1}{2}\left(\sqrt{\frac{\pi}{2\lambda_{\ell+\frac{1}{2},i}}} J_{\ell+1+\frac{1}{2}}\left(\lambda_{\ell+\frac{1}{2},i}\right)\right)^2 \\ &= \frac{1}{2} j_{\ell+1}^2(\kappa_{\ell i}). \end{aligned}$$

From this it follows, as before, that $\{\mathbf{e}_{m\ell i}\}$ is a complete orthogonal set on the unit ball B with respect to the inner product

$$\begin{aligned} (f(r,\theta,\phi), g(r,\theta,\phi)) &= \int_0^1 \int_0^\pi \int_0^{2\pi} f(r,\theta,\phi)\overline{g(r,\theta,\phi)}\, d\phi \sin\theta d\theta\, r^2 dr \\ &= \int_0^1 \int_0^\pi \int_0^{2\pi} f(r,\theta,\phi)\overline{g(r,\theta,\phi)} r^2 \sin\theta\, d\phi\, d\theta\, dr. \end{aligned}$$

(Note that the quantity $r^2 \sin\theta\, d\phi\, d\theta\, dr$ is just the volume element dV in spherical coordinates; in other words, the integral above is simply $\iiint_B f\overline{g} dV$.) This allows us to solve Poisson's equation and the heat equation on B, as we did with the unit cube Q and the cylinder C before.

EXAMPLE. Solve the following problem on B:

$$\nabla^2 u = r\sin\theta \sin\phi, \quad u|_{\partial B} = 0.$$

We begin, as usual, be expanding the function on the right-hand side in the basis of eigenfunctions $\{\mathbf{e}_{m\ell i}\}$ appropriate to the problem; thus we write

$$r\sin\theta\sin\phi = \sum_{m=0}^\infty \sum_{\ell=m}^\infty \sum_{i=1}^\infty j_\ell(\kappa_{\ell i} r) P_{\ell m}(\cos\theta)\left(a_{m\ell i}\cos m\phi + b_{m\ell i}\sin m\phi\right),$$

where

$$\begin{aligned} b_{m\ell i} &= \frac{(r\sin\theta\sin\phi, j_\ell(\kappa_{\ell i} r) P_{\ell m}(\cos\theta)\sin m\phi)}{(j_\ell(\kappa_{\ell i} r) P_{\ell m}(\cos\theta)\sin m\phi, j_\ell(\kappa_{\ell i} r) P_{\ell m}(\cos\theta)\sin m\phi)} \\ a_{m\ell i} &= \frac{(r\sin\theta\sin\phi, j_\ell(\kappa_{\ell i} r) P_{\ell m}(\cos\theta)\cos m\phi)}{(j_\ell(\kappa_{\ell i} r) P_{\ell m}(\cos\theta)\cos m\phi, j_\ell(\kappa_{\ell i} r) P_{\ell m}(\cos\theta)\cos m\phi)}. \end{aligned}$$

Since $(\sin\phi, \cos m\phi) = 0$ for all m, we see that we have $a_{m\ell i} = 0$ for all m, ℓ, i; similarly, $b_{m\ell i} = 0$ for all ℓ and i unless $m = 1$, in the which case we may compute (recalling that $\{P_{\ell m}(x)\}_{\ell=m}^\infty$ is a complete orthogonal set on $[-1,1]$ for all $m \geq 0$, and that $P_{11} = \sin\theta$)

$$\begin{aligned} (r\sin\theta\sin\phi, j_\ell(\kappa_{\ell i} r) P_{\ell 1}(\cos\theta)\sin\phi) &= \int_0^1 \int_0^\pi \int_0^{2\pi} r\sin\theta\sin\phi j_\ell(\kappa_{\ell i} r) P_{\ell 1}(\cos\theta)\sin\phi\, d\phi \sin\theta d\theta\, r^2 dr \\ &= \int_0^1 r^3 j_\ell(\kappa_{\ell i} r)\, dr \int_0^\pi \sin\theta P_{\ell 1}(\cos\theta)\sin\theta\, d\theta \int_0^{2\pi} \sin^2\phi\, d\phi \end{aligned}$$

which is zero unless $\ell = 1$, while if $\ell = 1$ it is (using the normalisation $\int_{-1}^{1} P_{\ell m}^2(x)\,dx = \frac{(\ell+m)!}{(\ell-m)!}\frac{2}{2\ell+1}$, which in our case becomes $\int_{-1}^{1} P_{11}^2(x)\,dx = \frac{4}{3}$, and remembering that $\kappa_{1i} = \lambda_{\frac{3}{2},i}$)

$$\frac{4\pi}{3}\int_0^1 r^3 j_1(\kappa_{1i} r)\,dr = \frac{4\pi}{3}\sqrt{\frac{\pi}{2\kappa_{1i}}}\int_0^1 r^{\frac{5}{2}} J_{\frac{3}{2}}\left(\lambda_{\frac{3}{2},i} r\right)dr$$
$$= \frac{4\pi}{3}\sqrt{\frac{\pi}{2\kappa_{1i}}}\frac{J_{\frac{5}{2}}\left(\lambda_{\frac{3}{2},i}\right)}{\lambda_{\frac{3}{2},i}} = \frac{4\pi}{3\kappa_{1i}} j_2(\kappa_{1i})$$

whence using the normalisation integrals for j_1, P_{11}, and $\sin\phi$ we obtain

$$b_{11i} = \frac{2}{\kappa_{1i} j_2(\kappa_{1i})},$$

while $b_{m\ell i} = 0$ unless $m = \ell = 1$. (Note the similarity of the above form to that derived for ordinary (nonspherical) Bessel functions when expanding expressions like ρ^m on a cylinder.) Thus we have finally the expansion

$$r\sin\theta\sin\phi = \sum_{i=1}^{\infty} \frac{2}{\kappa_{1i} j_2(\kappa_{1i})} j_1(\kappa_{1i} r)\sin\theta\sin\phi.$$

Writing now

$$u = \sum_{m=0}^{\infty}\sum_{\ell=m}^{\infty}\sum_{i=1}^{\infty} j_\ell(\kappa_{\ell i} r) P_{\ell m}(\cos\theta)\left(c_{m\ell i}\cos m\phi + d_{m\ell i}\sin m\phi\right),$$

we see that

$$\nabla^2 u = \sum_{m=0}^{\infty}\sum_{\ell=m}^{\infty}\sum_{i=1}^{\infty} -\kappa_{\ell i}^2 j_\ell(\kappa_{\ell i} r) P_{\ell m}(\cos\theta)\left(c_{m\ell i}\cos m\phi + d_{m\ell i}\sin m\phi\right);$$

equating this to the expansion for the function $r\sin\theta\sin\phi$ obtained above gives, as usual, $c_{m\ell i} = -\frac{1}{\kappa_{\ell i}^2} a_{m\ell i} = 0$ for all m, ℓ, i, while $d_{m\ell i} = -\frac{1}{\kappa_{\ell i}^2} b_{m\ell i}$ is zero unless $m = \ell = 1$, in the which case

$$d_{11i} = -\frac{2}{\kappa_{1i}^3 j_2(\kappa_{1i})},$$

and we have finally

$$u = \sum_{i=1}^{\infty} -\frac{2}{\kappa_{1i}^3 j_2(\kappa_{1i})} j_1(\kappa_{1i} r)\sin\theta\sin\phi.$$

A similar example could clearly be worked for the heat equation, along the lines of the pair of examples given in cylindrical coordinates above; we leave the formulation and solution of such a problem to the reader.

INHOMOGENEOUS BOUNDARY CONDITIONS. Consider now the problem (say on B)

$$\nabla^2 u = f, \quad u|_{\partial B} = g,$$

where neither f nor g is identically zero. This problem may be solved by first solving the two ancillary problems

$$\nabla^2 u_1 = f, \quad u_1|_{\partial B} = 0,$$
$$\nabla^2 u_2 = 0, \quad u_2|_{\partial B} = g,$$

the second of which may be solved using the methods developed for solving Laplace's equation on a ball, and the first of which may be solved using the eigenfunctions just derived. If we then set $u = u_1 + u_2$, we see that

$$\nabla^2 u = \nabla^2 u_1 + \nabla^2 u_2 = f + 0 = f,$$
$$u|_{\partial B} = u_1|_{\partial B} + u_2|_{\partial B} = 0 + g = g;$$

in other words, $u = u_1 + u_2$ is a solution to our original problem.

This method clearly applies to any of the regions Q, C, B we have studied.

In the hope that the foregoing is sufficiently clear as it stands, we skip giving any examples to talk about a similar method for the heat equation. In this case, we are given the problem

$$\frac{\partial u}{\partial t} = \nabla^2 u, \quad u|_{t=0} = f, \quad u|_{(0,+\infty)\times\partial B} = g.$$

As in the case of Poisson's equation just considered, this may be solved by decomposing u as a sum of solutions to two ancillary problems. The decomposition is a bit more subtle in this case. We first give some motivation. Recall from our previous work that solutions to the heat equation with homogeneous boundary data converge to 0 as $t \to +\infty$. A more careful consideration of the series solutions given above shows that in fact also $\frac{\partial u}{\partial t} \to 0$ as $t \to +\infty$.[2] Now if $\frac{\partial u}{\partial t} = 0$, then the heat equation becomes simply $\nabla^2 u = 0$, i.e., it becomes Laplace's equation. Noting that $u = 0$ is the unique solution to Laplace's equation on B satisfying $u|_{\partial B} = 0$, we see that in this (admittedly very special!) case the solution to the heat equation with boundary data $u|_{\partial B} = 0$ converges to the solution to *Laplace's* equation on B with the same boundary data.

It turns out that this is true for inhomogeneous boundary data also, as we shall now show. Thus let U_1 be the solution to the problem on B

$$\nabla^2 U_1 = 0, \quad U_1|_{\partial B} = g$$

(which is just a boundary-value problem for Laplace's equation on the unit ball, and hence is a problem we know how to solve). Now let us define $u_1 : (0, +\infty) \times B \to \mathbf{R}^1$ by $u_1(t, x, y, z) = U_1(x, y, z)$; then we see that u_1 is a solution to the problem

$$\frac{\partial u_1}{\partial t} = \nabla^2 u_1, \quad u_1|_{t=0} = U_1, \quad u_1|_{(0,+\infty)\times\partial B} = g,$$

since in this case $\frac{\partial u_1}{\partial t} = 0$. (Note that the initial condition is a bit silly since in fact $u_1 = U_1$ for all t; but it is certainly true nonetheless.) Since the problem we wish to solve is

$$\frac{\partial u}{\partial t} = \nabla^2 u, \quad u|_{t=0} = f, \quad u|_{(0,+\infty)\times\partial B} = g,$$

this suggests taking the other part of the solution to be the function u_2 satisfying

$$\frac{\partial u_2}{\partial t} = \nabla^2 u_2, \quad u_2|_{t=0} = f - U_1, \quad u|_{(0,+\infty)\times\partial B} = 0,$$

which we can solve using the eigenfunction methods developed earlier. Letting u_1 and u_2 be these two solutions, and taking $u = u_1 + u_2$, we see that

$$\begin{gathered}
\frac{\partial u}{\partial t} = \frac{\partial u_1}{\partial t} + \frac{\partial u_2}{\partial t} = 0 + \nabla^2 u_2 = \nabla^2 u_1 + \nabla^2 u_2 = \nabla^2 u, \\
u|_{t=0} = u_1|_{t=0} + u_2|_{t=0} = U_1 + f - U_1 = f, \\
u|_{(0,+\infty)\times\partial B} = u_1|_{(0,+\infty)\times\partial B} + u_2|_{(0,+\infty)\times\partial B} = g + 0 = g,
\end{gathered}$$

so that $u = u_1 + u_2$ is indeed a solution to our original problem, as desired.

EXAMPLE. We give a simple example of the foregoing to illustrate the procedure. Consider the following problem on B:

$$\frac{\partial u}{\partial t} = \nabla^2 u, \quad u|_{t=0} = r \sin\theta \sin\phi, \quad u|_{(0,+\infty)\times\partial B} = \cos\theta.$$

---

[2]Note that this does *not* follow from the preceding statement: consider, for example, the function $f(t) = \frac{1}{t}\sin t^3$; we have clearly $f(t) \to 0$ as $t \to +\infty$, while $f'(t) = -\frac{1}{t^2}\sin t^3 + 3t \sin t^3$, which does not converge to any limit as $t \to +\infty$.

We first solve the problem
$$\nabla^2 U_1 = 0, \quad U_1|_{\partial B} = \cos\theta;$$
now on ∂B we have $\cos\theta = z$, since $\partial B = \{(r,\theta,\phi)|r=1\}$; since z satisfies $\nabla^2 z = 0$, we see that the solution to this equation is just $U_1 = z = r\cos\theta$. Thus we are left with the problem

$$\frac{\partial u_2}{\partial t} = \nabla^2 u_2, \quad u|_{t=0} = r\sin\theta\sin\phi - r\cos\theta, \quad u|_{(0,+\infty)\times\partial B} = 0.$$

Now some reflection[3] indicates that the initial data here may be expanded as

$$\sum_{i=1}^{\infty} \frac{2}{\kappa_{1i} j_2(\kappa_{1i})} j_1(\kappa_{1i} r)(\sin\theta\sin\phi - \cos\theta)$$

(the point is that the sum above is just to expand the function r in the basis $\{j_1(\kappa_{1i} r)\}$, and hence is insensitive to which combination of $\{P_{1m}\cos m\phi, P_{1m}\sin m\phi\}$ the function r is multiplied by). Thus by standard methods (whose details we invite the reader to fill in as an exercise!) we have

$$u_2(t,x,y,z) = \sum_{i=1}^{\infty} \frac{2}{\kappa_{1i} j_2(\kappa_{1i})} e^{-\kappa_{1i}^2 t} j_1(\kappa_{1i} r)(\sin\theta\sin\phi - \cos\theta),$$

and thus we have finally the solution

$$\begin{aligned} u = u_1 + u_2 &= r\cos\theta + \sum_{i=1}^{\infty} \frac{2}{\kappa_{1i} j_2(\kappa_{1i})} e^{-\kappa_{1i}^2 t} j_1(\kappa_{1i} r)(\sin\theta\sin\phi - \cos\theta) \\ &= \sum_{i=1}^{\infty} \frac{2}{\kappa_{1i} j_2(\kappa_{1i})} j_1(\kappa_{1i} r)\left(e^{-\kappa_{1i}^2 t}\sin\theta\sin\phi + \left(1 - e^{-\kappa_{1i}^2 t}\right)\cos\theta\right). \end{aligned}$$

We note that this solution does indeed converge to the solution $u_1 = r\cos\theta$ to Laplace's equation with the given inhomogeneous boundary conditions, as claimed. We also note the nice interpolation that occurs term-by-term in the above sum between the initial data (for which the angular dependence is $\sin\theta\sin\phi$) and the final value (for which the angular dependence is $\cos\theta$).

[3]The author is reminded of a comment in the aforementioned textbook *Classical Electrodynamics* by J. D. Jackson to the effect that 'adroit use of the recurrence relation leads to ...', and of the exasperated reaction of his electrodynamics instructor upon finding this sentence: 'Oh, J. D.!' The author apologises for making a slightly similar remark here. He hopes that working out the details is somewhat more straightforward than for the corresponding result in Jackson!

APM 346, Homework 9. Due Monday, July 22, at 8.00 AM EDT. To be marked completed/not completed.

1. Using the eigenfunctions and eigenvalues for the Laplacian on the cylinder $C = \{(\rho, \phi, z) | \rho < 1, 0 \le z \le 1\}$ derived in class, solve the following problem on C:

$$\nabla^2 u = z \begin{cases} 0, & \rho < \frac{1}{2} \\ \rho^3 \cos 3\phi, & \frac{1}{2} < \rho < 1 \end{cases}, \quad u|_{\partial C} = 0.$$

Let us denote the right-hand side of Poisson's equation above by f. Then expanding

$$f = \begin{cases} 0, & \rho < \frac{1}{2} \\ \rho^3 \cos 3\phi, & \frac{1}{2} < \rho < 1 \end{cases} = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \sum_{i=1}^{\infty} J_m(\lambda_{mi}\rho) \sin n\pi z \left(a_{nmi} \cos m\phi + b_{nmi} \sin m\phi\right),$$

we see as usual that $b_{nmi} = 0$ for all n, m, and i, while $a_{nmi} = 0$ unless $m = 3$, and in that case (using the standard normalisation integrals for $J_3(\lambda_{3i}\rho)$ and $\sin n\pi z$ on $[0, 1]$)

$$\begin{aligned} a_{n3i} &= \frac{4}{J_4^2(\lambda_{3i})} \int_{\frac{1}{2}}^{1} \int_0^1 \rho^3 z J_3(\lambda_{3i}\rho) \sin n\pi z \, dz \, \rho d\rho = \frac{4}{J_4^2(\lambda_{3i})} \int_{\frac{1}{2}}^{1} \rho^4 J_3(\lambda_{3i}\rho) \, d\rho \int_0^1 z \sin n\pi z \, dz \\ &= \frac{4(-1)^{n+1}}{\lambda_{3i} J_4^2(\lambda_{3i}) n\pi} \left[J_4(\lambda_{3i}) - \frac{1}{16} J_4\left(\frac{1}{2}\lambda_{3i}\right)\right], \end{aligned}$$

whence

$$u = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \frac{4(-1)^n}{\lambda_{3i} J_4^2(\lambda_{3i}) n\pi (\lambda_{3i}^2 + n^2\pi^2)} \left(J_4(\lambda_{3i}) - \frac{1}{16} J_4\left(\frac{1}{2}\lambda_{3i}\right)\right) J_3(\lambda_{3i}\rho) \sin n\pi z \cos 3\phi.$$

2. Using the eigenfunctions and eigenvalues for the Laplacian on the unit ball $B = \{(r, \theta, \phi) | r < 1\}$ derived in class, solve the following problem on B:

$$\nabla^2 u = 3\sin^2\theta \cos 2\phi \begin{cases} r^2, & r < \frac{1}{2} \\ 0, & \frac{1}{2} < r < 1 \end{cases}, \quad u|_{\partial B} = 0.$$

Again, we expand the right-hand side:

$$\begin{cases} r^2, & r < \frac{1}{2} \\ 0, & \frac{1}{2} < r < 1 \end{cases} = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \sum_{i=1}^{\infty} j_\ell(\kappa_{\ell i} r) P_{\ell m}(\cos\theta) \left(a_{\ell m i} \cos m\phi + b_{\ell m i} \sin m\phi\right),$$

whence as before we have $b_{\ell mi} = 0$ for all ℓ, m, and i, while $a_{\ell mi} = 0$ unless $m = 2$. Now $P_{22}(\cos\theta) = 3\sin^2\theta$, so since $\{P_{\ell 2}(\cos\theta)\}_{\ell=2}^{\infty}$ is a complete orthogonal set on $[0, \pi]$, we must also have $a_{\ell 2i} = 0$ unless $\ell = 2$. Finally, denoting the above right-hand side by f,

$$\begin{aligned} a_{22i} &= \frac{(f, j_2(\kappa_{2i} r) P_{22}(\cos\theta) \cos 2\phi)}{(j_2(\kappa_{2i} r) P_{22}(\cos\theta) \cos 2\phi, j_2(\kappa_{2i} r) P_{22}(\cos\theta) \cos 2\phi)} \\ &= \frac{2}{j_3^2(\kappa_{2i})} \int_0^{\frac{1}{2}} r^4 j_2(\kappa_{2i} r) \, dr = \frac{2}{j_3^2(\kappa_{2i})} \int_0^{\frac{1}{2}} r^{\frac{7}{2}} \sqrt{\frac{\pi}{2}} J_{\frac{5}{2}}(\kappa_{2i} r) \, dr \\ &= \frac{2}{\kappa_{2i} j_3^2(\kappa_{2i})} \sqrt{\frac{\pi}{2}} \left(\frac{1}{\sqrt{128}} J_{\frac{7}{2}}\left(\frac{1}{2}\kappa_{2i}\right)\right) = \frac{1}{8\sqrt{\kappa_{2i}} j_3^2(\kappa_{2i})} j_3\left(\frac{1}{2}\kappa_{2i}\right), \end{aligned}$$

so (since the eigenvalue corresponding to $j_m(\kappa_{mi} r) P_{\ell m}(\cos\theta) \cos m\phi$ is simply $-\kappa_{mi}^2$)

$$u = \sum_{i=1}^{\infty} -\frac{1}{8 j_3^2(\kappa_{2i}) \kappa_{2i}^{\frac{5}{2}}} j_3\left(\frac{1}{2}\kappa_{2i}\right) j_2(\kappa_{2i}\rho) P_{22}(\cos\theta) \cos 2\phi.$$

3. Solve the following problem on the unit cube Q:

$$\nabla^2 u = 0, \quad u|_{x=0} = u|_{x=1} = u|_{y=0} = u|_{y=1} = 0, \quad u|_{z=0} = \sin \pi x \sin 2\pi y, \quad u|_{z=1} = 0.$$

One way of doing this (which is not really detailed enough to count as a full solution on a test!) is to note that the solution will be a linear combination of $\sin \pi x \sin 2\pi y \cosh \pi\sqrt{5}z$ and $\sin \pi x \sin 2\pi y \sinh \pi\sqrt{5}z$, after which a little thought shows that the solution is exactly

$$\sin \pi x \sin 2\pi y \left(\cosh \pi\sqrt{5}z - \coth \pi\sqrt{5} \sinh \pi\sqrt{5}z\right).$$

More systematically, we note that the solution can be written in the form

$$u = \sum_{\ell,m=1}^{\infty} \sin \ell\pi x \sin m\pi y \left(a_{\ell m} \cosh \pi\sqrt{\ell^2+m^2}z + b_{\ell m} \sinh \pi\sqrt{\ell^2+m^2}z\right);$$

then the boundary conditions give that for $(\ell, m) \neq (1, 2)$

$$a_{\ell m} = 0, \quad a_{\ell m} \cosh \pi\sqrt{\ell^2+m^2} + b_{\ell m} \sinh \pi\sqrt{\ell^2+m^2} = 0,$$

whence it is easily seen that $a_{\ell m} = b_{\ell m} = 0$ for all $(\ell, m) \neq (1, 2)$; further,

$$a_{12} = 1, \quad a_{12} \cosh \pi\sqrt{5} + b_{12} \sinh \pi\sqrt{5} = 0,$$

so $b_{12} = -\coth \pi\sqrt{5}$ and we obtain $u = \sin \pi x \sin 2\pi y \left(\cosh \pi\sqrt{5}z - \coth \pi\sqrt{5} \sinh \pi\sqrt{5}z\right)$, as claimed.

4. Recall the function χ defined in problem 1 of assignment 8:

$$\chi(x) = \begin{cases} 0, & 0 \le x < \frac{1}{2} \\ 1, & \frac{1}{2} < x \le 1 \end{cases}.$$

Let u_0 denote the solution to problem 3. Solve the following problem on the unit cube Q:

$$\frac{\partial u}{\partial t} = \nabla^2 u, \quad u|_{\partial Q} = u_0|_{\partial Q}, \quad u|_{t=0} = \chi(x)\chi(y)\chi(z).$$

[Optional: compute the coefficients in the series for u for two choices of ℓ, m, and n, one small (say $\ell = m = n = 1$) and another large (say $\ell, m, n > 10$). Compare the ratio of these coefficients for $t = 0$ and $t = 10$.]

Does the function u have a limit as $t \to +\infty$?

By what we did in class, this reduces to solving the two problems

$$\nabla^2 U_1 = 0, \quad U_1|_{\partial Q} = u_0|_{\partial Q},$$
$$\frac{\partial u_2}{\partial t} = \nabla^2 u_2, \quad u_2|_{\partial Q} = 0, \quad u_2|_{t=0} = \chi(x)\chi(y)\chi(z) - U_1;$$

now since u_0 satisfies $\nabla^2 u_0 = 0$, the first problem gives clearly $U_1 = u_0$, whence we need only to satisfy the problem

$$\frac{\partial u_2}{\partial t} = \nabla^2 u_2, \quad u_2|_{\partial Q} = 0, \quad u_2|_{t=0} = \chi(x)\chi(y)\chi(z) - u_0.$$

From problem 1 of assignment 8, we have the expansion

$$\chi(x)\chi(y)\chi(z) = \sum_{\ell,m,n=1}^{\infty} \frac{8}{\pi^3 \ell m n} \left((-1)^{\ell+1} + \cos\frac{1}{2}\ell\pi\right)\left((-1)^{m+1} + \cos\frac{1}{2}m\pi\right)\left((-1)^{n+1} + \cos\frac{1}{2}n\pi\right)$$
$$\cdot \sin \ell\pi x \sin m\pi y \sin n\pi z.$$

Now it is necessary to expand u_0 in the basis $\{\sin \ell\pi x \sin m\pi y \sin n\pi z\}$; the only tricky part of this is the expansion in the z direction. For this we note the following integral:

$$\begin{aligned}\int_0^1 e^{az} \sin n\pi z\, dz &= -\frac{\cos n\pi z}{n\pi} e^{az}\Big|_0^1 + \frac{a}{n\pi}\int_0^1 \cos n\pi z e^{az}\, dz \\ &= \frac{1}{n\pi}\left[1-(-1)^n e^a\right] + \frac{a}{n\pi}\left[\frac{\sin n\pi z}{n\pi} e^{az}\Big|_0^1 - \frac{a}{n\pi}\int_0^1 \sin n\pi z e^{az}\, dz\right]\end{aligned}$$

whence

$$\int_0^1 e^{az} \sin n\pi z\, dz = \frac{\frac{1}{n\pi}\left[1-(-1)^n e^a\right]}{1+\frac{a^2}{n^2\pi^2}} = \frac{n\pi\left[1-(-1)^n e^a\right]}{n^2\pi^2+a^2}.$$

From this we see easily that (since $\cosh x = \frac{1}{2}\left(e^x+e^{-x}\right)$ and $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$)

$$\begin{aligned}\int_0^1 \cosh \pi\sqrt{5}z \sin n\pi z\, dz &= \frac{n\pi\left[1-(-1)^n \cosh \pi\sqrt{5}\right]}{5\pi^2+n^2\pi^2}, \\ \int_0^1 \sinh \pi\sqrt{5}z \sin n\pi z\, dz &= -\frac{n\pi(-1)^n \sinh \pi\sqrt{5}}{5\pi^2+n^2\pi^2},\end{aligned}$$

and

$$\begin{aligned}\int_0^1 &\left(\cosh \pi\sqrt{5}z - \coth \pi\sqrt{5} \sinh \pi\sqrt{5}z\right)\sin n\pi z\, dz \\ &= \frac{n\pi}{5\pi^2+n^2\pi^2}\left(\left[1-(-1)^n\cosh \pi\sqrt{5}\right] + \coth\pi\sqrt{5}(-1)^n \sinh \pi\sqrt{5}\right) \\ &= \frac{n\pi}{5\pi^2+n^2\pi^2}.\end{aligned}$$

From this we see that

$$u_0 = \frac{2}{\pi}\sin \pi x \sin 2\pi y \sum_{n=1}^\infty \frac{n}{5+n^2}\sin n\pi z,$$

so that

$$\begin{aligned}\chi(x)\chi(y)\chi(z) - u_0 = \sum_{\substack{\ell,m,n=1\\(\ell,m)\neq(1,2)}}^\infty &\frac{8}{\pi^3\ell mn}\left((-1)^{\ell+1}+\cos\frac{1}{2}\ell\pi\right)\left((-1)^{m+1}+\cos\frac{1}{2}m\pi\right)\left((-1)^{n+1}+\cos\frac{1}{2}n\pi\right) \\ &\cdot \sin\ell\pi x \sin m\pi y \sin n\pi z \\ &+ \sin\pi x \sin 2\pi y \sum_{n=1}^\infty \left[\frac{8}{\pi^3 n}\left((-1)^{n+1}+\cos\frac{1}{2}n\pi\right) - \frac{2n}{\pi\left(5+n^2\right)}\right]\sin n\pi z\end{aligned}$$

and thus, by our usual method,

$$\begin{aligned}u_2 = \sum_{\substack{\ell,m,n=1\\(\ell,m)\neq(1,2)}}^\infty &\frac{8}{\pi^3\ell mn}\left((-1)^{\ell+1}+\cos\frac{1}{2}\ell\pi\right)\left((-1)^{m+1}+\cos\frac{1}{2}m\pi\right)\left((-1)^{n+1}+\cos\frac{1}{2}n\pi\right) \\ &\cdot \sin\ell\pi x \sin m\pi y \sin n\pi z e^{-\pi^2\left(\ell^2+m^2+n^2\right)t} \\ &+ \sin\pi x \sin 2\pi y \sum_{n=1}^\infty \left[\frac{8}{\pi^3 n}\left((-1)^{n+1}+\cos\frac{1}{2}n\pi\right) - \frac{2n}{\pi\left(5+n^2\right)}\right]\sin n\pi z e^{-\pi^2\left(5+n^2\right)t}\end{aligned}$$

and the solution to our original problem is

$$
\begin{aligned}
u = {} & \sin \pi x \sin 2\pi y \left(\cosh \pi\sqrt{5}z - \coth \pi\sqrt{5} \sinh \pi\sqrt{5}z\right) \\
& + \sum_{\substack{\ell,m,n=1 \\ (\ell,m)\neq(1,2)}}^{\infty} \frac{8}{\pi^3 \ell m n} \left((-1)^{\ell+1} + \cos\frac{1}{2}\ell\pi\right)\left((-1)^{m+1} + \cos\frac{1}{2}m\pi\right)\left((-1)^{n+1} + \cos\frac{1}{2}n\pi\right) \\
& \qquad \cdot \sin \ell\pi x \sin m\pi y \sin n\pi z e^{-\pi^2\left(\ell^2+m^2+n^2\right)t} \\
& + \sin \pi x \sin 2\pi y \sum_{n=1}^{\infty} \left[\frac{8}{\pi^3 n}\left((-1)^{n+1} + \cos\frac{1}{2}n\pi\right) - \frac{2n}{\pi\left(5+n^2\right)}\right] \sin n\pi z e^{-\pi^2\left(5+n^2\right)t}.
\end{aligned}
$$

Clearly, $u \to \sin \pi x \sin 2\pi y \left(\cosh \pi\sqrt{5}z - \coth \pi\sqrt{5} \sinh \pi\sqrt{5}z\right)$ as $t \to +\infty$. We leave the optional part of this exercise to the reader.

Summary:
- We clarify and review a couple points from last week.
- We then show how our work with Poisson's equation gives a series expression for the so-called *Green's function*.
- Using this, we derive other formulas relating to the Green's function, and indicate its conceptual import.
- We then introduce two new integral operations: the *Fourier transform*, which is an integral operator, and *convolution*, which is a generalised product of functions.
- We derive various properties of these operations and provide an indication of their use in solving partial differential equations.

INNER PRODUCTS FOR SPHERICAL BESSEL FUNCTIONS. In the previous week's lectures we derived the orthogonality relation

$$\int_0^1 x^2 j_\ell\left(\kappa_{\ell i} x\right) j_\ell\left(\kappa_{\ell j} x\right)\, dx = \begin{cases} 0, & i \neq j \\ \frac{1}{2} j_{\ell+1}^2\left(\kappa_{\ell i}\right), & i = j \end{cases}.$$

This indicates that when expanding functions in series of spherical Bessel functions on the interval $[0,1]$, the inner product we should use is

$$(f,g) = \int_0^1 f(x)\overline{g(x)} x^2\, dx.$$

This should be compared to the inner product

$$(f,g) = \int_0^1 f(x)\overline{g(x)} x\, dx$$

used when expanding functions in series of ordinary Bessel functions $J_m\left(\lambda_{mi} x\right)$ (there we typically used ρ instead of x).

With this inner product in the r coordinate, we noted that the full inner product used when expanding functions in series of the eigenfunctions of the Laplacian on a ball is

$$(f,g) = \int_0^1 \int_0^\pi \int_0^{2\pi} f(r,\theta,\phi)\overline{g(r,\theta,\phi)}\, d\phi \sin\theta\, d\theta\, r^2\, dr = \iiint_B f(\mathbf{x})\overline{g(\mathbf{x})}\, dV,$$

where $\mathbf{x}$ denotes an arbitrary point in three-dimensional space and V is the usual volume element in three-dimensional space. An examination of the inner products used for the various coordinates in the other coordinate systems (rectangular and cylindrical) in which we have constructed eigenfunctions for the Laplacian shows that the same formula holds; specifically, we have respectively

$$(f,g) = \int_Q f(\mathbf{x})\overline{g(\mathbf{x})}\, dV, \quad (f,g) = \int_C f(\mathbf{x})\overline{g(\mathbf{x})}\, dV$$

when expanding in rectangular and cylindrical coordinates, respectively. Thus while the inner products used in the different individual coordinates differ, the inner product on the full set is always given by integrating $f(\mathbf{x})\overline{g(\mathbf{x})}$ over the full set. While this does not add much computationally, it is helpful for remembering the individual inner products we have learned so far.

GREEN'S FUNCTIONS. Recall (see p. 10 of the lecture notes for July 2 – 4) the following manipulations. If $\{\mathbf{e}_I\}$ is a complete set of eigenfunctions for the Laplacian on a set D, say satisfying homogeneous Dirichlet boundary conditions on ∂D, with corresponding eigenvalues λ_I (which we assume to be all nonzero), then the solution to the problem

$$\nabla^2 u = f, \quad u|_{\partial D} = 0$$

has the series solution

$$u = \sum_I \frac{1}{\lambda_I} \frac{(f,\mathbf{e}_I)}{(\mathbf{e}_I,\mathbf{e}_I)} \mathbf{e}_I.$$

Here I represents an abstract index which may contain multiple separate indices; e.g., in the case of the eigenfunctions on the unit ball I will represent the triple (ℓ, m, i). For simplicity, let us assume that $(\mathbf{e}_I, \mathbf{e}_I) = 1$; this can always be achieved by rescaling the eigenfunctions $\mathbf{e}_I$ if necessary. Then the above formula can be expanded as follows:

$$\begin{aligned} u(\mathbf{x}) &= \sum_I \frac{1}{\lambda_I}(f, \mathbf{e}_I)\mathbf{e}_I(\mathbf{x}) \\ &= \sum_I \frac{1}{\lambda_I} \int_D f(\mathbf{x}')\overline{\mathbf{e}_I(\mathbf{x}')}\, d\mathbf{x}' \mathbf{e}_I(\mathbf{x}) \\ &= \int_D \left(\sum_I \frac{\mathbf{e}_I(\mathbf{x})\overline{\mathbf{e}_I(\mathbf{x}')}}{\lambda_I} \right) f(\mathbf{x}')\, d\mathbf{x}', \end{aligned} \tag{1}$$

where we assume that the sum is such that we may interchange sum and integral. The (negative[1] of the) function in parentheses above is called the *Green's function* for the problem. We denote it by $G(\mathbf{x}, \mathbf{x}')$, noting that both $\mathbf{x}$ and $\mathbf{x}'$ are actually points in D, hence (at least for $D \subset \mathbf{R}^3$) in $\mathbf{R}^3$. Note that, since G is expressed in terms of the eigenfunctions, it depends in principle upon everything that they depend on, namely (1) the operator (the Laplacian); (2) the region D; (3) the boundary conditions (here, homogeneous Dirichlet). If any of these change, the Green's function will in principle change also.

If the above were all there were to the Green's function, it would not be clear why the above formula is useful: it is not clear that the series expansion above should be summable to anything simpler, and if it isn't then the only real use for the above formula would be to run the above derivation backwards to obtain the series expansion for u from which we started. It turns out, though, that the notion of a Green's function can be discussed profitably independent of any expansion in eigenfunctions, and to this we now turn.

(Before doing this, it is probably helpful to say a few words about the general direction of the course moving forwards. So far we have been focussed almost exclusively on obtaining expansions in complete orthogonal sets which provide formal solutions to our problems. Going forwards, what we shall do gives more directly *integral* representations (rather than series expansions) for the solutions to our problems. Philosophically, though, the two parts are not all that different: in both cases we are seeking representation formulas for solutions.)

Let us return to equation (1) above. First, since for us the eigenfunctions $\mathbf{e}_I$ are all real, we see that we may drop the complex conjugate on $\mathbf{e}_I(\mathbf{x}')$, meaning that we have the series expansion

$$G(\mathbf{x}, \mathbf{x}') = -\sum_I \frac{\mathbf{e}_I(\mathbf{x})\mathbf{e}_I(\mathbf{x}')}{\lambda_I},$$

from which we see easily that $G(\mathbf{x}, \mathbf{x}') = G(\mathbf{x}', \mathbf{x})$, i.e., the Green's function is *symmetric* in its arguments. (This will be important below.) Now if we formally take the Laplacian of G with respect to $\mathbf{x}$, we obtain (denoting this by $\nabla^2_{\mathbf{x}}$, and keeping the complex conjugate since the symmetry of G is not important here)

$$\begin{aligned} \nabla^2_{\mathbf{x}} G(\mathbf{x}, \mathbf{x}') &= -\sum_I \frac{1}{\lambda_I} \nabla^2_{\mathbf{x}} \mathbf{e}_I(\mathbf{x})\overline{\mathbf{e}_I(\mathbf{x}')} = -\sum_I \frac{1}{\lambda_I} \lambda_I \mathbf{e}_I(\mathbf{x})\overline{\mathbf{e}_I(\mathbf{x}')} \\ &= -\sum_I \mathbf{e}_I(\mathbf{x})\overline{\mathbf{e}_I(\mathbf{x}')}. \end{aligned}$$

Now this last sum generally does not converge in any usual sense; however, we can make some sense out of it by integrating it against a sufficiently smooth function f and then proceeding formally:

$$\begin{aligned} \int_D \left(\sum_I \mathbf{e}_I(\mathbf{x})\overline{\mathbf{e}_I(\mathbf{x}')} \right) f(\mathbf{x}')\, d\mathbf{x}' &= \sum_I \mathbf{e}_I(\mathbf{x}) \int_D f(\mathbf{x}')\overline{\mathbf{e}_I(\mathbf{x}')}\, d\mathbf{x}' \\ &= \sum_I (f, \mathbf{e}_I)\mathbf{e}_I(\mathbf{x}) = f(\mathbf{x}), \end{aligned}$$

[1]This negative sign – which was not used in the lecture – seems to be standard, for some reason, but is also extremely annoying from our perspective. We shall, however, include it for ease of reference both to the textbook and to other external sources.

since $\{\mathbf{e}_I\}$ is a complete orthonormal set by assumption. This shows that, at least formally, $\sum_I \mathbf{e}_I(\mathbf{x})\overline{\mathbf{e}_I(\mathbf{x}')} = \delta(\mathbf{x}-\mathbf{x}')$, where $\delta(\mathbf{x})$ is the celebrated *Dirac delta function*, beloved of physicists probably long before it was understood by mathematicians.[2] Heuristically, this is described as a 'function' possessing the following two properties:

1. $\delta(\mathbf{x}) = 0$ unless $\mathbf{x} = 0$;
2. $\displaystyle\int_{\mathbf{R}^3} f(\mathbf{x})\delta(\mathbf{x})\,d\mathbf{x} = f(0)$ for all functions f.

Putting these two conditions together, we see that in effect $\delta(\mathbf{x})$ is zero everywhere except at the origin, where it has an infinitely high peak. We note that the second property implies in particular that $\int_{\mathbf{R}^3} \delta(\mathbf{x})\,d\mathbf{x} = 1$, and also that for any (suitable[3]) f

$$\int_{\mathbf{R}^3} f(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')\,d\mathbf{x}' = f(\mathbf{x});$$

this may be seen by doing a simple substitution with $\mathbf{u} = \mathbf{x} - \mathbf{x}'$. While there is no actual function which satisfies the above two properties, there are many sequences of functions which satisfy them *in the limit*, in the following sense.

DEFINITION. A sequence of functions $\{\delta_n\}$ on some $\mathbf{R}^m$ is said to *have the properties of a delta function in the limit*, or to be an *approximate identity*,[4] if the following two properties hold:

1. $\displaystyle\int_{\mathbf{R}^m} \delta_n(\mathbf{x})\,d\mathbf{x} = 1$ for all n;
2. $\displaystyle\lim_{n\to\infty}\int_{\mathbf{R}^m} f(\mathbf{x})\delta_n(\mathbf{x})\,d\mathbf{x} = f(0)$ for all suitable functions $f(\mathbf{x})$

(see our commentary about suitable functionsn in the footnotes).

The point of this definition is that, while the sequence $\{\delta_n\}$ itself need not have a limit in any normal sense, the functionals (linear maps to $\mathbf{R}$) it induces on spaces of functions have the delta function as a 'limit';[5] more intuitively, while it doesn't make any sense to put the limit above inside the integral, everything works well if we keep it outside the integral. One could in fact probably work out most of the theory using sequences which are approximate identities without mentioning the delta function itself at all; but we shall prefer to take the delta function as something which exists by itself, and only use approximate identities for cases of illustration and in dealing with fine points.

We shall now give several examples of approximate identities, beginning with a simple one to illustrate the idea and then proceeding to more complicated ones which shall be useful in our future work with the

[2]Mathematicians, take note! Just because a physicist fails to give an object a precise mathematical formulation does not mean that one doesn't exist. I heard a quote from a well-known mathematician (I have unfortunately forgotten who) to the effect that, If physicists have been using something consistently for years, mathematicians ought to study how it works (or something like that, I don't remember exactly how the second half went).

[3]We shall be vague about what is meant by 'suitable'. In the mathematically rigorous formulation of delta functions, one actually restricts f to be C^∞, and generally either of compact support – meaning that it vanishes outside of a bounded set – or of 'rapid decrease' (falling to zero faster than any polynomial function) at infinity. It is sometimes appropriate to work with more general functions, though; and the functions which we shall say *have the properties of a delta function in the limit* (see the definition immediately following) satisfy this property for fairly general classes of functions, cf. [1], Theorem 8.15.

[4]This terminology comes from the equation just given, which is seen – cf. the definition of convolution below – to show that the delta function is an identity for the convolution operation. See [1], Theorem 8.15, and ensuing commentary.

[5]Those interested in seeing how to make this notion of limit rigorous may see the appendix [to be added soon], but be warned that it assumes a fairly detailed understanding of point-set toplogy and a high level of mathematical maturity.

Fourier transform. We shall begin by considering delta functions on $\mathbf{R}^1$ and then use these to construct them on $\mathbf{R}^3$.

(Before beginning the examples, we mention the one example we have constructed so far of something which 'looks like a delta function in the limit', namely $\sum_I \mathbf{e}_I(\mathbf{x})\overline{\mathbf{e}_I(\mathbf{x}')}$ (this is a limit since it is an infinite sum). The above calculation can be written more carefully as follows (using the notation $I \to \infty$ to indicate that all of the indices in I go to infinity; convergence of multiply-indexed sequences is a tricky business and we elide the details here):

$$\lim_{I\to\infty}\int_D \sum_J^I \mathbf{e}_J(\mathbf{x})\overline{\mathbf{e}_J(\mathbf{x}')}f(\mathbf{x}')d\mathbf{x}' = \lim_{I\to\infty}\sum_J^I \mathbf{e}_J(\mathbf{x})\int_D \overline{\mathbf{e}_J(\mathbf{x}')}f(\mathbf{x}')d\mathbf{x}' = \lim_{I\to\infty}\sum_J^I (\mathbf{e}_J, f)\mathbf{e}_J(\mathbf{x}),$$

and we know that this limit equals $f(\mathbf{x})$ if we take it to be in the L^2 sense. Thus, while our discussion of complete orthogonal sets is insufficient to conclude that $\sum_I \mathbf{e}_I(\mathbf{x})\overline{\mathbf{e}_I(\mathbf{x}')}$ is an approximate identity in the sense just given (or more precisely, that it would be if we replaced $\mathbf{x}$ by 0), it is somehow one in an L^2 sense (whatever that means[6]). Generally speaking, series such as the foregoing converge pointwisely at points of continuity of f, so that if we restrict to continuous functions then the sum should be an approximate identity. Perhaps the take-home lesson here is that, at least for our purposes, general statements of when a certain sequence is or is not an approximate identity are probably less important than understanding the general idea and specific cases.)

EXAMPLES. (a) For each $n \in \mathbf{Z}$, $n > 0$, define a function $\chi_n : \mathbf{R}^1 \to \mathbf{R}^1$ as follows:

$$\chi_n(x) = \begin{cases} \frac{n}{2}, & x \in [-\frac{1}{n}, \frac{1}{n}] \\ 0, & \text{otherwise} \end{cases}.$$

Then we see that $\int_{\mathbf{R}^1} \chi_n(x)\,dx = 1$ for all n, and that $\chi_n(x) \to 0$ for all $x \neq 0$ as $n \to \infty$ (to see this, let $x \in \mathbf{R}$, $x \neq 0$, and let $N \in \mathbf{Z}$, $N > \frac{1}{|x|}$; then for all $n > N$ we have $\frac{1}{n} < |x|$, so $x \notin [-\frac{1}{n}, \frac{1}{n}]$ and $\chi_n(x) = 0$). Now let $f : \mathbf{R}^1 \to \mathbf{R}^1$ be continuous at $x = 0$. We claim that

$$\int_{\mathbf{R}^1} f(x)\chi_n(x)\,dx \to f(0) \quad \text{as} \quad n \to \infty.$$

To prove this, we first rewrite the left-hand side as follows (recall that $\int_{\mathbf{R}^1} \chi_n(x)\,dx = 1$ for all n):

$$\int_{\mathbf{R}^1} f(x)\chi_n(x)\,dx = \int_{\mathbf{R}^1} (f(x) - f(0) + f(0))\chi_n(x)\,dx = \int_{\mathbf{R}^1} (f(x) - f(0))\chi_n(x)\,dx + f(0).$$

Now let $\epsilon > 0$, let $\delta > 0$ be such that $|f(x) - f(0)| < \epsilon$ when $|x| < \delta$, and let $N \in \mathbf{Z}$ be such that $\frac{1}{N} < \delta$. Then for all $n > N$, we have $\frac{1}{n} < \delta$, so for such n

$$\int_{\mathbf{R}^1} f(x)\chi_n(x)\,dx = f(0) + \int_{\mathbf{R}^1} (f(x) - f(0))\chi_n(x)\,dx = f(0) + \int_{-\frac{1}{n}}^{\frac{1}{n}} (f(x) - f(0))\frac{n}{2}\,dx,$$

and

$$\left|\int_{\mathbf{R}^1} f(x)\chi_n(x)\,dx - f(0)\right| \leq \frac{n}{2}\int_{-\frac{1}{n}}^{\frac{1}{n}} |f(x) - f(0)|\,dx < \frac{n}{2}\frac{2\epsilon}{n} = \epsilon,$$

which shows that $\int_{\mathbf{R}^1} f(x)\chi_n(x)\,dx \to f(0)$, as claimed. While $\lim \chi_n$ does not exist in any normal sense, we see that, in some sense, the sequence χ_n has the properties of the delta function in the limit as $n \to \infty$.

(Intuitively, the idea behind the above ϵ-δ proof is as follows. Since f is continuous at $x = 0$, $f(x) - f(0)$ will be small if x is close to zero. Now if n is large, then $\chi_n(x)$ is zero unless $|x| \leq \frac{1}{n}$; thus $\chi_n(x)$ will be zero

[6]The author suspects that someone more talented than he has already made this precise, but even if that is the case he is not aware of it; for the which ignorance, he apologises.

unless x is small, in the which case $f(x)-f(0)$ will also be small. This suggests that $\int_{\mathbf{R}^1}(f(x)-f(0))\chi_n(x)\,dx$ will be small in this case. Unfortunately, while $f(x)-f(0)$ is small, $\chi_n(x)$ will be large (since $\chi_n(0)\to\infty$ as $n\to\infty$), so this does not follow immediately; but since the interval is also getting small, it turns out – and the calculations above prove rigorously – that in fact this integral is small, as desired. Just for the sake of thouroughness, we give another proof (which in rigour is halfway between the full proof and the intuitive description just given) in the special case where f is continuous on some interval containing 0: in this case, by taking n large enough we may assume that f is continuous on $[\frac{1}{n},\frac{1}{n}]$. Thus $|f(x)-f(0)|$ must also be continuous on this interval, and hence must have a maximum there, call it M_n. Then we may write

$$\int_{\mathbf{R}^1}|f(x)-f(0)|\chi_n(x)\,dx = \int_{-\frac{1}{n}}^{\frac{1}{n}}|f(x)-f(0)|\frac{n}{2}\,dx \le M_n\frac{n}{2}\frac{2}{n} = M_n.$$

But since f is continuous at 0, the quantity M_n must become small as $n\to\infty$, so that the integral $\int_{\mathbf{R}^1}(f(x)-f(0))\chi_n(x)\,dx$ must become small as well, as claimed.)

(b) Again, for each $n\in\mathbf{Z}$, $n>0$, define a function $\phi_n:\mathbf{R}^1\to\mathbf{R}^1$ by

$$\phi_n(x)=\sqrt{n}\pi e^{-nx^2}.$$

We recall the Gaussian integral: for any $a>0$,

$$\int_{\mathbf{R}^1} e^{-ax^2}\,dx=\sqrt{\frac{\pi}{a}};$$

thus we have (as for χ_n) $\int_{-\infty}^{\infty}\phi_n(x)\,dx=1$ for all n. Moreover, $\phi_n(x)\to 0$ for all $x\neq 0$ as $n\to\infty$, as for χ_n. We claim again that for any function f which is continuous at 0 and (in this case) bounded on $\mathbf{R}^1$

$$\int_{\mathbf{R}^1} f(x)\phi_n(x)\,dx\to f(0)\quad\text{as}\quad n\to\infty.$$

The intuition is very similar to that in the previous proof (note that ϕ_n, like χ_n, becomes infinitely sharply peaked at 0 in the limit as $n\to\infty$) and we give only the ϵ-δ proof. Thus let $M=\sup_{x\in\mathbf{R}^1}|f(x)|+1$, $\epsilon>0$, let $\delta>0$ be such that $|f(x)-f(0)|<\epsilon$ for $|x|<\delta$, and let $K\in\mathbf{Z}$, $K>0$ be such that

$$\left|1-\frac{1}{\sqrt{\pi}}\int_{-K}^{K}e^{-x^2}\,dx\right|<\frac{\epsilon}{2M}$$

(such a K certainly exists since $\int_{-\infty}^{+\infty}e^{-x^2}\,dx=\sqrt{\pi}$). Let $N\in\mathbf{Z}$, $N>0$ be such that $\delta\sqrt{N}>K$. Then for $n>N$ we have, doing a change of variables with $u=x\sqrt{n}$,

$$\int_{-\delta}^{\delta}\phi_n(x)\,dx=\sqrt{\frac{n}{\pi}}\int_{-\delta}^{\delta}e^{-nx^2}\,dx=\frac{1}{\sqrt{pi}}\int_{-\delta\sqrt{n}}^{\delta\sqrt{n}}e^{-u^2}\,du,$$

from which we see that

$$\left|\int_{\mathbf{R}^1\setminus[-\delta,\delta]}\phi_n(x)\,dx\right|=\left|1-\int_{-\delta}^{\delta}\phi_n(x)\,dx\right|<\frac{\epsilon}{2M}$$

and moreover that

$$\begin{aligned}\left|f(0)-\int_{-\infty}^{\infty}f(x)\phi_n(x)\,dx\right| &\le \int_{\mathbf{R}^1}|f(x)-f(0)|\phi_n(x)\,dx\\ &=\int_{\mathbf{R}^1\setminus[-\delta,\delta]}|f(x)-f(0)|\phi_n(x)\,dx+\int_{-\delta}^{\delta}|f(x)-f(0)|\phi_n(x)\,dx\\ &\le 2M\int_{\mathbf{R}^1\setminus[-\delta,\delta]}\phi_n(x)\,dx+\epsilon\int_{-\delta}^{\delta}\phi_n(x)\,dx\\ &<\epsilon+\epsilon=2\epsilon,\end{aligned}$$

which shows that $\int_{\mathbf{R}^1} f(x)\phi_n(x)\,dx \to f(0)$ as $n \to \infty$, as desired.[7]

(c) [This example can be skipped at a first reading; in that case, replace ψ_n with ϕ_n in (d) below, and references to (c) with references to (b). For a similar but more careful and general result, see the aforementioned Theorem 8.15 of [1].] Given the foregoing, we are now ready to posit the following general result: suppose that $\psi : \mathbf{R}^1 \to \mathbf{R}^1$ satisfies $\int_{\mathbf{R}^1} \psi(x)\,dx = 1$, $\int_{\mathbf{R}^1} |\psi(x)|\,dx < \infty$ (this latter means that ψ is *in* L^1), and for all $n \in \mathbf{Z}$, $n > 0$ define

$$\psi_n(x) = n\psi(nx).$$

(χ_n in example (a) is certainly of this form; in example (b), we have basically this same form except that we scale by $\sqrt{n}$ instead of n.) Assume now that $\psi_n(x) \to 0$ as $n \to \infty$.[8] Then we claim that for any $f : \mathbf{R}^1 \to \mathbf{R}^1$ which is continuous at $x = 0$ and bounded on $\mathbf{R}^1$,

$$\int_{\mathbf{R}^1} f(x)\psi_n(x)\,dx \to f(0) \quad \text{and} \quad n \to \infty.$$

To see this, we first note that for any n, using a change of variables $u = nx$,

$$\int_{\mathbf{R}^1} \psi_n(x)\,dx = \int_{-\infty}^{\infty} n\psi(nx)\,dx = \int_{-\infty}^{\infty} \psi(u)\,du = 1.$$

Now let $M = \sup_{x\in\mathbf{R}^1} |f(x)| + 1$, let $\epsilon > 0$, let $\delta > 0$ be such that $|f(x) - f(0)| < \epsilon$ when $|x| < \delta$, and let $K \in \mathbf{Z}$, $K > 0$ be such that

$$\int_{\mathbf{R}^1\setminus[-K,K]} |\psi(x)|\,dx < \frac{\epsilon}{2M};$$

such a K clearly exists since $\int_{-\infty}^{\infty} |\psi(x)|\,dx < \infty$. Now choose $N \in \mathbf{Z}$, $N > 0$, such that $N\delta > K$, and let $n > N$. Then we have

$$\begin{aligned}\left|\int_{\mathbf{R}^1\setminus[-\delta,\delta]} \psi_n(x)\,dx\right| &\le \int_{\mathbf{R}^1\setminus[-\delta,\delta]} |\psi_n(x)|\,dx \\ &= \int_{\mathbf{R}^1\setminus[-n\delta,n\delta]} |\psi(u)|\,du < \frac{\epsilon}{2M},\end{aligned}$$

so

$$\begin{aligned}\left|f(0) - \int_{-\infty}^{\infty} f(x)\psi_n(x)\,dx\right| &\le \int_{\mathbf{R}^1} |f(x) - f(0)||\psi_n(x)|\,dx \\ &= \int_{\mathbf{R}^1\setminus[-\delta,\delta]} |f(x) - f(0)||\psi_n(x)|\,dx + \int_{-\delta}^{\delta} |f(x) - f(0)||\psi_n(x)|\,dx \\ &\le 2M \int_{\mathbf{R}^1\setminus[-\delta,\delta]} |\psi_n(x)|\,dx + \epsilon \int_{-\infty}^{\infty} |\psi_n(x)|\,dx < 2\epsilon,\end{aligned}$$

and $\int_{\mathbf{R}^1} f(x)\psi_n(x)\,dx \to f(0)$ as $n \to \infty$, as claimed.

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[7]Note that we did not really need f to be bounded; we just needed f to be such that the tails

$$\int_{\mathbf{R}^1\setminus[-\delta,\delta]} f(x)\phi_n(x)\,dx$$

would go to zero as $n \to \infty$. Since $\phi_n(x)$ goes to zero like e^{-nx^2}, it is sufficient, for example, that f go to infinity no more than exponentially fast as $x \to \pm\infty$. This material is interesting, but we pass over it for now.

[8]It is possible to give sufficient conditions for this to hold: for example, suppose that there were $M > 0$, $\alpha > 1$ such that $|\psi(x)| < \frac{M}{x^\alpha}$ for all x (or even for all x sufficiently large); then clearly $\psi_n(x) < n\frac{M}{(nx)^\alpha} = \frac{M}{x}n^{1-\alpha} \to 0$ as $n \to \infty$, since $\alpha > 1$ – this is the condition used in [1], Theorem 8.15. Other conditions could presumably also be found.

(d) We would now like to construct a delta function on $\mathbf{R}^3$. We may construct it as the improper limit of a sequence like those just given in the following way. Let ψ be any function satisfying the requirements given in (c), and define

$$\Psi_{\ell mn}(x,y,z) = \psi_\ell(x)\psi_m(y)\psi_n(z).$$

(Using three different indices is just for convenience in making the calculation below simpler; it could probably be done without it, at least for f uniformly continuous on some neighborhood of the origin.) Then if $f : \mathbf{R}^3 \to \mathbf{R}^1$ is any function which is bounded on $\mathbf{R}^3$ and continuous at $x = 0$, we have

$$\int_{\mathbf{R}^3} f(\mathbf{x})\Psi_{\ell mn}(\mathbf{x})\,d\mathbf{x} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\psi_\ell(x)\psi_m(y)\psi_n(z)\,dx\,dy\,dz.$$

Now the hypotheses allow us to interchange the limit on n with the integral over z;[9] taking this limit, the above integral becomes

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,0)\psi_\ell(x)\psi_m(y)\,dx\,dy.$$

We may then similarly take the limit on m, and finally on ℓ, to see that

$$\int_{\mathbf{R}^3} f(\mathbf{x})\Psi_{\ell mn}(\mathbf{x})\,d\mathbf{x} \to f(0,0,0) \quad \text{as} \quad \ell, m, n \to \infty.$$

From this it can be shown that the sequence

$$\hat{\Psi}_n(\mathbf{x}) = \Psi_{nnn}(\mathbf{x})$$

behaves also like a delta function in the limit as $n \to \infty$. In general, then, if we have a sequence in $\mathbf{R}^1$ of the form given in (c) which behaves like a delta function in the limit, then we may obtain a sequence in $\mathbf{R}^n$ (for any n) which behaves like a delta function in the limit by taking a product of n copies of the sequence in $\mathbf{R}^1$, one in each variable separately. We write this symbolically in $\mathbf{R}^3$ as

$$\delta(\mathbf{x}) = \delta(x)\delta(y)\delta(z)$$

(one tends to write δ for the delta function in any dimension, and even for multiple different dimensions in one single equation as above, without regards to niceties of notation!).

(e) Finally, for use in a moment we would like to find an expression for the delta function in $\mathbf{R}^3$ which is adapted to spherical coordinates in the way the expressions in (d) are adapted to rectangular coordinates. We shall apply the method of (d) to the function from (b). We have

$$\phi_n(x) = \sqrt{\frac{n}{\pi}}e^{-nx^2},$$

so

$$\Phi_n(\mathbf{x}) = \left(\frac{n}{\pi}\right)^{\frac{3}{2}} e^{-n(x^2+y^2+z^2)}$$

will behave like a delta function in the limit as $n \to \infty$. In spherical coordinates, this can be written as

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{+\infty} f(r,\theta,\phi)\left(\frac{n}{\pi}\right)^{\frac{3}{2}} e^{-nr^2} r^2 \sin\theta\,dr\,d\theta\,d\phi;$$

if we extend f to be even in r on $\mathbf{R}^1$, then this may be rewritten as

$$\frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_{-\infty}^{\infty} f(r,\theta,\phi)\left(\frac{n}{\pi}\right)^{\frac{3}{2}} e^{-nr^2} r^2 \sin\theta\,dr\,d\theta\,d\phi$$
$$= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_{-\infty}^{\infty} f(r,\theta,\phi) 2\sqrt{\frac{n}{\pi}}e^{-nr^2}\frac{nr^2}{2\pi r^2} r^2 \sin\theta\,dr\,d\theta\,d\phi.$$

[9]One might need the dominated convergence theorem and Lebesgue integration. The deponent verb may have all the forms of the gerund.

Now we note that

$$\int_{-\infty}^{\infty} x^2 e^{-x^2}\,dx = -\frac{d}{da}\int_{-\infty}^{\infty} e^{-ax^2}\,dx\Big|_{a=1} = -\frac{d}{da}\sqrt{\frac{\pi}{a}}\Big|_{a=1} = \frac{1}{2}\sqrt{\pi}$$

thus the function $\psi = \frac{2}{\sqrt{\pi}}x^2e^{-x^2}$ is of the form covered by (c), so that (scaling by $\sqrt{n}$ instead of n, as in (b)) the sequence $\psi_n(x) = 2\sqrt{\frac{n}{\pi}}nx^2e^{-nx^2}$ will behave like a delta function in the limit $n \to \infty$. But the above integral is exactly

$$\int_0^{2\pi}\int_0^{\pi}\int_{-\infty}^{\infty} f(r,\theta,\phi)\frac{1}{4\pi r^2}\psi_n(r)r^2\,dr\sin\theta\,d\theta\,d\phi.$$

Thus we identify the three-dimensional delta function $\delta(\mathbf{x})$ with the function

$$\frac{1}{4\pi r^2}\delta(r)$$

in spherical coordinates.[10]

With our understanding of delta functions (hopefully!) increased by these examples, we return to our study of Green's functions. We recall that we have derived the relation (which holds in some sort of L^2 sense, perhaps not in the precise sense of our definition of approximate identity above)

$$\nabla^2_{\mathbf{x}}G(\mathbf{x},\mathbf{x}') = -\delta(\mathbf{x}-\mathbf{x}').$$

For the rest of our work with Green's functions, we shall take this equation (and not the expansion in eigenfunctions) as the starting point; in other words, for us a *Green's function* will be any function on $\mathbf{R}^m$ (generally we have $m = 3$) which satisfies the above equation, in the sense that for any suitable function f

$$\int_{\mathbf{R}^m} -\left(\nabla^2_{\mathbf{x}}G(\mathbf{x},\mathbf{x}')\right)f(\mathbf{x}')\,d\mathbf{x}' = f(\mathbf{x}).$$

We shall show later how to take boundary conditions into account. We show how it may be used to prove the above representation formula for solutions to Poisson's equation. Suppose that $f : \mathbf{R}^3 \to \mathbf{R}$ is a function such that $\int_{\mathbf{R}^3} G(\mathbf{x},\mathbf{x}')f(\mathbf{x}')\,d\mathbf{x}'$ exists for all $\mathbf{x} \in \mathbf{R}^3$; then, assuming that we may interchange integration and differentiation, we have

$$\nabla^2_{\mathbf{x}}\int_{\mathbf{R}^3} G(\mathbf{x},\mathbf{x}')f(\mathbf{x}')\,d\mathbf{x}' = \int_{\mathbf{R}^3}\nabla^2_{\mathbf{x}}G(\mathbf{x},\mathbf{x}')f(\mathbf{x}')\,d\mathbf{x}' = \int_{\mathbf{R}^3}\delta(\mathbf{x}-\mathbf{x}')f(\mathbf{x}')\,d\mathbf{x}' = -f(\mathbf{x})$$

[10]There are several objections which could legitimately be brought up to this derivation and formula. One of them involves the fact that $\delta(r) = 0$ unless $r = 0$, which means that we should be able to replace r by 0 in the above expression: but this would involve dividing by 0, which is meaningless. This can be answered by noting that delta functions *proper* (as opposed to the sequences we have been constructing which lead to them) only really exist when appearing under integral signs, and the one here only exists when appearing under a three-dimensional integral; in such a case, there will always be the factor r^2 from the volume element in spherical coordinates to cancel the r^2 in the denominator here. Another, more subtle, objection is that the final integral we just derived involved integrating over r from $-\infty$ to ∞, whereas the integral we started with only involved an integral from 0 to $+\infty$; thus we seem to have counted things twice. This can be answered (though not, I admit, entirely resolved; to entirely resolve either of these objections we would probably have to work in a much more rigorous setting) by the following observation: note that the normalisation we used for ψ was also obtained by integrating from $-\infty$ to ∞; thus we have effectively divided by an extra factor of 2, which should cancel the problematic one. It should perhaps also be pointed out that the delta function requires integrating over an interval containing zero, not just on half of such an interval, so that the integral from 0 to ∞ might be said, in some sense, to collect only half of the delta function (though I do not think this can be made precise in any real sense).

This shows that the function

$$u = -\int_{\mathbf{R}^3} G(\mathbf{x}, \mathbf{x}')f(\mathbf{x}')\, d\mathbf{x}' \tag{1}$$

satisfies Poisson's equation on $\mathbf{R}^3$. (See Theorem 8.3 of the textbook for a more careful treatment by another method of this result; but note that the textbook's definition of a Green's function interchanges $\mathbf{x}$ and $\mathbf{x}'$ compared with ours – symmetry of the Green's function (see below) can be used to turn the result back into something closer to what we have here.)

It might be helpful to give some intuitive content to this relation. Consider Poisson's equation. The Green's function $G(\mathbf{x}, \mathbf{x}')$ gives the solution at the point $\mathbf{x}$ due to a (negative) unit (i.e., negative delta function) source at the point $\mathbf{x}'$. (In physical terms, if we are solving for the electrostatic potential, so that the right-hand side of Poisson's equation is essentially the charge density, then the Green's function gives the electrostatic potential at $\mathbf{x}$ due to a point charge of unit size at $\mathbf{x}'$; if we are solving for the steady-state temperature distribution in a body with internal sources, then the Green's function gives the temperature at a point $\mathbf{x}$ due to a single point source of unit strength at $\mathbf{x}'$; and so on.) Since Poisson's equation is *linear*, we expect that the solution for a sum of such sources, say at the points $\mathbf{x}_i'$ should be the sum $-\sum_i f_i G(\mathbf{x}, \mathbf{x}_i')$, where f_i represents the size of the source at $\mathbf{x}_i'$. Now if we are given a continuous source, then it makes sense[11] that the sum should become an integral, and that the whole solution should be

$$-\int_{\mathbf{R}^3} G(\mathbf{x}, \mathbf{x}')f(\mathbf{x}')\, d\mathbf{x}',$$

exactly as we showed just now. In other words, the representation formula for the solution to Poisson's equation using the Green's function is just what is obtained by taking a superposition of solutions due to individual point sources.

To go back to our more formal investigations for a moment, if we look at the relation

$$\nabla^2_{\mathbf{x}} \int_{\mathbf{R}^3} G(\mathbf{x}, \mathbf{x}')f(\mathbf{x}')\, d\mathbf{x}' = -f(\mathbf{x})$$

more carefully, we see that it means that the operator

$$f(\mathbf{x}) \mapsto -\int_{\mathbf{R}^3} G(\mathbf{x}, \mathbf{x}')f(\mathbf{x}')\, d\mathbf{x}'$$

is a left inverse to the Laplacian, in the sense that, denoting the map by $\mathcal{G}$, we have $\nabla^2 \mathcal{G}[f] = f$ for all suitable functions f. If we assume that G is symmetric (for the Green's function on all of $\mathbf{R}^3$, this will follow from the calculation we give in a moment; for the Green's function on a bounded region, this was already noted as following from the expansion in eigenfunctions, or see Theorem 8.4 in the textbook), then we can show that $\mathcal{G}$ is also a right inverse:

$$\begin{aligned}
-\int_{\mathbf{R}^3} G(\mathbf{x}, \mathbf{x}')\nabla^2_{\mathbf{x}'} f(\mathbf{x}')\, d\mathbf{x}' &= -\int_{\mathbf{R}^3} \nabla_{\mathbf{x}'} \cdot (G(\mathbf{x}, \mathbf{x}')\nabla_{\mathbf{x}'} f(\mathbf{x}')) - \nabla_{\mathbf{x}'} G(\mathbf{x}, \mathbf{x}') \cdot \nabla_{\mathbf{x}'} f(\mathbf{x}')\, d\mathbf{x}' \\
&= \int_{\mathbf{R}^3} \nabla_{\mathbf{x}'} \cdot (\nabla_{\mathbf{x}'} G(\mathbf{x}, \mathbf{x}') f(\mathbf{x}')) - \nabla^2_{\mathbf{x}'} G(\mathbf{x}, \mathbf{x}') f(\mathbf{x}')\, d\mathbf{x}' \\
&= -\int_{\mathbf{R}^3} \nabla^2_{\mathbf{x}'} G(\mathbf{x}, \mathbf{x}') f(\mathbf{x}')\, d\mathbf{x}' \\
&= -\int_{\mathbf{R}^3} \nabla^2_{\mathbf{x}'} G(\mathbf{x}', \mathbf{x}) f(\mathbf{x}')\, d\mathbf{x}' = \int_{\mathbf{R}^3} \delta(\mathbf{x}' - \mathbf{x}) f(\mathbf{x}')\, d\mathbf{x}' = f(\mathbf{x}), \qquad (1')
\end{aligned}$$

where we have used the fact that the delta function is even, and also that $G(\mathbf{x}, \mathbf{x}') \to 0$ as $\mathbf{x}$ or $\mathbf{x}' \to \infty$.

GREEN'S FUNCTION ON $\mathbf{R}^3$. The Green's function on $\mathbf{R}^3$ can most conveniently be computed by using the expression for the delta function derived in Example (d) above. We have the equation

$$\nabla^2_{\mathbf{x}} G(\mathbf{x}, \mathbf{x}') = -\delta(\mathbf{x} - \mathbf{x}').$$

[11] One could derive this in more detail, but we shall pass over it here because of space and time considerations.

Now in this equation $\mathbf{x}'$ is considered to be a fixed parameter. We now introduce spherical coordinates on $\mathbf{R}^3$ centred at $\mathbf{x}'$; i.e., let (r, θ, ϕ) of a point $\mathbf{x}$ satisfy (writing $\mathbf{x} = (x, y, z)$, $\mathbf{x}' = (x', y', z')$)

$$r\sin\theta\cos\phi = x - x', \quad r\sin\theta\sin\phi = y - y', \quad r\cos\theta = z - z'.$$

Then the delta function $\delta(\mathbf{x} - \mathbf{x}')$ can be written, by Example (d) above, as

$$\delta(\mathbf{x} - \mathbf{x}') = \frac{\delta(r)}{4\pi r^2}.$$

This suggests that the Green's function will only depend on r (at least as far as its x dependence is concerned); thus the equation for G reduces to

$$\frac{\partial^2 G}{\partial r^2} + \frac{2}{r}\frac{\partial G}{\partial r} = -\frac{\delta(r)}{4\pi r^2}.$$

Now

$$\frac{\partial^2 G}{\partial r^2} + \frac{2}{r}\frac{\partial G}{\partial r} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial G}{\partial r}\right),$$

so multiplying by r^2 we obtain

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial G}{\partial r}\right) = -\frac{\delta(r)}{4}\pi.$$

Integrating from 0 to some value r gives

$$r^2\frac{\partial G}{\partial r} = -\frac{1}{4\pi},$$

whence

$$\frac{\partial G}{\partial r} = -\frac{1}{4\pi r^2}, \quad G = \frac{1}{4\pi r} + C$$

for some constant C. Requiring G to vanish as $\mathbf{x} \to \infty$ gives $C = 0$. Thus we have the Green's function

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi r} = \frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|},$$

which is seen to be symmetric in $\mathbf{x}$ and $\mathbf{x}'$, as claimed.

GREEN'S FUNCTION ON A FINITE REGION. Suppose that we are now interested in solving Poisson's equation on a bounded region D; in other words, consider the equation

$$\nabla^2 u = f, \quad u|_{\partial D} = 0.$$

We claim that an appropriate Green's function for this problem is given by the solution to the problem

$$\nabla^2_{\mathbf{x}} G(\mathbf{x}, \mathbf{x}') = -\delta(\mathbf{x}, \mathbf{x}'), \quad G|_{\mathbf{x}\in\partial D} = 0;$$

in other words, we claim that the solution to this problem is given by

$$u(\mathbf{x}) = -\int_D G(\mathbf{x}, \mathbf{x}') f(\mathbf{x}')\, d\mathbf{x}'.$$

(Note the relation of this to our discussion of the intuitive content of the representation formula (1) above: in the case at hand, the solution u must also satisfy a homogeneous Dirichlet boundary condition – in the context of the two physical examples we gave above, this amounts to saying that the region D is surrounded by a grounded conductor or by a substance of constant temperature 0 – and so we require the solutions to the point source problems – the Green's function – to satisfy the same condition.) While we could give a proof of this using definition of G in terms of an orthogonal expansion given initially, we would like to prove it using just the definition of G given here. In order to do this, we shall first derive a calculus formula known as *Green's second identity* (see pp. 490 – 491 in the textbook). Suppose that f and g are suitably smooth functions on a domain D. Then

$$\int_D f\nabla^2 g - g\nabla^2 f \, d\mathbf{x} = \int_{\partial D} f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} dS,$$

where $\frac{\partial f}{\partial n} = \mathbf{n} \cdot \nabla f$ denotes the outwards normal derivative of f at ∂D.

This is basically an integration-by-parts formula and may be derived as follows. We have

$$\begin{aligned}\int_D f\nabla^2 g \, d\mathbf{x} &= \int_D \nabla \cdot (f\nabla g) - \nabla f \cdot \nabla g \, d\mathbf{x} = \int_{\partial D} \mathbf{n} \cdot (f\nabla g) \, dS - \int_D \nabla f \cdot \nabla g \, d\mathbf{x} \\ &= \int_{\partial D} f\frac{\partial g}{\partial n} \, dS - \int_D \nabla f \cdot \nabla g \, d\mathbf{x};\end{aligned}$$

interchanging f and g and subtracting then gives

$$\begin{aligned}\int_D f\nabla^2 g - g\nabla^2 f \, d\mathbf{x} &= \int_{\partial D} f\frac{\partial g}{\partial n} - g\frac{\partial g}{\partial n} \, dS - \int_D \nabla f \cdot \nabla g - \nabla g \cdot \nabla f \, d\mathbf{x} \\ &= \int_{\partial D} f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} \, dS,\end{aligned}$$

as claimed.

Now, formally, if we pretend that Green's identity applies also to the Green's function in $\mathbf{x}'$ (which it won't, because of the singularity; see Theorem 8.3 in the textbook for a more careful treatment of the result here), then we may write, applying Green's identity (integrating in $\mathbf{x}'$) with $f = u(\mathbf{x}')$ and $g = G(\mathbf{x}, \mathbf{x}')$ (please note that the results we derived above, giving $\mathcal{G}\nabla^2 = \nabla^2\mathcal{G} =$ identity, were on $\mathbf{R}^3$ and hence do not apply in the present case; in essence, we are trying to re-derive at least one of them in the present case),

$$\begin{aligned}\int_D u(\mathbf{x}')\nabla^2_{\mathbf{x}'} G(\mathbf{x}, \mathbf{x}') - G(\mathbf{x}, \mathbf{x}')\nabla^2_{\mathbf{x}'} u(\mathbf{x}') \, d\mathbf{x}' &= -u(\mathbf{x}) - \int_D G(\mathbf{x}, \mathbf{x}')\nabla^2_{\mathbf{x}'} u(\mathbf{x}') \, d\mathbf{x}' \\ &= \int_{\partial D} u(\mathbf{x}')\frac{\partial G}{\partial n'} - G(\mathbf{x}, \mathbf{x}')\frac{\partial u}{\partial n'} \, dS'\end{aligned}$$

where primes denote derivatives and integrals with respect to $\mathbf{x}'$. Thus we obtain

$$u(\mathbf{x}) = -\int_D G(\mathbf{x}, \mathbf{x}')\nabla^2_{\mathbf{x}'} u(\mathbf{x}') \, d\mathbf{x}' + \int_{\partial D} G(\mathbf{x}, \mathbf{x}')\frac{\partial u}{\partial n'} - u(\mathbf{x}')\frac{\partial G}{\partial n'} \, dS'. \qquad (2)$$

Now in the case of the Poisson equation above,

$$\nabla^2 u = f, \quad u|_{\partial D} = 0,$$

we see that the volume integral becomes simply

$$-\int_D G(\mathbf{x}, \mathbf{x}')\nabla^2_{\mathbf{x}'} u(\mathbf{x}') \, d\mathbf{x}',$$

while the surface integral vanishes, since $u|_{\partial D} = 0$ and $G|_{\mathbf{x}' \in \partial D} = 0$. Thus we have

$$u(\mathbf{x}) = -\int_D G(\mathbf{x}, \mathbf{x}')f(\mathbf{x}') \, d\mathbf{x}',$$

as claimed.

The above formula (2) above is of wider applicability. We pause for a moment to derive from it an important result about solutions to Laplace's equation. Suppose that D is some open set in $\mathbf{R}^3$, and that u satisfies $\nabla^2 u = 0$ on D. Let G denote the Green's function on $\mathbf{R}^3$ derived in the previous section. Let

$\mathbf{x} \in D$, and let $r > 0$ be such that $B_r(\mathbf{x}) \subset D$; we can do this since D is open (more intuitively, $\mathbf{x}$ cannot be on the boundary of D so it has to be inside D). Then applying formula (2) using G as our Green's function, $B_r(\mathbf{x})$ as our region (in place of D), and $\mathbf{x}$ as our evaluation point, we obtain

$$u(\mathbf{x}) = -\int_{B_r(\mathbf{x})} G(\mathbf{x}, \mathbf{x}')\nabla^2_{x'} u(\mathbf{x}')\, d\mathbf{x}' + \int_{\partial B_r(\mathbf{x})} G(\mathbf{x}, \mathbf{x}')\frac{\partial u}{\partial n'} - u(\mathbf{x}')\frac{\partial G}{\partial n'}\, dS'.$$

Now on $B_r(\mathbf{x})$ we have $\nabla^2 u = 0$, so the first integral vanishes; also, for $\mathbf{x}' \in \partial B_r(\mathbf{x})$ we have

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|} = \frac{1}{4\pi r},$$

which is a constant, so that the middle term above becomes

$$\begin{aligned}\int_{\partial B_r(\mathbf{x})} G(\mathbf{x}, \mathbf{x}')\frac{\partial u}{\partial n'}\, dS' &= \frac{1}{4\pi r}\int_{\partial B_r(\mathbf{x})} \mathbf{n} \cdot \nabla u\, dS' \\ &= \frac{1}{4\pi r}\int_{B_r(\mathbf{x})} \nabla^2 u\, dS' = 0,\end{aligned}$$

where we have used the divergence theorem in the penultimate inequality. Thus we are left with only the last term, i.e.,

$$u(\mathbf{x}) = -\int_{\partial B_r(\mathbf{x})} u(\mathbf{x}')\frac{\partial G}{\partial n'}\, dS'.$$

Now to calculate $\frac{\partial G}{\partial n'}$ we may proceed geometrically as follows. This derivative is the derivative in the direction normal to the sphere $\partial B_r(\mathbf{x})$; alternatively, if we set up a spherical coordinate system (r'', θ'', ϕ'') for $\mathbf{x}'$ centred at the point $\mathbf{x}$, then $\frac{\partial}{\partial n'}$ will simply be the radial derivative $\frac{\partial}{\partial r''}$. But we have also

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|} = \frac{1}{4\pi r''},$$

so that

$$\frac{\partial G}{\partial n'} = \frac{\partial G}{\partial r''} = -\frac{1}{4\pi r''^2} = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|^2},$$

and we obtain finally

$$u(\mathbf{x}) = \int_{\partial B_r(\mathbf{x})} \frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|^2} u(\mathbf{x}')\, dS' = \frac{1}{4\pi r^2}\int_{\partial B_r(\mathbf{x})} u(\mathbf{x}')\, dS',$$

i.e., the value of u at any point $\mathbf{x}$ is equal to the average of u over a sphere centred at $\mathbf{x}$, as long as the corresponding ball bounded by the sphere is entirely contained in the region D on which u satisfies Laplace's equation. This implies that u cannot have a local maximum or local minimum in D, unless it is constant. The main idea is as follows: if u had a local maximum at some point $\mathbf{x}$, and r were any number for which the above formula applied and small enough that $\mathbf{x}$ was a maximum point for u on $B_r(\mathbf{x})$, then u on $\partial B_r(\mathbf{x})$ would have to be equal everywhere to $u(\mathbf{x})$, as it cannot anywhere be greater so were it somewhere less its average would also be less, which would contradict the above equation. The same logic holds for a local minimum.

The foregoing shows that if u is a continuous solution to Laplace's equation on a bounded region, then it must take its extreme values on the boundary of the region (since it must take them somewhere, and if it took either of them at an interior point then it would have a local extremum; and in that case it would be constant, so its maximum value on D would equal its minimum value there, and both would be taken on the boundary – and everything would be quite trivial, of course!).

Returning to our main topic, now, we note the similarity of (2) to formula (1′) above: the only additional terms are those for the boundary conditions. In other words, formula (2) allows us to pass backwards from the Laplacian of u to obtain u, as long as we are given suitable information about u on the boundary. Note though (see [2], p. 37, on which the following discussion is based) that as it stands the information which seems to be required is too much: for example, suppose that we are trying to solve Laplace's equation; we know from our previous studies that giving just the value of u on the boundary suffices to obtain a unique solution to Laplace's equation, and hence to obtain also the normal derivatives $\frac{\partial u}{\partial n}$ on the boundary. But the formula above seems to require us to give both; since they cannot be specified independently, the formula does not seem to be much use as it stands. The resolution lies in the manipulations following equation (2): we are free to impose boundary conditions on G also, and may impose them to make either of the terms in the surface integral vanish. Thus, if we are given u on the boundary, we shall use a Green's function which vanishes on the boundary, so that the term involving $\frac{\partial u}{\partial n}$ does not appear; if we are instead given $\frac{\partial u}{\partial n}$ on the boundary, then we shall use a Green's function which satisfies $\frac{\partial G}{\partial n}\big|_{x\in\partial D} = 0$, so that (using symmetry of G) the term involving u on the boundary vanishes.

As an example, consider now the problem

$$\nabla^2 u = f, \quad u|_{\partial D} = g;$$

if G is the Green's function defined above, then by (2) we may write

$$u(\mathbf{x}) = -\int_D G(\mathbf{x}, \mathbf{x}')f(\mathbf{x}')\,d\mathbf{x}' - \int_{\partial D} g(\mathbf{x}')\frac{\partial G}{\partial n'}\,dS'.$$

This gives an alternative representation of the solution to the above problem (which we already know how to solve in terms of orthogonal expansions, at least for $D = Q, C, B$). In the case where $f = 0$, we obtain the formula

$$u(\mathbf{x}) = -\int_{\partial D} g(\mathbf{x}')\frac{\partial G}{\partial n'}\,dS'$$

for the solution to the boundary-value problem for Laplace's equation

$$\nabla^2 u = 0, \quad u|_{\partial D} = g.$$

Returning to general theory, we may now ask how we are to find a Green's function satisfying an appropriate boundary condition on ∂D. This may be done by a method very similar to that used to solve Poisson's equation with nonhomogeneous boundary conditions. We first set $G_0(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi|\mathbf{x}-\mathbf{x}'}$, which is just the Green's function for $\mathbf{R}^3$ and hence satisfies $\nabla^2_{\mathbf{x}} G_0 = -\delta(\mathbf{x} - \mathbf{x}')$. Then we let $u(\mathbf{x}, \mathbf{x}')$ solve the boundary-value problem for Laplace's equation given by

$$\nabla^2_{\mathbf{x}} u(\mathbf{x}, \mathbf{x}') = 0, \quad u|_{\mathbf{x}\in\partial D} = -G_0(\mathbf{x}, \mathbf{x}');$$

if we let $G(\mathbf{x}, \mathbf{x}') = G_0(\mathbf{x}, \mathbf{x}') + u(\mathbf{x}, \mathbf{x}')$, then we see that G satisfies

$$\nabla^2_{\mathbf{x}} G(\mathbf{x}, \mathbf{x}') = -\delta(\mathbf{x} - \mathbf{x}'), \quad G|_{\mathbf{x}\in\partial D} = 0,$$

and hence that G is the desired Green's function.

If we wished instead to solve Poisson's equation with Neumann boundary conditions, we would use the function u satisfying instead the problem

$$\nabla^2_{\mathbf{x}} u(\mathbf{x}, \mathbf{x}') = 0, \quad \left.\frac{\partial u}{\partial n}\right|_{\mathbf{x}\in\partial D} = -\frac{\partial G_0}{\partial n};$$

then $G = G_0 + u$ would be the desired Green's function.

All of this, of course, is just empty air unless we have a method to actually calculate a Green's function; in other words, unless we can actually solve the boundary-value problems for Laplace's equation given above. In general, of course, the only methods we know for solving Laplace's equation involve expansions in orthogonal sets, so it seems that the only formulas we can obtain for Green's functions at the moment are still as expansions in orthogonal sets, and it isn't clear that we have gained much. It turns out, though, that for specific geometries there are other techniques for finding a Green's function; for example, for a sphere one can calculate the Green's function using the so-called *method of images* – see Example 8.2.1 in the textbook. (Regardless of this, the foregoing is important to know from a theoretical point of view. And actually we have gained something practical by the manipulations above, since the solutions to Laplace's equation are written in terms of orthogonal sets on sets of dimension one less than the space in which we work: for example, if we use an orthogonal expansion to find the Green's function on the sphere, we would be expanding in the basis $\{P_{\ell m}(\cos\theta)\cos m\phi, P_{\ell m}(\cos\theta)\sin m\phi\}$ rather than the full basis of eigenfunctions of the Laplacian on the ball. This is a gain in simplicity, at least.)

While it would be beneficial and interesting to take some time to give concrete examples involving Green's functions, considerations of time and space impel us to pass over this and continue to our next topic, Fourier transforms. We shall try to come back and give an example or two of the above theory at some point in the future.

FOURIER TRANSFORMS. We first say a few words about orthogonal expansions in general. Consider an expansion in the basis of eigenfunctions for the Laplacian on Q obeying homogeneous Dirichlet boundary conditions: this is a series of the form $\sum_{\ell mn} a_{\ell mn} \sin \ell\pi x \sin m\pi y \sin n\pi z$, where ℓ, m and n range over all positive integers. Now if we use instead periodic boundary conditions then we would obtain expressions of the above form but with both sin and cos terms; if we were to express everything in terms of complex exponentials, we would get a sum of the form

$$\sum_{\ell,m,n=-\infty}^{\infty} a_{\ell mn} e^{2i\ell\pi x} e^{2im\pi y} e^{2in\pi z} = \sum_{\ell,m,n=-\infty}^{\infty} a_{\mathbf{l}} e^{2\pi i \mathbf{l}\cdot\mathbf{x}},$$

where $\mathbf{l} = (\ell, m, n)$. The point here is that we can express any function on the bounded region Q as a *series* in a *discrete* set of functions $\{e^{2\pi i\mathbf{l}\cdot\mathbf{x}}\}$. This is related to the fact that the eigenvalues of the Laplacian on Q (with suitable boundary conditions) form a discrete set (for periodic boundary conditions, $4\pi^2\left(\ell^2 + m^2 + n^2\right)$). Now it turns out that, in general, the set of eigenvalues of the Laplacian on a bounded set is discrete. (For those who have or will study quantum mechanics, this is closely related to the statement that a bound partical has only a discrete set of energy levels.) On an unbounded set, though, the set of eigenvalues (more properly, in this case, the *spectrum*[12]) of the Laplacian becomes continuous, and the sum over eigenvalues in the above expression must be replaced by an integral. (See section 5.1.1 in the textbook for a more detailed explanation of this crossover from series to integral.) In the case of the complex exponential basis, this gives rise to the *Fourier transform*.

We begin by recall the complex exponential basis on $[0, 1]$: if f is any suitably well-behaved function on $[0, 1]$, then we have

$$f = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{2\pi ikx},$$

where

$$\hat{f}_k = \int_0^1 f(x) e^{-2\pi ikx}\, dx$$

(the signs in the exponents differ since when we take an inner product we always take the conjugate of the second function). This may clearly be extended to the unit cube, giving

$$f(\mathbf{x}) = \sum_{\ell,m,n=-\infty}^{\infty} \hat{f}_{\mathbf{k}} e^{2\pi i\mathbf{k}\cdot\mathbf{x}},$$

where

[12]If A is a linear operator on a vector space V, the *spectrum* of A is defined as the set of numbers λ such that $A - \lambda I$ is not invertible. If V is finite-dimensional, this is the same as the set of eigenvalues of A; but in infinite-dimensional spaces, such as those we work with here, this is not necessarily the case.

$$\hat{f}_{\mathbf{k}} = \int_Q f(x)e^{-2\pi i\mathbf{k}\cdot\mathbf{x}}\,d\mathbf{x}$$

where we now write $\mathbf{k} = (\ell, m, n)$. This motivates the following definition of the Fourier transform. We first make one comment about terminology. We shall be dealing mainly with functions f which have the property that $\int_{\mathbf{R}^m} |f(\mathbf{x})|\,d\mathbf{x} < \infty$; such functions are said to be in L^1 (on $\mathbf{R}^m$); in symbols,

$$L^1(\mathbf{R}^m) = \{f : \mathbf{R}^m \to \mathbf{C} | \int_{\mathbf{R}^m} |f(\mathbf{x})|\,d\mathbf{x} < \infty\}.$$

DEFINITION. Suppose that $f \in L^1(\mathbf{R}^m)$. Then we define the *Fourier transform* of f, which we denote $\hat{f}$ or $\mathcal{F}[f]$, to be the function from $\mathbf{R}^m$ to $\mathbf{C}$ given by

$$\hat{f}(\mathbf{k}) = \int_{\mathbf{R}^m} f(\mathbf{x})e^{-2\pi i\mathbf{k}\cdot\mathbf{x}}\,d\mathbf{x}.$$

(Note that this definition differs from that in the textbook by the factor of 2π in the exponent. As we shall see, factors of 2π appear in various places in the study of Fourier transforms; there is no way to get rid of all of them, and the different conventions just push them around into different locations. From a purely mathematical point of view, there are many reasons why the above definition recommends itself. See [1], p. 278, for discussion.)

The Fourier transform enjoys the following properties. We write $\mathbf{x} = (x^1, \ldots, x^m)$, $\mathbf{k} = (k^1, \ldots, k^m)$; the superscripts are just that – superscripts – not powers; think of them like subscripts, but written up, not down.[13]

PROPERTIES OF THE FOURIER TRANSFORM. (a) The Fourier transform is linear:

$$\mathcal{F}[af + bg](\mathbf{k}) = a\mathcal{F}[f](\mathbf{k}) + b\mathcal{F}[g](\mathbf{k}).$$

(b) If f is differentiable and $\frac{\partial f}{\partial x^j} \in L^1$ for some j (note that this latter does not follow from $f \in L^1$!), then

$$\mathcal{F}[\partial_j f](\mathbf{k}) = 2\pi i k^j \mathcal{F}[f](\mathbf{k}).$$

(c) If $x^j f \in L^1$ for some j, then

$$\mathcal{F}[2\pi i x^j f](\mathbf{k}) = -\frac{\partial \hat{f}}{\partial k^j}.$$

(d) For any $\boldsymbol{\alpha} \in \mathbf{R}^m$,

$$\mathcal{F}[f(\mathbf{x} - \boldsymbol{\alpha})](\mathbf{k}) = e^{-2\pi i\mathbf{k}\cdot\boldsymbol{\alpha}}\hat{f}(\mathbf{k}).$$

(e) Similarly,

$$\mathcal{F}[e^{2\pi i\boldsymbol{\alpha}\cdot\mathbf{x}}f(\mathbf{x})](\mathbf{k}) = \hat{f}(\mathbf{k} - \boldsymbol{\alpha}).$$

Proof. (a) This is entirely straightforward, almost trivial:

$$\begin{aligned}\mathcal{F}[af + bg](\mathbf{k}) &= \int_{\mathbf{R}^m} [af(\mathbf{x}) + bg(\mathbf{x})]\,e^{-2\pi i\mathbf{k}\cdot\mathbf{x}}\,d\mathbf{x}\\ &= a\int_{\mathbf{R}^m} f(\mathbf{x})e^{-2\pi i\mathbf{k}\cdot\mathbf{x}}\,d\mathbf{x} + b\int_{\mathbf{R}^m} g(\mathbf{x})e^{-2\pi i\mathbf{k}\cdot\mathbf{x}}\,d\mathbf{x} = a\mathcal{F}[f](\mathbf{k}) + b\mathcal{F}[g](\mathbf{k}).\end{aligned}$$

(b) We first note a technical point. Contrary to what I said in class, $f \in L^1$ does *not* imply that $f \to 0$ as $\mathbf{x} \to \infty$ (think of the function on $\mathbf{R}^1$ defined by

$$f(x) = \begin{cases} 1, & x \in [n, n + \frac{1}{n^2}]\text{ for some } n \in \mathbf{Z} \\ 0, & \text{otherwise} \end{cases};$$

[13]The reason for this is that from a differential-geometric perspective, quantities with indices *up* are distinct from quantities with indices *down*. This is related to the first notion of 'tensor' which I mentioned in class on Thursday, as a collection of numbers transforming in a certain way under a coordinate transformation.

it is simple to see that $f \in L^1$, but clearly f has no limit as $x \to \infty$). However, $\frac{\partial f}{\partial x^j} \in L^1$ implies that $\lim_{x^j \to \infty} f(\mathbf{x})$ exists: for

$$\lim_{x^j \to \infty} f(\mathbf{x}) = f(0) + \lim_{x^j \to \infty} \int_0^{x^j} \frac{\partial f}{\partial x^j}(x^1, \cdots, y^j, \cdots, x^n)\, dy^j,$$

and since the latter integral is absolutely convergent (as $\partial_j f \in L^1$), it must be convergent, meaning that $\lim_{x^j \to \infty} f(\mathbf{x})$ exists. Since $f \in L^1$, this limit must be zero. The result may now be proven using integration by parts:

$$\begin{aligned}\mathcal{F}[\partial_j f](\mathbf{k}) &= \int_{\mathbf{R}^m} \frac{\partial f}{\partial x^j} e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x} = \int_{\mathbf{R}^m} \frac{\partial}{\partial x^j}\left(f e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\right) + 2\pi i k^j f(\mathbf{x}) e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x} \\ &= 2\pi i k^j \mathcal{F}[f](\mathbf{k}),\end{aligned}$$

where the boundary term vanishes since, as just noted, $\lim_{x^j \to \infty} f(\mathbf{x}) = 0$.

(c) We have

$$\begin{aligned}\mathcal{F}[2\pi i x^j f](\mathbf{k}) &= \int_{\mathbf{R}^m} 2\pi i x^j f(\mathbf{x}) e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x} = -\int_{\mathbf{R}^m} f(\mathbf{x}) \frac{\partial}{\partial k^j} e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x} \\ &= -\frac{\partial}{\partial k^j} \int_{\mathbf{R}^m} f(\mathbf{x}) e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x} = -\frac{\partial \hat{f}}{\partial k^j},\end{aligned}$$

where we may interchange differentiation and integration since $x^j f \in L^1$[14]

(d) This and (e) are very straightforward calculations. Here we do a change of variables $\mathbf{y} = \mathbf{x} - \boldsymbol{\alpha}$:

$$\begin{aligned}\mathcal{F}[f(\mathbf{x} - \boldsymbol{\alpha})](\mathbf{k}) &= \int_{\mathbf{R}^m} f(\mathbf{x} - \boldsymbol{\alpha}) e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x} \\ &= \int_{\mathbf{R}^m} f(\mathbf{y}) e^{-2\pi i \mathbf{k}\cdot(y + \boldsymbol{\alpha})}\, d\mathbf{x} = e^{-2\pi i \mathbf{k}\cdot\boldsymbol{\alpha}} \hat{f}(\mathbf{k}).\end{aligned}$$

(e)
$$\begin{aligned}\mathcal{F}[e^{2\pi i \boldsymbol{\alpha}\cdot\mathbf{x}} f(\mathbf{x})](\mathbf{k}) &= \int_{\mathbf{R}^m} e^{2\pi i \boldsymbol{\alpha}\cdot\mathbf{x}} f(\mathbf{x}) e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x} = \int_{\mathbf{R}^m} f(\mathbf{x}) e^{-2\pi i (\mathbf{k} - \boldsymbol{\alpha})\cdot\mathbf{x}}\, d\mathbf{x} \\ &= \hat{f}(\mathbf{k} - \boldsymbol{\alpha}).\end{aligned}$$

Parts (a) and (b) are probably the most important for us at the moment. To show how (b) is used more generally, suppose that $P(\mathbf{k})$ is some polynomial on $\mathbf{R}^m$, i.e., $P(\mathbf{k})$ is a linear combination of monomial terms of the form

$$\left(k^1\right)^{\alpha_1} \left(k^2\right)^{\alpha_2} \cdots \left(k^m\right)^{\alpha_m},$$

where $\alpha_1, \ldots, \alpha_m$ are all nonnegative integers. Suppose that $Q(\mathbf{k})$ is the above monomial. Then we define the differential operator $Q(\nabla)$ by

$$Q(\nabla)(f)(\mathbf{x}) = \frac{\partial^{\alpha_1} f}{\partial {x^1}^{\alpha_1}} \cdots \frac{\partial^{\alpha_m} f}{\partial {x^m}^{\alpha_m}}.$$

Part (b) can then be used to show that for any polynomial $P(\mathbf{k})$,

$$\mathcal{F}\left[P(\nabla) f\right](\mathbf{k}) = P(2\pi i \mathbf{k}) \hat{f}(\mathbf{k});$$

i.e., the Fourier transform turns differential operators into multiplication operators. Let us consider a few examples.

[14] Again, to get things in this generality one probably needs the Lebesgue integral and dominated convergence theorem. We apologise.

EXAMPLES. (a) Let us consider an example in $\mathbf{R}^1$ for simplicity. If f is such that all of the relevant Fourier transforms are defined, then

$$\mathcal{F}[f'(x)](k) = -2\pi i k \hat{f}(k),$$
$$\mathcal{F}[f''(x)](k) = -4\pi^2 k^2 \hat{f}(k).$$

(b) Let us consider how the Fourier transform acts on the Laplacian of a function in $\mathbf{R}^3$. Thus suppose that f is a function on $\mathbf{R}^3$ and is such that all of the relevant Fourier transforms are defined. Then

$$\begin{aligned}\mathcal{F}[\nabla^2 f](\mathbf{k}) &= \mathcal{F}\left[\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right] = (-2\pi i k^1)^2 \hat{f}(\mathbf{k}) + (-2\pi i k^2)^2 \hat{f}(\mathbf{k}) + (-2\pi i k^3)^2 \hat{f}(\mathbf{k}) \\ &= -4\pi^2 (k^1)^2 \hat{f}(\mathbf{k}) - 4\pi^2 (k^2)^2 \hat{f}(\mathbf{k}) - 4\pi^2 (k^3)^2 \hat{f}(\mathbf{k}) = -4\pi^2 |\mathbf{k}|^2 \hat{f}(\mathbf{k}),\end{aligned}$$

where $|\mathbf{k}| = (k^1)^2 + (k^2)^2 + (k^3)^2$ is the Euclidean norm of $\mathbf{k}$.

The above three examples will be the most important examples of this kind of thing for us going forwards.

CONVOLUTION. To derive the next property satisfied by the Fourier transform, we need to define another operation on functions known as *convolution*. Unlike the Fourier transform, which takes a single function to a single function, convolution is a product, which takes two functions to one function. It is defined as follows: suppose that $f, g \in L^1(\mathbf{R}^m)$; then we define their *convolution* $f * g$ (note that this is a star, an asterisk, with *six points*, and may not be properly written with less!) to be the function on $\mathbf{R}^m$

$$(f * g)(\mathbf{x}) = \int_{\mathbf{R}^m} f(\mathbf{x} - \mathbf{x}')g(\mathbf{x}')\, d\mathbf{x}'.$$

Convolution satisfies the following properties:

(a) It is *bilinear*:

$$([af + bh] * g)(\mathbf{x}) = a(f * g)(\mathbf{x}) + b(h * g)(\mathbf{x}), \quad (f * [ag + bh])(\mathbf{x}) = a(f * g)(\mathbf{x}) + b(f * h)(\mathbf{x}).$$

(b) It is commutative:

$$(f * g)(\mathbf{x}) = (g * f)(\mathbf{x})$$

and associative:

$$[(f * g) * h](\mathbf{x}) = [f * (g * h)](\mathbf{x}).$$

Proof. (a) This is again almost trivial; we show only the first one as the second is identical:

$$\begin{aligned}([af + bh] * g)(\mathbf{x}) &= \int_{\mathbf{R}^m} (af + bh)(\mathbf{x} - \mathbf{x}')g(\mathbf{x}')\, d\mathbf{x}' = a\int_{\mathbf{R}^m} f(\mathbf{x} - \mathbf{x}')g(\mathbf{x}')\, d\mathbf{x}' + b\int_{\mathbf{R}^m} h(\mathbf{x} - \mathbf{x}')g(\mathbf{x}')\, d\mathbf{x}' \\ &= a(f * g)(\mathbf{x}) + b(h * g)(\mathbf{x}).\end{aligned}$$

(b) The first of these is straightforward, doing a change of variables to $\mathbf{y} = \mathbf{x} - \mathbf{x}'$:

$$(f * g)(\mathbf{x}) = \int_{\mathbf{R}^m} f(\mathbf{x} - \mathbf{x}')g(\mathbf{x}')\, d\mathbf{x}' = \int_{R^m} f(\mathbf{y})g(\mathbf{x} - \mathbf{y})\, d\mathbf{y} = (g * f)(\mathbf{x}).$$

The second may be shewn as follows, using the change of variables $\mathbf{y} = \mathbf{x}' + \mathbf{x}''$, $\mathbf{y}' = \mathbf{x}'$; we note that this has unit determinant. We ignore the difficulties in rewriting the iterated integrals as a single integral.

$$\begin{aligned}[(f * g) * h](\mathbf{x}) &= \int_{\mathbf{R}^m} (f * g)(\mathbf{x} - \mathbf{x}')h(\mathbf{x}')\, d\mathbf{x}' = \int_{\mathbf{R}^m}\int_{\mathbf{R}^m} f(\mathbf{x} - \mathbf{x}' - \mathbf{x}'')g(\mathbf{x}'')h(\mathbf{x}')\, d\mathbf{x}''\, d\mathbf{x}' \\ &= \int_{\mathbf{R}^m \times \mathbf{R}^m} f(\mathbf{x} - \mathbf{y})g(\mathbf{y} - \mathbf{y}')h(\mathbf{y}')\, d\mathbf{y}'\, d\mathbf{y} = \int_{\mathbf{R}^m} f(\mathbf{x} - \mathbf{y})(g * h)(\mathbf{y})\, d\mathbf{y} = [f * (g * h)](\mathbf{x}),\end{aligned}$$

as desired.

For us, the main interest in convolution is not the above algebraic properties (though it is important to know about these), but rather the way in which convolution interacts with the Fourier transform. This is given in the following theorem.

THEOREM. Suppose that $f, g \in L^1(\mathbf{R}^m)$, $f * g \in L^1(\mathbf{R}^m)$. Then

$$\mathcal{F}[f * g](\mathbf{k}) = \hat{f}(\mathbf{k})\hat{g}(\mathbf{k});$$

i.e., the Fourier transform turns convolutions into products.

Proof. Assuming (as in the proof of associativity above) that we may combine the iterated integrals appearing here into a single integral over the product space, we have, using the change of variables $\mathbf{y} = \mathbf{x} - \mathbf{x}'$, $\mathbf{y}' = \mathbf{x}'$,

$$\begin{aligned}\mathcal{F}[f * g](\mathbf{k}) &= \int_{\mathbf{R}^m} (f * g)(\mathbf{x}) e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x} = \int_{\mathbf{R}^m}\int_{\mathbf{R}^m} f(\mathbf{x}-\mathbf{x}')g(\mathbf{x}')e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x}'\, d\mathbf{x} \\ &= \int_{\mathbf{R}^m\times\mathbf{R}^m} f(\mathbf{y})g(\mathbf{y}')e^{-2\pi i \mathbf{k}\cdot(\mathbf{y}+\mathbf{y}')}\, d\mathbf{y}'\, d\mathbf{y} = \int_{\mathbf{R}^m} f(\mathbf{y})e^{-2\pi i \mathbf{k}\cdot\mathbf{y}}\, d\mathbf{y} \int_{\mathbf{R}^m} g(\mathbf{y}')e^{-2\pi i \mathbf{k}\cdot\mathbf{y}'}\, d\mathbf{y}' \\ &= \hat{f}(\mathbf{k})\hat{g}(\mathbf{k}),\end{aligned}$$

as claimed.

We note in passing that an analogous result would hold if we replaced $e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}$ in $\mathcal{F}$ by $e^{2\pi i \mathbf{k}\cdot\mathbf{x}}$; this will be used shortly.

One use of the Fourier transform for solving partial differential equations can now be indicated briefly. Suppose that we are interested in solving Poisson's equation on $\mathbf{R}^m$; i.e., that we have the problem $\nabla^2 u = f$. If $f \in L^1$, then assuming $u \in L^1$ we may take the Fourier transform of both sides to obtain

$$-4\pi^2|\mathbf{k}|^2\hat{u}(\mathbf{k}) = \hat{f}(\mathbf{k}),$$

whence

$$\hat{u} = -\frac{1}{4\pi^2|\mathbf{k}|^2}\hat{f}(\mathbf{k}).$$

By the foregoing theorem, then, if we can find a function g whose Fourier transform is $-\frac{1}{4\pi^2|\mathbf{k}|^2}$, then we will have

$$u(\mathbf{x}) = (g * f)(\mathbf{x}) = \int_{\mathbf{R}^m} g(\mathbf{x}-\mathbf{x}')f(\mathbf{x}')\, d\mathbf{x}'.$$

Our work with Green's functions suggests that in $\mathbf{R}^3$ we have $g(\mathbf{x}) = \frac{1}{4\pi|\mathbf{x}|}$. Note, though, that the Fourier transform result above is independent of m.

Even more generally, suppose that we were interested in the equation

$$P(\nabla)u = f$$

for some polynomial P; if $f \in L^1$, then taking the Fourier transform gives $P(\mathbf{k})u = f$, or $u(\mathbf{k}) = \frac{1}{P(\mathbf{k})}f(\mathbf{k})$. Hence, if $\frac{1}{P(\mathbf{k})}$ is the Fourier transform of a function g, then $u(\mathbf{x}) = (g * f)(\mathbf{x})$ will be the solution to the original problem. Actually calculating g, however, is quite another matter. We shall try to say more about this later.

FOURIER INVERSION THEOREM. Our study of the basic theory of the Fourier transform will be essentially completed once we have established the *Fourier inversion theorem*, which tells us how to invert the Fourier transform. Its statement is as follows; a proof will be given next week.

THEOREM. Suppose that $f \in L^1$ is such that f is bounded on $\mathbf{R}^m$ (this condition can be removed by doing more careful calculations in Example (b) on p. 5) and $\hat{f} \in L^1$. Then we have[15]

$$f(\mathbf{x}) = \int_{\mathbf{R}^m} \hat{f}(\mathbf{k})e^{2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k}.$$

[15] This formula, like so many of the other formulas we have given in this course, may not hold at every point. It will, however, hold at points at which f is continuous; it will, in fact, hold *almost everywhere*, meaning everywhere except on a set of *measure zero*. For a related result, see Theorem 5.1 in the textbook.

REFERENCES

1. Folland, G. B. Real Analysis: Modern Techniques and Their Applications, 2nd ed. New York: John Wiley and Sons, 1999.
2. Jackson, J. D. Classical Electrodynamics, 3rd ed. [New York?] John Wiley and Sons, 1999.

APPENDIX I. Let us take $X = \mathbf{R}^n$ for some n, and consider the set $C_c^\infty(X)$ of all C^∞ functions on X which have *compact support*, i.e., which vanish outside of a compact set. (The *support* of a function is defined as

$$\operatorname{supp} f = \overline{\{x | f(x) \neq 0\}},$$

where $\overline{U}$ indicates the closure of the set U.) We may define a toplogy on $C_c^\infty(X)$ which gives it the structure of a topological vector space by means of an infinite family of *seminorms*, as follows. (Recall that a *norm* on a vector space V is a map $\|\cdot\| : V \to \mathbf{R}_+$ which satisfies the following properties:

$$\begin{aligned} &1. \quad \|x\| \geq 0 \,\text{and}\, \|x\| = 0 \,\text{implies}\, x = 0 \\ &2. \quad \|\alpha x\| = |\alpha| \|x\| \,\text{for all}\, \alpha \in \mathbf{R} \\ &3. \quad \|x + y\| \leq \|x\| + \|y\|; \end{aligned}$$

a *seminorm* is a map $|\cdot| : V \to \mathbf{R}_+$ which satisfies all of these except that $|x| = 0$ does not imply $x = 0$.) Let $K \subset X$ be compact, and let $\ell \geq 0$. We let I denote a *multiindex*, i.e., $I = (I_1, \ldots, I_n)$, where $I_1, \ldots, I_n \in \mathbf{Z}$, $I_1, \ldots, I_n \geq 0$, and for such an I we define $|I| = \sum_i I_i$ and for $f \in C_c^\infty(X)$

$$\partial^I f = \frac{\partial^{|I|} f}{\partial x_1^{I_1} \partial x_2^{I_2} \cdots \partial x_n^{I_n}}.$$

Then for $f \in C_c^\infty(X)$ we define

$$|f|_{K,\ell} = \sup_{x \in K} \sup_{|I| \leq \ell} |\partial^I f(x)|.$$

It is straightforward to show that $|\cdot|_{K,\ell}$ is indeed a seminorm on $C_c^\infty(X)$ for all K and ℓ as above. This family of seminorms may be used to define a topology by taking the corresponding balls

$$B_{\epsilon,K,\ell}(f_0) = \{f \,|\, |f - f_0|_{K,\ell} < \epsilon\}$$

as a basis: i.e., we define a set $U \subset C_c^\infty(X)$ to be open if and only if for all $f_0 \in U$ there is an $\epsilon > 0$, $K \subset X$ compact, and $\ell \geq 0$ such that $B_{\epsilon,K,\ell}(f_0) \subset U$. It is simple to show that with this definition $C_c^\infty(X)$ becomes a topological space. (It can also be shown that the vector-space operations of addition and scalar multiplication are continuous with respect to this topology.) We may then ask which linear functionals $\phi : C_c^\infty(X) \to \mathbf{R}$ are continuous with respect to this topology. Such linear functionals are called *distributions*, and the set of all distributions is denoted $D'(X)$ (after the notation $D(X)$ for $C_c^\infty(X)$, a prime denoting the so-called dual space of continuous linear functionals).

APM 346, Homework 10. Due Monday, July 29, at 6.00 AM EDT. To be marked completed/not completed.

1. Starting from separation of variables, give the series expansion to the solution for the following problem in terms of an appropriate set of eigenfunctions of the Laplacian on the unit cube $Q = \{(x, y, z) | 0 \leq x, y, z \leq 1\}$:

$$\nabla^2 u = \begin{cases} 1, & 0 \leq z < \frac{1}{2} \\ -1, & \frac{1}{2} < z \leq 1 \end{cases}, \quad \partial_\nu u|_{\partial Q} = 0, \quad u(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = 0,$$

where ∂_ν denotes the outward normal derivative on the surface (e.g., on the surface $\partial Q \cap \{z = 0\}$, it is $-\frac{\partial}{\partial z}$).

We begin by finding the eigenfunctions of the Laplacian on Q appropriate to the given boundary conditions. (The last condition $u(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = 0$ is a condition on the solution, not the eigenfunctions, and will be dealt with at the end.) We shall look as usual for separated eigenfunctions; thus we seek functions $u = X(x)Y(y)Z(z)$ and numbers λ satisfying

$$\nabla^2 u = \lambda u, \quad \partial_\nu u|_{\partial Q} = 0;$$

now substituting $u = X(x)Y(y)Z(z)$ into the first equation and dividing through by u (since we assume that u, as an eigenfunction, is not identically zero), we have as usual the equation

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = \lambda. \tag{1}$$

Now we need to determine how the boundary conditions are to be implemented in terms of X, Y, and Z. Now the boundary ∂Q of Q has six parts, which lie in the planes $z = 0$, $z = 1$, $x = 0$, $x = 1$, $y = 0$, $y = 1$; since ∂_ν is the unit outward normal derivative on the boundary of Q, we see that on the plane $z = 0$, $\partial_\nu = -\frac{\partial}{\partial z}$, while on $z = 1$ we have $\partial_\nu = \frac{\partial}{\partial z}$; thus the parts of the boundary condition $\partial_\nu u|_{\partial Q} = 0$ corresponding to the top and bottom surfaces of the cube are

$$X(x)Y(y)\left(-Z'(0)\right) = 0, \quad X(x)Y(y)Z'(1) = 0,$$

i.e., $X(x)Y(y)Z'(0) = X(x)Y(y)Z'(1) = 0$ for all x and y. Since X and Y are not identically zero, we conclude that $Z'(0) = Z'(1) = 0$. Analogously, the boundary conditions on the other sides of the cube give $X'(0) = X'(1) = 0$, $Y'(0) = Y'(1) = 0$, and we thus have in addition to (1) the boundary conditions

$$X'(0) = X'(1) = Y'(0) = Y'(1) = Z'(0) = Z'(1) = 0.$$

From these we see as usual (since the derivative of a linear combination of exponentials is still a linear combination of exponentials) that X, Y, and Z must all be oscillatory; thus $\frac{X''}{X}$, $\frac{Y''}{Y}$, $\frac{Z''}{Z} < 0$, so we may write

$$X'' = -\lambda_1^2 X, \quad Y'' = -\lambda_2^2 Y, \quad Z'' = -\lambda_3^2 Z$$

(note that we do not yet know what the λ_i are since the boundary conditions are not the homogeneous Dirichlet conditions we have met previously; in other words, we cannot just directly write $\lambda_1 = \ell\pi$, etc.). Let us consider the problem for X:

$$X'' = -\lambda_1^2 X, \quad X'(0) = X'(1) = 0.$$

From the equation, we have

$$X = a\cos\lambda_1 x + b\sin\lambda_1 x,$$

whence the boundary conditions give

$$X'(0) = -\lambda_1 b = 0, \quad X'(1) = -a\lambda_1 \sin\lambda_1 + b\lambda_1\cos\lambda_1 = 0;$$

the first gives either $\lambda_1 = 0$, in the which case $X = a$ is constant, or $b = 0$; in the first case the second boundary condition is satisfied automatically, while in the second case ($\lambda_1 \neq 0$, $b = 0$) it gives

$$a\lambda_1 \sin\lambda_1 = 0,$$

so (since $a \neq 0$ as $X \neq 0$, and $\lambda_1 \neq 0$ by assumption) we must have $\lambda_1 = \ell\pi$, $\ell \in \mathbf{Z}$, $\ell > 0$, as before. Thus we have two separate cases: either $X = a$ or $X = a\cos\ell\pi x$, $\ell \in \mathbf{Z}$, $\ell > 0$. Clearly, we may combine these two cases; dropping the arbitrary constant a, we may write

$$X = \cos\ell\pi x, \quad \ell \in \mathbf{Z}, \ell \geq 0.$$

Similar logic clearly applies also to Y and Z, so we have

$$\begin{aligned} Y &= \cos m\pi y, \quad m \in \mathbf{Z}, m \geq 0, \\ Z &= \cos n\pi z, \quad n \in \mathbf{Z}, n \geq 0, \end{aligned}$$

and we have finally the eigenfunctions

$$\mathbf{e}_{\ell mn} = \cos\ell\pi x \cos m\pi y \cos n\pi z, \quad \ell, m, n \in \mathbf{Z}, \ell, m, n \geq 0,$$

while the corresponding eigenvalues are

$$\lambda_{\ell mn} = -\pi^2\left(\ell^2 + m^2 + n^2\right).$$

Note that $\lambda_{000} = 0$, i.e., we have a zero eigenvalue; this is because the constant function satisfies the boundary condition in this case. This will create some extra wrinkles in our solution, one of which is obvious while one is less so, as we shall see shortly.

We note that the set $\{\cos\ell\pi x\}_{\ell=0}^\infty$ is complete on $[0, 1]$; this can be shown in a way similar to that by which we showed $\{\sin\ell\pi x\}_{\ell=1}^\infty$ complete on $[0, 1]$: if $f : [0, 1] \to \mathbf{R}^1$ is any suitable function, then we may extend it to $[-1, 1]$ by requiring it to be *even*, i.e., we may define a new function

$$f^* : [-1, 1], \quad f^*(x) = \begin{cases} f(x), & x \geq 0 \\ f(-x), & x \leq 0 \end{cases};$$

since $\{\cos\ell\pi x, \sin\ell\pi x\}_{\ell=0}^\infty$ is complete on $[-1, 1]$, we may expand f^* in a aeries in $\cos\ell\pi x$ and $\sin\ell\pi x$; but since f^* is even, all of the coefficients for the $\sin\ell\pi x$ terms vanish, meaning that f^* can be written in a series

$$f^* = \sum_{\ell=0}^\infty a_\ell \cos\ell\pi x$$

on $[-1, 1]$. But from this it follows that on $[0, 1]$ we have the series

$$f = \sum_{\ell=0}^\infty a_\ell \cos\ell\pi x,$$

meaning that $\{\cos\ell\pi x\}_{\ell=0}^\infty$ is complete on $[0, 1]$, as desired. By standard logic, it follows that the set of eigenfunctions $\{\mathbf{e}_{\ell mn}\}_{\ell,m,n=0}^\infty$ is complete on Q.

We may now proceed as usual to solve the equation. We begin by expanding the right-hand side of the given Poisson equation in terms of the above basis of eigenfunctions. Thus let

$$g(x, y, z) = \begin{cases} 1, & 0 \leq z < \frac{1}{2} \\ -1, & \frac{1}{2} < z \leq 1 \end{cases};$$

then we may write

$$g(x,y,z) = \sum_{\ell,m,n=0}^{\infty} a_{\ell mn} \cos \ell\pi x \cos m\pi y \cos n\pi z.$$

To write out a formula for the $a_{\ell mn}$, we need to determine the normalisation constants for the $\mathbf{e}_{\ell mn}$. Now

$$\int_0^1 \cos^2 \ell\pi x \, dx = \begin{cases} 1, & \ell = 0 \\ \frac{1}{2}, & \ell \neq 0 \end{cases};$$

if we denote this quantity by N_ℓ, then we may write

$$\int_Q \mathbf{e}_{\ell mn}^2(x,y,z) \, dV = N_\ell N_m N_n.$$

Thus we may write the coefficients $a_{\ell mn}$ in the above expansion as

$$\begin{aligned} a_{\ell mn} &= \frac{1}{N_\ell N_m N_n} \int_Q g(x,y,z) \mathbf{e}_{\ell mn} \, dV = \frac{1}{N_\ell N_m N_n} \int_0^1 \int_0^1 \int_0^1 g(x,y,z) \cos \ell\pi x \cos m\pi y \cos n\pi z \, dz \, dy \, dx \\ &= \frac{1}{N_\ell N_m N_n} \int_0^1 \cos \ell\pi x \, dx \int_0^1 \cos m\pi y \, dy \int_0^1 g \cos n\pi z \, dz, \end{aligned}$$

where we have used the fact that g depends only on z. Now we may write

$$\int_0^1 \cos \ell\pi x \, dx = (1, \cos \ell\pi x) = \begin{cases} 1, & \ell = 0 \\ 0, & \ell \neq 0 \end{cases},$$

since both $1 = \cos 0\pi x$ is an element of the orthogonal set $\{\cos \ell\pi x\}_{\ell=0}^{\infty}$. From this, we see that $a_{\ell mn} = 0$ unless $\ell = m = 0$. Further, we see that

$$\int_0^1 g \cos 0\pi z \, dz = \int_0^1 g \, dz = 0,$$

so that $a_{000} = 0$, while if $n \neq 0$

$$\begin{aligned} a_{00n} &= 2 \int_0^1 g \cos n\pi z \, dz = 2 \left(\int_0^{\frac{1}{2}} \cos n\pi z \, dz - \int_{\frac{1}{2}}^1 \cos n\pi z \, dz \right) \\ &= 2 \left(\frac{1}{n\pi} \sin n\pi z \Big|_0^{\frac{1}{2}} - \frac{1}{n\pi} \sin n\pi z \Big|_{\frac{1}{2}}^1 \right) = \frac{4}{n\pi} \sin \frac{n\pi}{2}. \end{aligned}$$

Thus we have finally

$$g(x,y,z) = \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin \frac{n\pi}{2} \cos n\pi z.$$

Now we assume that the solution u to $\nabla^2 u = g$ may be expanded in the basis $\{\mathbf{e}_{\ell mn}\}_{\ell,m,n=0}^{\infty}$ as

$$u = \sum_{\ell,m,n=0}^{\infty} b_{\ell mn} \cos \ell\pi x \cos m\pi y \cos n\pi z;$$

substituting this in, and using the series expansion for g above, we have

$$\sum_{\ell,m,n=0}^{\infty} \lambda_{\ell mn} b_{\ell mn} \mathbf{e}_{\ell mn} = \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin \frac{n\pi}{2} \cos n\pi z;$$

from this we see, first of all, that

$$\lambda_{000} b_{000} = a_{000} = 0;$$

but since $\lambda_{000} = 0$, this tells us nothing about b_{000}. Thus b_{000} is not determined by the boundary conditions on u. We note also that had g been such that $a_{000} \neq 0$ – which, unravelling everything, amounts to saying $\int_Q g\, dV \neq 0$ – then the above equation would become

$$\lambda_{000} b_{000} = 0 = a_{000} \neq 0,$$

which has no solution. If we recall our abstract formula for the solution to Poisson's equation $nabla^2 u = g$,

$$u = \sum_I \frac{1}{\lambda_I} \frac{(g, e_I)}{(e_I, e_I)} e_I,$$

we see that this is exactly the condition that $(g, e_I) = 0$ for all I for which λ_I vanishes, while also the coefficients in the series for u corresponding to such I are undetermined. These are common difficulties when the Laplacian has a zero eigenvalue.

Proceeding to the nonzero eigenvalues, we see that $b_{\ell mn} = 0$ unless $\ell = m = 0$, while for $n \neq 0$

$$b_{00n} = \frac{1}{\lambda_{\ell mn}} \frac{4}{n\pi} \sin \frac{n\pi}{2} = -\frac{4}{n^3\pi^3} \sin \frac{n\pi}{2}.$$

Thus we have the series solution

$$u = b_{000} - \sum_{n=1}^{\infty} \frac{4}{n^3\pi^3} \sin \frac{n\pi}{2} \cos n\pi z.$$

To determine b_{000}, we apply the final condition, noting that $\sin \frac{n\pi}{2} \cos \frac{n\pi}{2} = \frac{1}{2} \sin n\pi = 0$ for all $n \in \mathbf{Z}$:

$$u(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = b_{000} - \sum_{n=1}^{\infty} \frac{4}{n^3\pi^3} \sin \frac{n\pi}{2} \cos \frac{n\pi}{2} = b_{000} = 0,$$

so that finally we have the solution

$$u(x, y, z) = -\sum_{n=1}^{\infty} \frac{4}{n^3\pi^3} \sin \frac{n\pi}{2} \cos n\pi z.$$

2. Compute the Fourier transforms of the following functions:

$$f(x) = \begin{cases} 1, & x \in [-1, 1] \\ 0, & \text{otherwise} \end{cases}.$$

$$f(x) = \begin{cases} 1 - |x|, & x \in [-1, 1] \\ 0, & \text{otherwise} \end{cases}.$$

$$f(r, \theta, \phi) = \begin{cases} 1, & r \leq 1 \\ 0, & \text{otherwise} \end{cases}.$$

$$f(x) = e^{-ax^2}, \quad a \in \mathbf{R}, a > 0.$$

$$f(r, \theta, \phi) = e^{-ar^2}, \quad a \in \mathbf{R}, a > 0.$$

$$f(x) = xe^{-ax^2}, \quad a \in \mathbf{R}, a > 0.$$

[For the fifth of these, it may be simpler to change to rectangular coordinates.]

We take these one by one:

$$\begin{aligned}\mathcal{F}[f](k) &= \int_{-\infty}^{\infty} f(x)e^{-2\pi ikx}\,dx = \int_{-1}^{1} e^{-2\pi ikx}\,dx = \frac{i}{2\pi k}e^{-2\pi ikx}\Big|_{-1}^{1} \\ &= \frac{i}{2\pi k}\left(e^{-2\pi ik} - e^{2\pi ik}\right) = \frac{\sin 2\pi k}{\pi k}.\end{aligned}$$

The above calculation only works for $k \neq 0$; but for $k = 0$ we have clearly $\mathcal{F}[f](0) = 2$, which is the limit of the above function as $k \to 0$. Thus we have

$$\mathcal{F}[f](k) = \begin{cases} 2, & k = 0 \\ \frac{\sin 2\pi k}{\pi k}, & k \neq 0 \end{cases}.$$

This function of k is closely related to the so-called sinc function, which is useful in many different places. We shall typically just write it as $\frac{\sin 2\pi k}{\pi k}$, with the understanding that its value at $k = 0$ is taken to be 2. (We note that with this definition it is actually an analytic function of k with a power series expansion convergent on the entire real line, or complex plane.)

Next, we have

$$\mathcal{F}[f](k) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ikx}\,dx = \int_{-1}^{1} (1 - |x|)\,e^{-2\pi ikx}\,dx.$$

To compute this integral, we note that for $k \neq 0$

$$\int xe^{-2\pi ikx}\,dx = -\frac{1}{2\pi ik}xe^{-2\pi ikx} + \frac{1}{2\pi ik}\int e^{-2\pi ikx}\,dx = \left(-\frac{1}{2\pi ik}x + \frac{1}{4\pi^2k^2}\right)e^{-2\pi ikx} + C,$$

while when $k = 0$

$$\int xe^{-2\pi ikx}\,dx = \int x\,dx = \frac{1}{2}x^2 + C.$$

Thus the above integrals become

$$\int_{-1}^{1} (1 - |x|)\,e^{-2\pi ikx}\,dx = \int_{-1}^{1} e^{-2\pi ikx}\,dx + \int_{-1}^{0} xe^{-2\pi ikx}\,dx - \int_{0}^{1} xe^{-2\pi ikx}\,dx;$$

the first of these is just $\frac{\sin 2\pi k}{\pi k}$, while the second two give

$$\begin{aligned}&\left(-\frac{1}{2\pi ik}x + \frac{1}{4\pi^2k^2}\right)e^{-2\pi ikx}\Big|_{-1}^{0} - \left(-\frac{1}{2\pi ik}x + \frac{1}{4\pi^2k^2}\right)e^{-2\pi ikx}\Big|_{0}^{1} \\ &\qquad = \frac{1}{4\pi^2k^2} - \left(\frac{1}{2\pi ik} + \frac{1}{4\pi^2k^2}\right)e^{2\pi ik} - \left(\left(-\frac{1}{2\pi ik} + \frac{1}{4\pi^2k^2}\right)e^{-2\pi ik} - \frac{1}{4\pi^2k^2}\right) \\ &\qquad = \frac{1}{2\pi^2k^2}\left(1 - \frac{1}{2}\left(e^{2\pi ik} + e^{-2\pi ik}\right)\right) - \frac{1}{\pi k}\frac{1}{2i}\left(e^{2\pi ik} - e^{-2\pi ik}\right) \\ &\qquad = \frac{1}{2\pi^2k^2}(1 - \cos 2\pi k) - \frac{\sin 2\pi k}{\pi k},\end{aligned}$$

in the case that $k \neq 0$; when $k = 0$, they give simply

$$\int_{-1}^{0} x\,dx - \int_{0}^{1} x\,dx = \frac{1}{2}x^2\Big|_{-1}^{0} - \frac{1}{2}x^2\Big|_{0}^{1} = -1,$$

which is seen to be the limiting value of the above expression as $k \to 0$. Taking it to have this value at $k = 0$ (as we did with $\frac{\sin 2\pi k}{\pi k}$ above), we have finally

$$\begin{aligned}\mathcal{F}[f](k) &= \frac{\sin 2\pi k}{\pi k} + \frac{1}{2\pi^2k^2}(1 - \cos 2\pi k) - \frac{\sin 2\pi k}{\pi k} \\ &= \frac{1}{2\pi^2k^2}(1 - \cos 2\pi k).\end{aligned}$$

(We note that, defining this function to have its limiting value at $k = 0$, it is also analytic.)

Proceeding, we have

$$\mathcal{F}[f](\mathbf{k}) = \int_{\mathbf{R}^3} f(r, \theta, \phi) e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x}.$$

Fix some $\mathbf{k} \in \mathbf{R}^3$. Now since f is spherically symmetric, we may assume that our spherical coordinate system (r, θ, ϕ) is such that in it $\mathbf{k} = (k, 0, 0)$, i.e., that $\mathbf{k}$ points along the positive z axis. In this case, $\mathbf{k} \cdot \mathbf{x} = kr\cos\theta$, and the above integral may be written

$$\int_0^{2\pi} \int_0^{\pi} \int_0^1 e^{-2\pi i kr\cos\theta} r^2 \sin\theta \, dr\, d\theta\, d\phi$$

which may be evaluated as

$$\begin{aligned}
2\pi \int_0^1 \frac{1}{2\pi i k} r e^{-2\pi i kr\cos\theta}\Big|_0^{\pi} dr &= 2\pi \int_0^1 \frac{1}{2\pi i k} r\left(e^{2\pi i kr} - e^{-2\pi i kr}\right) dr = 2\pi \int_0^1 \frac{r\sin 2\pi kr}{\pi k}\, dr \\
&= \frac{2}{k}\left(-r\frac{\cos 2\pi kr}{2\pi k}\Big|_0^1 + \frac{1}{2\pi k}\int_0^1 \cos 2\pi kr\, dr\right) \\
&= \frac{2}{k}\left(-\frac{\cos 2\pi k}{2\pi k} + \frac{1}{4\pi^2 k^2}\sin 2\pi kr\Big|_0^1\right) = \frac{2}{k}\left(-\frac{\cos 2\pi k}{2\pi k} + \frac{\sin 2\pi k}{4\pi^2 k^2}\right) \\
&= \frac{1}{2\pi^2 k^3}\left(-2\pi k\cos 2\pi k + \sin 2\pi k\right),
\end{aligned}$$

for $k \neq 0$, while if $k = 0$ it is clearly just $\frac{4}{3}\pi$, the volume of the unit sphere; and we note that this is just the limit of the above expression as $k \to 0$:

$$\begin{aligned}
\frac{1}{2\pi^2 k^3}\left(-2\pi k\cos 2\pi k + \sin 2\pi k\right) &= \frac{1}{2\pi^2 k^3}\left(-2\pi k + \pi k\,(2\pi k)^2 - \cdots + 2\pi k - \frac{1}{6}(2\pi k)^3 + \cdots\right) \\
&= \frac{1}{2\pi^2 k^3}\left(4\pi^3 k^3 - \frac{4}{3}\pi^3 k^3 + \cdots\right) = \frac{\frac{8}{3}\pi^3 k^3 + \cdots}{2\pi^2 k^3} = \frac{4}{3}\pi + \cdots,
\end{aligned}$$

where $\cdots$ indicates terms of order in k higher than those preceding. This expression thus clearly approaches $\frac{4}{3}\pi$ as $k \to 0$, as claimed.

Continuing with fortitude, we have, noting the Gaussian integral

$$\int_{\mathbf{R}^1} e^{-ax^2}\, dx = \sqrt{\frac{\pi}{a}}$$

(which holds for all complex a with $\Re a > 0$)

$$\begin{aligned}
\mathcal{F}[f](k) &= \int_{-\infty}^{\infty} e^{-ax^2} e^{-2\pi i kx}\, dx \\
&= \int_{-\infty}^{\infty} e^{-a\left(x + \frac{\pi i k}{a}\right)^2 - \frac{\pi^2 k^2}{a}}\, dx = e^{-\frac{\pi^2 k^2}{a}} \int_{-\infty}^{\infty} e^{-a\left(x + \frac{\pi i k}{a}\right)^2}\, dx \\
&= \sqrt{\frac{\pi}{a}} e^{-\frac{\pi^2 k^2}{a}},
\end{aligned}$$

where we have used the substitution $u = x + \frac{\pi i k}{a}$ in the last equality. (This can be justified more rigorously in the context of complex variable theory by thinking of adjusting the contour $z = t$ to the contour $z = t + \frac{\pi i k}{a}$ bit by bit, and noting that the integrand rapidly goes to zero as $t \to \pm\infty$ along either contour.) We note that the width of the Gaussian giving the Fourier transform is proportional to the reciprocal of the width of the original Gaussian; this is a manifestation of the celebrated *uncertainty principle*, which is probably best known from quantum mechanics but can also be formulated as a theorem on Fourier transforms (since, we note for those who have seen some quantum mechanics, the momentum-space representation of the wavefunction is essentially just the Fourier transform of its position-space representation).

Continuing, and using the hint, we have

$$\begin{aligned}\mathcal{F}[f](\mathbf{k}) &= \int_{\mathbf{R}^3} e^{-ar^2} e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x} \\ &= \int_{\mathbf{R}^3} e^{-a\left(x^2+y^2+z^2\right)} e^{-2\pi i(k_1 x + k_2 y + k_3 z)}\, d\mathbf{x},\end{aligned}$$

which is easily seen to be a product of three transforms of Gaussians; in other words, we have

$$\mathcal{F}[f](\mathbf{k}) = \left(\frac{\pi}{a}\right)^{\frac{3}{2}} e^{-\frac{\pi^2\left(k_1^2+k_2^2+k_3^2\right)}{a}} = \left(\frac{\pi}{a}\right)^{\frac{3}{2}} e^{-\frac{\pi^2|\mathbf{k}|^2}{a}}.$$

For the final Fourier transform, we could proceed directly, but that would be quite a nuisance; instead we use a property of the Fourier transform to write

$$\begin{aligned}\mathcal{F}[xe^{-ax^2}](k) &= \mathcal{F}\left[-\frac{1}{2a}\frac{d}{dx}\left(e^{-ax^2}\right)\right](k) = -\frac{1}{2a} 2\pi i k \sqrt{\frac{\pi}{a}} e^{-\frac{\pi^2 k^2}{a}} \\ &= -ik\left(\frac{\pi}{a}\right)^{\frac{3}{2}} e^{-\frac{\pi^2 k^2}{a}}.\end{aligned}$$

This formula is related to the properties of the so-called *Hermite polynomials* discussed in section 5.2.8 of the textbook.

Summary:

- We use the notion of approximate identities introduced last week to prove a version of the Fourier inversion theorem.
- We then use Fourier transforms to study the heat equation, obtaining both integral formulas for solutions to the homogeneous and inhomogeneous equations as well as qualitative information.

A THEOREM ON APPROXIMATE IDENTITIES. We have the following generalisation of Example (c) from last week's lecture notes.

THEOREM 1. Let $\psi : \mathbf{R}^m \to \mathbf{R}^1$ satisfy $\int_{\mathbf{R}^m} |\psi(\mathbf{x})|\, d\mathbf{x} < \infty$, $\int_{\mathbf{R}^m} \psi(\mathbf{x})\, d\mathbf{x} = 1$. Then the sequence $\{\psi_n\}_{n=1}^{\infty}$ given by

$$\psi_n(\mathbf{x}) = n^m \psi(n\mathbf{x})$$

is an approximate identity, at least for continuous, bounded functions (i.e., elements of the space $C_b(\mathbf{R}^m)$ to be introduced momentarily).

Proof. The proof is almost identitical to that of the case $m = 1$. Let $f : \mathbf{R}^m \to \mathbf{R}^1$ be bounded and continuous. Then we see that

$$\int_{\mathbf{R}^m} f(\mathbf{x})\psi_n(\mathbf{x})\, d\mathbf{x} = \int_{\mathbf{R}^m} f(\mathbf{x})\psi(n\mathbf{x})n^m\, d\mathbf{x} = \int_{\mathbf{R}^m} f\left(\frac{\mathbf{u}}{n}\right)\psi(\mathbf{u})\, d\mathbf{u},$$

where we have made the change of variables $\mathbf{u} = n\mathbf{x}$, which gives $d\mathbf{u} = n^m\, d\mathbf{x}$ since we are working on $\mathbf{R}^m$. Now this integral can be broken down as follows:

$$\int_{\mathbf{R}^m} f\left(\frac{\mathbf{u}}{n}\right)\psi(\mathbf{u})\, d\mathbf{u} = \int_{\mathbf{R}^m} \left[f\left(\frac{\mathbf{u}}{n}\right) - f(0)\right]\psi(\mathbf{u})\, d\mathbf{u} + f(0),$$

since $\int_{\mathbf{R}^m} \psi(\mathbf{u})\, d\mathbf{u} = 1$. It thus suffices to show that the first term on the right-hand side above approaches 0 as $n \to \infty$. Let $M = \sup_{\mathbf{x}\in\mathbf{R}^m} |f(\mathbf{x})| + 1$ (where $\sup_{\mathbf{x}\in\mathbf{R}^m} |f(\mathbf{x})|$ denotes the least upper bound for $|f(\mathbf{x})|$ on $\mathbf{R}^m$), let $\epsilon > 0$, let $\delta > 0$ be such that $|f(\mathbf{x}) - f(0)| < \frac{\epsilon}{2\int_{\mathbf{R}^m} |\psi(\mathbf{u})|\, d\mathbf{u}}$ when $|\mathbf{x}| < \delta$, and let $K \in \mathbf{Z}$, $K > 0$ be such that

$$\int_{|\mathbf{x}|>K} |\psi(\mathbf{x})|\, d\mathbf{x} < \frac{\epsilon}{2M};$$

such a K clearly exists since $\int_{\mathbf{R}^m} |\psi(\mathbf{x})|\, d\mathbf{x} < \infty$. Furthermore, let $N \in \mathbf{Z}$, $N > 0$ be such that $N > \frac{K}{\delta}$, and let $n > N$. Now we have

$$\begin{aligned}
\left|\int_{\mathbf{R}^m} \left[f\left(\frac{\mathbf{u}}{n}\right) - f(0)\right]\psi(\mathbf{u})\, d\mathbf{u}\right| &\leq \int_{\mathbf{R}^m} \left|\left[f\left(\frac{\mathbf{u}}{n}\right) - f(0)\right]\right| |\psi(\mathbf{u})|\, d\mathbf{u} \\
&= \int_{|\mathbf{x}|<K} \left|\left[f\left(\frac{\mathbf{u}}{n}\right) - f(0)\right]\right| |\psi(\mathbf{u})|\, d\mathbf{u} + \int_{|x|>K} \left|\left[f\left(\frac{\mathbf{u}}{n}\right) - f(0)\right]\right| |\psi(\mathbf{u})|\, d\mathbf{u} \\
&\leq \frac{\epsilon}{2\int_{\mathbf{R}^m} |\psi(\mathbf{u})|\, d\mathbf{u}} \int_{|\mathbf{x}|<K} |\psi(\mathbf{u})|\, d\mathbf{u} + 2M \int_{|\mathbf{x}|>K} |\psi(\mathbf{u})|\, d\mathbf{u} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}$$

where we have used the fact that $n > N$ implies $\frac{K}{n} < \delta$, and replaced the integral over $|\mathbf{x}| < K$ one over $\mathbf{R}^m$ in the last line. This completes the proof. QED.

The basic idea here is that the function $f\left(\frac{\mathbf{u}}{n}\right)$ looks like a very 'zoomed-in' version of f, so that since ψ needs to be concentrated somewhere finite, if we zoom in f enough it will eventually cover essentially all of the places where ψ is not trivially small; and since f is continuous, zooming in like this makes it look very close to the single number $f(0)$, and since $\int_{\mathbf{R}^m} \psi(\mathbf{x})\, d\mathbf{x} = 1$, the resulting integral will be very close to $f(0)$. The foregoing ϵ-δ proof merely makes this rigorous.

A WORD ON FUNCTION SPACES, AND THE NATURE OF THE FOURIER TRANSFORM. We recall that we have defined the space

$$L^1(\mathbf{R}^m) = \{f : \mathbf{R}^m \to \mathbf{R}^1 | \int_{\mathbf{R}^m} |f(\mathbf{x})|\, d\mathbf{x} < \infty\}.$$

We now define the space of bounded continuous functions on $\mathbf{R}^m$:

$$C_b(\mathbf{R}^m) = \{f : \mathbf{R}^m \to \mathbf{R}^1 | f \text{ is bounded and continuous on } \mathbf{R}^m\}.$$

(Both spaces could also be defined with real-valued functions replaced by complex-valued ones; in that case, $|f(\mathbf{x})|$ in the definition of $L^1(\mathbf{R}^m)$ means the modulus of the complex number $f(\mathbf{x})$.)

We are now in a position to say more precisely what exactly the Fourier transform *is*. First of all, we recall that a *function* f from a set A to a set B is a rule which assigns to each element $a \in A$ an element $f(a) \in B$. Now the Fourier transform is a *function on functions*, in the sense that for every function in a certain class it gives another function in another class. We have shown how to define $\mathcal{F}[f]$ for any $f \in L^1(\mathbf{R}^m)$; the result is another function on $\mathbf{R}^m$ whose rule is

$$\mathcal{F}[f](\mathbf{k}) = \int_{\mathbf{R}^m} f(\mathbf{x}) e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x}.$$

Now we claim that for $f \in L^1(\mathbf{R}^m)$, $\mathcal{F}[f] \in C_b(\mathbf{R}^m)$. That $\mathcal{F}[f]$ is bounded can be seen easily: for any $\mathbf{k} \in \mathbf{R}^m$,

$$|\mathcal{F}[f](\mathbf{k})| = |\int_{\mathbf{R}^m} f(\mathbf{x}) e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x}| \le \int_{\mathbf{R}^m} |f(\mathbf{x}) e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}|\, d\mathbf{x} = \int_{\mathbf{R}^m} |f(\mathbf{x})|\, d\mathbf{x},$$

and this last quantity is finite since $f \in L^1(\mathbf{R}^m)$. Since it is independent of $\mathbf{k}$, we see that $\mathcal{F}[f]$ is indeed bounded on $\mathbf{R}^m$, as claimed. To see that it is also continuous on $\mathbf{R}^m$, we may proceed as follows: let $\mathbf{k}_0 \in \mathbf{R}^m$; then

$$\begin{aligned}\lim_{\mathbf{k}\to\mathbf{k}_0} \mathcal{F}[f](\mathbf{k}) &= \lim_{\mathbf{k}\to\mathbf{k}_0} \mathcal{F}[f](\mathbf{k}) = \lim_{\mathbf{k}\to\mathbf{k}_0} \int_{\mathbf{R}^m} f(\mathbf{x}) e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x} \\ &= \int_{\mathbf{R}^m} f(\mathbf{x}) \lim_{\mathbf{k}\to\mathbf{k}_0} e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x} = \int_{\mathbf{R}^m} f(\mathbf{x}) e^{-2\pi i \mathbf{k}_0\cdot\mathbf{x}}\, d\mathbf{x} = \mathcal{F}[f](\mathbf{k}_0),\end{aligned}$$

where we can interchange the limit with the integral since $f \in L^1(\mathbf{R}^m)$[1]. This shows that $\mathcal{F}[f]$ is continuous on $\mathbf{R}^m$, and hence that $\mathcal{F}[f] \in C_b(\mathbf{R}^m)$, as claimed.

The foregoing shows that we may think of the Fourier transform $\mathcal{F}$ as a *function on functions*, or perhaps better put, a *transformation* or *map* on functions which takes elements of $L^1(\mathbf{R}^m)$ to elements of $C_b(\mathbf{R}^m)$.[2] It can be shown that the Fourier transform actually maps into the subspace of $C_b(\mathbf{R}^m)$ consisting of those functions which *go to zero at infty* in a certain sense, but we shall not show that here.

FOURIER INVERSION THEOREM. A version of the Fourier inversion theorem was stated at the end of last week's notes; here we shall prove the following slightly modified version.

THEOREM 2. Suppose that $f \in L^1(\mathbf{R}^m) \cap C_b(\mathbf{R}^m)$ (i.e., that f is in both L^1 and C^b), and that $\hat{f} \in L^1$. Then we have

$$f(\mathbf{x}) = \int_{\mathbf{R}^m} \hat{f}(\mathbf{k}) e^{2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k}.$$

Proof. This may be shown by using a particular approximate identity. (The one we shall use here is not the only option, incidentally; actually there is a very broad range of possibilities.) For convenience, if $\mathbf{k} \in \mathbf{R}^m$ we shall write $k = |\mathbf{k}|$ for the norm of $\mathbf{k}$. We work from the right-hand side to the left-hand side. Now[3] since $\hat{f} \in L^1(\mathbf{R}^m)$, we may write

$$\int_{\mathbf{R}^m} \hat{f}(\mathbf{k}) e^{2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k} = \lim_{n\to\infty} \int_{\mathbf{R}^m} \hat{f}(\mathbf{k}) e^{-\frac{k^2}{n^2}} e^{2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k},$$

[1]Again, this technically requires that we use the dominated convergence theorem for the Lebesgue integral.
[2]We note in passing that the inequality $|\mathcal{F}[f](\mathbf{k})| \le \int_{\mathbf{R}^m} |f(\mathbf{x})|\, d\mathbf{x}$ implies that $\mathcal{F}$ is in fact a *continuous* map from L^1 to C_b, at least if we use appropriate norms to give these spaces topologies.
[3]Applying again the dominated convergence theorem of Lebesgue integration theory!

since in the limit the quantity $-\frac{k^2}{n^2} \to 0$, so the exponential approaches 1. Substituting in the definition of $\hat{f}(\mathbf{k})$, we have

$$\int_{\mathbf{R}^m} \hat{f}(\mathbf{k}) e^{-\frac{k^2}{n^2}} e^{2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k} = \int_{\mathbf{R}^m} \left[\int_{\mathbf{R}^m} f(\mathbf{x}') e^{-2\pi i \mathbf{k}\cdot\mathbf{x}'}\, d\mathbf{x}' \right] e^{-\frac{k^2}{n^2}} e^{2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k};$$

now because of the factor $e^{-\frac{k^2}{n^2}}$, the integrand is in fact integrable over the product $\mathbf{R}^m \times \mathbf{R}^m$, which implies that we can interchange the order of integration, obtaining

$$\int_{\mathbf{R}^m} \left[\int_{\mathbf{R}^m} e^{-\frac{k^2}{n^2}} e^{-2\pi i \mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})}\, d\mathbf{k} \right] f(\mathbf{x}')\, d\mathbf{x}'.$$

Now the integral in brackets is seen to be the Fourier transform of the Gaussian function $e^{-\frac{k^2}{n^2}}$, evaluated at the point $\mathbf{x}' - \mathbf{x}$. From the results on homework 10, this is seen to be

$$\left(\pi n^2\right)^{\frac{m}{2}} e^{-\pi^2 n^2 |\mathbf{x}'-\mathbf{x}|^2}. \tag{1}$$

Thus the full integral above becomes

$$\int_{\mathbf{R}^m} \left(\pi n^2\right)^{\frac{m}{2}} f(\mathbf{x}') e^{-\pi^2 n^2 |\mathbf{x}-\mathbf{x}'|^2}\, d\mathbf{x}'.$$

Now setting

$$\psi(\mathbf{x}) = \pi^{\frac{m}{2}} e^{-\pi^2 |\mathbf{x}|^2},$$

and noting that $\psi \in L^1(\mathbf{R}^m)$, $\int_{\mathbf{R}^m} \psi(\mathbf{x})\, d\mathbf{x} = 1$, and that the function in (1) above is just $\psi_n(\mathbf{x})$ as defined in Theorem 1, we see that, by Theorem 1, we have finally

$$\int_{\mathbf{R}^m} \hat{f}(\mathbf{k}) e^{2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k} = \lim_{n\to\infty} \int_{\mathbf{R}^m} \left(\pi n^2\right)^{\frac{m}{2}} e^{-\pi^2 n^2 |\mathbf{x}-\mathbf{x}'|^2} f(\mathbf{x}')\, d\mathbf{x}' = f(\mathbf{x}),$$

as desired. QED.

The transformation on functions which takes a function $f(\mathbf{k})$ in $L^1(\mathbf{R}^m)$ to the function

$$\int_{\mathbf{R}^m} f(\mathbf{k}) e^{2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k}$$

is called the *inverse Fourier transform* and is denoted $\mathcal{F}^{-1}[f]$. The foregoing shows that, if $f \in L^1(\mathbf{R}^m) \cap C_b(\mathbf{R}^m)$, then $\mathcal{F}^{-1}[\mathcal{F}[f]] = f$, i.e., that $\mathcal{F}^{-1}$ is indeed a left inverse to $\mathcal{F}$. Identical arguments to those in the proof just given show that also $\mathcal{F}[\mathcal{F}^{-1}[f]] = f$ for such f. These formulas are also correct much more generally: in fact, if $f \in L^1(\mathbf{R}^m)$ is any function satisfying $\int_{\mathbf{R}^m} |f(\mathbf{x})|^2\, d\mathbf{x} < \infty$, then these relations still hold for f. We shall, however, not pursue such questions here but merely regard the above result as being an example of the results which can be obtained. For the most part we shall work with Fourier transforms and their inverses rather more formally.

HEAT EQUATION ON R^m. Consider the following problem on $(0, +\infty) \times \mathbf{R}^m$ (points of which we shall denote as $(t, \mathbf{x})$):

$$\frac{\partial u}{\partial t} = \nabla^2 u, \quad u|_{t=0} = f.$$

Suppose that $f \in L^1(\mathbf{R}^m)$, and suppose that u and all of its derivatives up to second order are in $L^1(\mathbf{R}^m)$[4]; then, taking the Fourier transform of the above equation, we obtain (assuming that we may interchange the order of integration and differentiation with respect to t)

$$\frac{\partial \hat{u}}{\partial t} = -4\pi^2 |\mathbf{k}|^2 \hat{u}, \quad \hat{u}|_{t=0} = \hat{f}.$$

[4]All we really need, of course, is for u to be such that we can take the Fourier transforms needed below. The given conditions are sufficient but probably not necessary.

Now the first equation is almost identical to the one we found when solving the heat equation on the unit cube, and has the solution

$$\hat{u}(t,\mathbf{k}) = \hat{u}(0,\mathbf{k})e^{-4\pi^2|\mathbf{k}|^2 t} = \hat{f}(\mathbf{k})e^{-4\pi^2|\mathbf{k}|^2 t}.$$

Assuming that we may apply Fourier inversion, this gives rise immediately to the integral expression

$$u = \int_{\mathbf{R}^m} \hat{f}(\mathbf{k})e^{-4\pi^2|\mathbf{k}|^2 t}e^{2\pi i\mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k}.$$

This expression, however, is rather unsatisfactory, since calculating $\hat{f}$ requires us to perform a rather difficult integral, and then we are still faced with evaluating the above integral in order to finally obtain u; in other words, the above expression requires two integrations. We may use properties of the Fourier transform to reduce this to one, as follows. First, we note that

$$\begin{aligned}\mathcal{F}^{-1}[e^{-4\pi^2|\mathbf{k}|^2 t}](\mathbf{x}) &= \int_{\mathbf{R}^m} e^{-4\pi^2|\mathbf{k}|^2 t}e^{2\pi i\mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k} = \int_{\mathbf{R}^m} e^{-4\pi^2|\mathbf{k}|^2 t}e^{-2\pi i\mathbf{k}\cdot(-\mathbf{x})}\, d\mathbf{k} \\ &= \left(\frac{\pi}{4\pi^2 t}\right)^{\frac{m}{2}} e^{-\frac{\pi^2|-\mathbf{x}|^2}{4\pi^2 t}} = \frac{1}{(4\pi t)^{\frac{m}{2}}}e^{-\frac{|\mathbf{x}|^2}{4t}}.\end{aligned}$$

This last expression is called the *heat kernel*; let us denote it by $K(t,\mathbf{x})$. Thus we see that $\mathcal{F}[K](t,\mathbf{k}) = e^{-4\pi^2|\mathbf{k}|^2 t}$, so that

$$\hat{u}(t,\mathbf{k}) = \mathcal{F}[f](\mathbf{k})\mathcal{F}[K](t,\mathbf{k}) = \mathcal{F}[f * K],$$

where the convolution is performed only on the spatial variables. Fourier inversion then implies that we have

$$\begin{aligned}u(t,\mathbf{x}) = (f*K)(t,\mathbf{x}) = (K*f)(t,\mathbf{x}) &= \int_{\mathbf{R}^m} K(t,\mathbf{x}-\mathbf{x}')f(\mathbf{x}')\, d\mathbf{x}' \\ &= \frac{1}{(4\pi t)^{\frac{m}{2}}}\int_{\mathbf{R}^m} e^{-\frac{|\mathbf{x}-\mathbf{x}'|^2}{4t}} f(\mathbf{x}')\, d\mathbf{x}'.\end{aligned}$$

This is the desired formula for u in terms of f.

In order to apply this formula to concrete examples, of course, we would need to find a function f for which the integral above is actually calculable. There are some examples in the textbook for which the above integral can be determined in terms of the error function; for now we shall just comment on some qualitative properties of solutions to the heat equation which emerge from it. The first of these is the result

$$\lim_{t\to\infty} u(t,\mathbf{x}) = 0;$$

this can be seen from the above formula since the quantity $\frac{1}{(4\pi t)^{\frac{m}{2}}} \to 0$ as $t\to\infty$, while the integral simply approaches $\int_{\mathbf{R}^m} f(\mathbf{x}')\, d\mathbf{x}'$, which is finite since $f \in L^1(\mathbf{R}^m)$. It can actually be seen even more clearly from the formula for the Fourier transform for u above, namely

$$\hat{u}(t,\mathbf{k}) = \hat{f}(\mathbf{k})e^{-4\pi^2|\mathbf{k}|^2 t}:$$

from this formula it is entirely obvious that $\hat{u} \to 0$ as $t \to \infty$, so assuming that the inverse Fourier transform is continuous in an appropriate sense, the same will be true also of u. Next we note that, at least assuming $f \in C_b$,

$$\lim_{t\to 0+} u(t,\mathbf{x}) = f(\mathbf{x}).$$

To prove this fully rigorously would require an extension of Theorem 1 to the case of nonintegral n; we shall content ourselves by investigating the limit[5]

$$\lim_{n\to\infty} u(\frac{1}{n^2},\mathbf{x}).$$

[5] If the limit above exists, it will certainly be equal to the limit below. However, the limit below can exist without the original limit existing (consider, for example, the function $sin(\frac{2\pi}{t})$, which is zero when $t = \frac{1}{n^2}$ but has no limit as $t \to 0$): this is similar to the fact we learned in multivariable calculus, that a function can have a limit at a point along a certain curve without having a full limit at that point.

This is seen to be

$$\lim_{n\to\infty} \frac{n^m}{(4\pi)^{\frac{m}{2}}} \int_{\mathbf{R}^m} e^{-\frac{n^2|\mathbf{x}-\mathbf{x}'|^2}{4}} f(\mathbf{x}')\, d\mathbf{x}';$$

since

$$\int_{\mathbf{R}^m} e^{-\frac{n^2|\mathbf{x}|^2}{4}}\, d\mathbf{x} = (4\pi)^{\frac{m}{2}},$$

we see that we may apply Theorem 1 to conclude that this limit is in fact $f(\mathbf{x})$, as desired.

The foregoing has the following curious consequence: any function $f \in L^1(\mathbf{R}^m) \cap C_b(\mathbf{R}^m)$ is the limit of a sequence of functions which have infinitely many derivatives. To see this, we need the following result about convolutions (which is worth knowing in its own right). Suppose that $f, g \in L^1(\mathbf{R}^m)$, and that $\partial_j f \in L^1(\mathbf{R}^m)$ for some j. Then we have

$$\partial_j (f * g)(\mathbf{x}) = \partial_j \int_{\mathbf{R}^m} f(\mathbf{x}-\mathbf{x}')g(\mathbf{x}')\, d\mathbf{x}' = \int_{\mathbf{R}^m} (\partial_j f)(\mathbf{x}-\mathbf{x}')g(\mathbf{x}')\, d\mathbf{x}' = ((\partial_j f) * g)(\mathbf{x});$$

in other words, $\partial_j (f * g) = (\partial_j f) * g$. Note that we did not need to assume anything about differentiability (or even continuity) of g here; thus this result shows that the convlution of two functions is at least as *smooth* (i.e., possesses at least as many derivatives) as the smoother of the two factors. Now the heat kernel

$$K(t, \mathbf{x}) = \frac{1}{(4\pi t)^{\frac{m}{2}}} e^{-\frac{|x|^2}{4t}}$$

clearly possesses derivatives of all orders in $\mathbf{x}$, for all $t > 0$; since any solution to the heat equation is just the convolution of K with the initial data f, we see that any such solution must have derivatives of all orders in $\mathbf{x}$ for all $t > 0$. In other words, the functions

$$u(\frac{1}{n^2}, \mathbf{x})$$

must have infinitely many derivatives in $\mathbf{x}$ for all n. But these functions converge to f, meaning that f is indeed a limit of functions with infinitely many derivatives, as claimed. We say that the heat equation *smooths out* its initial data. (This is a general property of the class of equations known as parabolic equations of which the heat equation is the simplest example. The wave equation, which we shall study next week, is a member of the class of *hyperbolic equations* and transports singularities rather than smoothing them out.)

Finally, we show how Fourier techniques can be used to solve the inhomogeneous heat equation. To this end, consider the following problem on $\mathbf{R}^m$:

$$\frac{\partial u}{\partial t} = \nabla^2 u + g, \quad u|_{t=0} = f.$$

If we assume as usual that all necessary Fourier transforms exist, then Fourier transforming gives

$$\frac{\partial \hat{u}}{\partial t} = -4\pi^2 |\mathbf{k}|^2 \hat{u} + \hat{g}, \quad \hat{u}|_{t=0} = \hat{f}.$$

The first equation again becomes a simple linear first-order ordinary differential equation which may be solved using the integrating factor $e^{4\pi^2|\mathbf{k}|^2 t}$. Multiplying both sides by this factor and rearranging, we obtain

$$e^{4\pi^2|\mathbf{k}|^2 t}\hat{g} = e^{4\pi^2|\mathbf{k}|^2 t}\frac{\partial \hat{u}}{\partial t} + 4\pi^2|\mathbf{k}|^2 e^{4\pi^2|\mathbf{k}|^2 t}\hat{u} = \frac{\partial}{\partial t}\left[e^{4\pi^2|\mathbf{k}|^2 t}\hat{u}\right],$$

so replacing t by s and integrating with respect to s from 0 to t,

$$\begin{aligned} e^{4\pi^2|\mathbf{k}|^2 s}\hat{u}(s, \mathbf{k})\Big|_{s=0}^{s=t} &= \int_0^t e^{4\pi^2|\mathbf{k}|^2 s}\hat{g}(s, \mathbf{k})\, ds \\ e^{4\pi^2|\mathbf{k}|^2 t}\hat{u}(t, \mathbf{k}) - \hat{u}(0, \mathbf{k}) &= \int_0^t e^{4\pi^2|\mathbf{k}|^2 s}\hat{g}(s, \mathbf{k})\, ds \\ \hat{u}(t, \mathbf{k}) &= e^{-4\pi^2|\mathbf{k}|^2 t}\hat{f}(\mathbf{k}) + \int_0^t e^{-4\pi^2|\mathbf{k}|^2 (t-s)}\hat{g}(s, \mathbf{k}), ds. \end{aligned} \tag{2}$$

The first term is just the expression we obtained before, as it should be since that is just the case $g = 0$, and in that case the second term vanishes. Now the second term looks somewhat like a convolution integral, though not quite because of the limits; it turns out that this type of integral is the kind of convolution appropriate for the so-called *Laplace transform* usually encountered in introductory classes on ordinary differential equations.[6] At any rate, assuming that we may interchange the order of the t integral with the $\mathbf{k}$ integral appearing in $\mathcal{F}^{-1}$, we may take the inverse Fourier transform of this expression as before to obtain

$$u(t, \mathbf{x}) = K(t, \mathbf{x}) * f(\mathbf{x}) + \int_0^t K(t-s, \mathbf{x}) * g(s, \mathbf{x})\, ds,$$

where all convolutions are with respect to the variable $\mathbf{x}$.

As with the case of the homogeneous heat equation above, for this formula to be useful in practice we would need functions f and g for which the above integrals are calculable. An (attempt at an) example of this sort is given in Homework 11. For the moment let us do what we did when we discussed the homogeneous heat equation and see what kinds of qualitative information we can determine from this solution. We see that the first term, which is just the solution of the homogeneous equation with the given initial data, goes to zero as $t \to \infty$ and to $f(\mathbf{x})$ as $t \to 0^+$, as before. Now let us consider the second term. Suppose that $g(t, \mathbf{x}) = g_0(\mathbf{x})$ for all $t \geq 0$. Then $\hat{g}(t, \mathbf{k}) = \hat{g}_0(\mathbf{k})$ for all $\mathbf{k}$. Returning now to the expression for the Fourier transform of u in equation (2) above, we see that

$$\begin{aligned}
4\pi^2|\mathbf{k}|^2\hat{u}(t, \mathbf{k}) &= 4\pi^2|\mathbf{k}|^2 \left[e^{-4\pi^2|\mathbf{k}|^2 t}\hat{f}(\mathbf{k}) + \hat{g}_0(\mathbf{k})e^{-4\pi^2|\mathbf{k}|^2 t}\int_0^t e^{4\pi^2|\mathbf{k}|^2 s}\, ds\right] \\
&= 4\pi^2|\mathbf{k}|^2 e^{-4\pi^2|\mathbf{k}|^2 t}\hat{f}(\mathbf{k}) + \hat{g}_0(\mathbf{k})e^{-4\pi^2|\mathbf{k}|^2 t}e^{4\pi^2|\mathbf{k}|^2 s}\Big|_0^t \\
&= 4\pi^2|\mathbf{k}|^2 e^{-4\pi^2|\mathbf{k}|^2 t}\hat{f}(\mathbf{k}) + \hat{g}_0(\mathbf{k})\left[1 - e^{-4\pi^2|\mathbf{k}|^2 t}\right],
\end{aligned}$$

from which it is clear that in the limit $t \to \infty$ we have

$$4\pi^2|\mathbf{k}|^2\hat{u} = \hat{g}_0(\mathbf{k}).$$

But (assuming that the functions involved are such that we can take the inverse Fourier transform of both sides) this is nothing but the equation $-\nabla^2 u = g_0$! From this we see that (at least for suitable functions f and g_0) in the limit as $t \to \infty$, u converges to the solution to the Poisson equation $\nabla^2 u = -g_0$ on $\mathbf{R}^m$. This should be compared with our earlier result, when working on a bounded region, that if the heat equation were solved with nonhomogeneous boundary conditions, in the limit as $t \to \infty$ the solution would converge to the solution to Laplace's equation on that region with the same boundary conditions. In the current case, since we are solving on the whole space $\mathbf{R}^m$, there are no real boundary conditions (the only relevant one are that u should be in L^1), but our work here shows that a similar result holds for the inhomogeneous heat equation.

[6] The Laplace transform takes account of initial conditions while the Fourier transform extends from $-\infty$ to $+\infty$, i.e., over the whole range of the variable. We might have a chance to say a little bit about the Laplace transform towards the end of the course.

APM 346, Homework 11, solutions. Due Monday, August 5, at 6.00 AM EDT. To be marked completed/not completed.

1. Using the eigenfunctions derived in homework 10, problem 1, construct the Green's function on Q satisfying

$$\nabla^2_{\mathbf{x}} G(\mathbf{x},\mathbf{x}') = -\delta(\mathbf{x}-\mathbf{x}'), \quad \left.\frac{\partial G}{\partial n}\right|_{\mathbf{x}\in\partial Q} = 0,$$

and use it to find a series expansion for the solution to the following problem on Q:

$$\nabla^2 u = \sin 2\pi x \sin 2\pi y \sin 2\pi z, \quad \left.\frac{\partial u}{\partial n}\right|_{\partial Q} = 1.$$

[The following question is worth considering: What would happen if we replaced $\sin 2\pi x \sin 2\pi y \sin 2\pi z$ by $\sin \pi x \sin \pi y \sin \pi z$ above?]

We have the eigenfunctions

$$\mathbf{e}_{\ell mn} = \cos \ell\pi x \cos m\pi y \cos n\pi z, \quad \ell, m, n \in \mathbf{Z}, \ell, m, n \geq 0,$$

with corresponding eigenvalues

$$\lambda_{\ell mn} = -\pi^2 \left(\ell^2 + m^2 + n^2\right).$$

Formally, then, we have the Gren's function

$$G(\mathbf{x},\mathbf{x}') = -\sum_{\ell,m,n} \frac{8}{\lambda_{\ell mn}} \cos \ell\pi x \cos m\pi y \cos n\pi z \cos \ell\pi x' \cos m\pi y' \cos n\pi z'.$$

However, in this case we have a zero eigenvalue $\lambda_{000} = 0$, and the corresponding term in the above sum is undefined. We shall show at the end of this document that the ordinary formulas work just as well in this case, if we drop the terms with $\lambda_{\ell mn} = 0$ from the above sum and assume that $(f, \mathbf{e}_{\ell mn}) = 0$ for all such ℓmn. For the moment we show how to apply this to solve the current problem. We have the Green's function

$$G(\mathbf{x},\mathbf{x}') = \sum_{(\ell,m,n)\neq(0,0,0)} \frac{8}{\pi^2 \left(\ell^2 + m^2 + n^2\right)} \cos \ell\pi x \cos m\pi y \cos n\pi z \cos \ell\pi x' \cos m\pi y' \cos n\pi z'.$$

Now applying the formula

$$u = -\int_Q G(\mathbf{x},\mathbf{x}')\nabla^2 u(\mathbf{x}')\, d\mathbf{x}' + \int_{\partial Q} G(\mathbf{x},\mathbf{x}')\frac{\partial u}{\partial n'} - u(\mathbf{x}')\frac{\partial G}{\partial n'}\, dS',$$

and using the fact that G satisfies homogeneous Neumann conditions on ∂Q, we have

$$u = -\int_Q G(\mathbf{x},\mathbf{x}') \sin 2\pi x' \sin 2\pi y' \sin 2\pi z'\, d\mathbf{x}' + \int_{\partial Q} G(\mathbf{x},\mathbf{x}')\, dS'.$$

Now we note that, for $\ell \neq 2$,

$$\begin{aligned}
\int_0^1 \cos \ell\pi x' \sin 2\pi x'\, dx' &= \frac{1}{2}\int_0^1 \sin\left[(\ell+2)\pi x'\right] - \sin\left[(\ell-2)\pi x'\right]\, dx' \\
&= -\frac{1}{2\pi}\left[\frac{1}{\ell+2}\cos\left[(\ell+2)\pi x'\right] - \frac{1}{\ell-2}\cos\left[(\ell-2)\pi x'\right]\right]\Bigg|_0^1 \\
&= \frac{1}{2\pi}\left[\frac{1}{\ell+2}(1-(-1)^\ell) - \frac{1}{\ell-2}(1-(-1)^\ell)\right] = -\frac{2}{\pi(\ell^2-4)}(1-(-1)^\ell),
\end{aligned}$$

while if $\ell = 2$ we have

$$\int_0^1 \cos 2\pi x' \sin 2\pi x'\, dx' = \frac{1}{2}\int_0^1 \sin 4\pi x'\, dx' = 0.$$

Thus the first integral above becomes

$$\begin{aligned}
-\int_Q & G(\mathbf{x},\mathbf{x}')\sin 2\pi x' \sin 2\pi y' \sin 2\pi z'\, d\mathbf{x}' \\
&= \sum_{\substack{(\ell,m,n)\neq(0,0,0)\\ \ell,m,n\neq 2}} \frac{8}{\pi^2(\ell^2+m^2+n^2)}\left(-\frac{1}{\pi^3(\ell^2-4)(m^2-4)(n^2-4)}\right) \\
&\qquad \cdot 8\cdot(1-(-1)^\ell)(1-(-1)^m)(1-(-1)^n)\cos \ell\pi x \cos m\pi y \cos n\pi z.
\end{aligned}$$

For the second integral, we note that ∂Q is a union of six squares, namely

$$\begin{aligned}
&\{(x,y,z)|x\in\{0,1\},(y,z)\in[0,1]^2\} \\
&\quad \cup \{(x,y,z)|y\in\{0,1\},(x,z)\in[0,1]^2\} \\
&\qquad \cup \{(x,y,z)|z\in\{0,1\},(x,y)\in[0,1]^2\};
\end{aligned}$$

these are the left and right, front and back, and top and bottom sides, respectively. Now the part of the second integral corresponding to the first of these would be

$$\begin{aligned}
\int_0^1\int_0^1 & G(\mathbf{x},\mathbf{x}')|_{x'=0}\, dy'\, dz' + \int_0^1\int_0^1 G(\mathbf{x},\mathbf{x}')|_{x'=1}\, dy'\, dz' \\
&= 8\sum_{\substack{(\ell,m,n)\neq(0,0,0)\\ m,n\neq 2}} \frac{\cos \ell\pi x \cos m\pi y \cos n\pi z}{\pi^2(\ell^2+m^2+n^2)}(1+(-1)^\ell)\int_0^1\int_0^1 \cos m\pi y' \cos n\pi z'\, dy'\, dz' \\
&= 8\sum_{\ell=1}^{\infty}\frac{1+(-1)^\ell}{\pi^2\ell^2}\cos \ell\pi x,
\end{aligned}$$

since the integral is zero unless $m = n = 0$ (and the final sum begins at $\ell = 1$ since we cannot have $\ell = m = n = 0$). Similar results would hold for the integrals over the other pairs of sides. Thus we would have finally the awe-inspiring (or perhaps, ahem, awe-ful!) expression

$$\begin{aligned}
u = -&\sum_{\substack{(\ell,m,n)\neq(0,0,0)\\ \ell,m,n\neq 2}} \frac{64(1-(-1)^\ell)(1-(-1)^m)(1-(-1)^n)}{\pi^5(\ell^2+m^2+n^2)(\ell^2-4)(m^2-4)(n^2-4)}\cos \ell\pi x \cos m\pi y \cos n\pi z \\
&+ 8\sum_{\ell=1}^{\infty}\frac{1+(-1)^\ell}{\pi^2\ell^2}\cos \ell\pi x + 8\sum_{m=1}^{\infty}\frac{1+(-1)^m}{\pi^2 m^2}\cos m\pi y + 8\sum_{n=1}^{\infty}\frac{1+(-1)^n}{\pi^2 n^2}\cos n\pi z.
\end{aligned}$$

[The above solution would not be unique since adding any constant to it will give another solution to the original problem. This constant could be fixed by giving another condition such as the condition $u(\frac{1}{2},\frac{1}{2},\frac{1}{2}) = 0$ which we had in problem 1 of homework 10.]

Unfortunately the above procedure fails to actually give a solution, partly because the problem as stated has in fact no solution (your instructor apparently failed to notice this somehow). The reason for this is easy to see once one thinks about it for a bit. We have the problem

$$\nabla^2 u = \sin 2\pi x \sin 2\pi y \sin 2\pi z, \qquad \left.\frac{\partial u}{\partial n}\right|_{\partial Q} = 1.$$

Let us integrate $\nabla^2 u$ over Q and apply the divergence theorem:

$$\begin{aligned}\int_Q \nabla^2 u\,d\mathbf{x} = \int_Q \nabla\cdot\nabla u\,d\mathbf{x}&\\ &= \int_{\partial Q} \mathbf{n}\cdot\nabla u\,d\mathbf{x} = \int_{\partial Q}\frac{\partial u}{\partial n}\,dS\\ &= \int_{\partial Q} 1\,dS = 6;\end{aligned}$$

but we have also

$$\begin{aligned}\int_Q \nabla^2 u\,d\mathbf{x} &= \int_Q \sin 2\pi x \sin 2\pi y \sin 2\pi z\,d\mathbf{x}\\ &= 0,\end{aligned}$$

a contradiction. In other words, the problem

$$\nabla^2 u = f,\quad \left.\frac{\partial u}{\partial n}\right|_{\partial Q} = g$$

must satisfy the consistency condition

$$\int_Q f\,d\mathbf{x} = \int_{\partial Q} g\,dS$$

in order to have a solution. (This is an extension of the condition $(f, e_I) = 0$ for I such that $\lambda_I = 0$ which we derive in the Appendix.) Since the given f and g do not satisfy this condition, this problem has no solution as stated. We apologise.

2. Using Fourier transforms in space, solve the problem on $(0,+\infty)\times\mathbf{R}^3$

$$\frac{\partial u}{\partial t} = \nabla^2 u + \frac{1}{\sqrt{t}}e^{-\frac{x^2}{4t}},\quad u|_{t=0} = 0.$$

[It is worth considering what would happen if a factor other than 4 were used in the exponent above; but the calculations would become far more involved.]

This problem is actually quite easy (particularly compared to the previous one!). First we recall (see the solutions to Homework 10) that on $\mathbf{R}^3$

$$\mathcal{F}\left[e^{-a|\mathbf{x}|^2}\right](\mathbf{k}) = \left(\frac{\pi}{a}\right)^{\frac{3}{2}} e^{-\frac{\pi^2|\mathbf{k}|^2}{a}};$$

an exactly analogous result holds for the inverse Fourier transform:

$$\mathcal{F}^{-1}\left[e^{-a|\mathbf{k}|^2}\right](\mathbf{x}) = \left(\frac{\pi}{a}\right)^{\frac{3}{2}} e^{-\frac{\pi^2|\mathbf{x}|^2}{a}}$$

(this can either be seen by turning the first result above backwards – i.e., by replacing a by $\frac{\pi^2}{a}$ and moving the multiplicative factor to the left-hand side – or by noting that for a real-valued function f

$$\mathcal{F}^{-1}[f](\mathbf{x}) = \overline{\mathcal{F}[f](\mathbf{x})}.)$$

Thus, assuming that u and sufficiently many of its derivatives have Fourier transforms, we may take the Fourier transform of the above problem to obtain

$$\frac{\partial \hat{u}}{\partial t} = -4\pi^2|\mathbf{k}|^2\hat{u} + \frac{1}{\sqrt{t}}\left(4\pi t\right)^{\frac{3}{2}} e^{-4\pi^2|\mathbf{k}|^2 t},\quad \hat{u}|_{t=0} = 0;$$

multiplying the first equation by $e^{4\pi^2|\mathbf{k}|^2 t}$ and rearranging gives

$$\frac{\partial}{\partial t}\left(e^{4\pi^2|\mathbf{k}|^2 t}\hat{u}\right) = (4\pi)^{\frac{3}{2}} t$$
$$\hat{u} = 4\pi^{\frac{3}{2}} t^2 e^{-4\pi^2|\mathbf{k}|^2 t},$$

so, taking the inverse Fourier transform, we obtain

$$u = 4\pi^{\frac{3}{2}} t^2 \left(\frac{\pi}{4\pi^2 t}\right)^{\frac{3}{2}} e^{-\frac{x^2}{4t}} = 4\pi^{\frac{3}{2}} t^2 t^{-\frac{3}{2}} (4\pi)^{-\frac{3}{2}} e^{-\frac{x^2}{4t}} = \frac{\sqrt{t}}{2} e^{-\frac{x^2}{4t}}.$$

If the coefficient in the exponent in the original problem were not $\frac{1}{4}$, the Gaussian factors would not cancel, but we would still be able to integrate because of the t factor. (Note that if we were working in any dimension other than 3 the t factor would become t^α for some $\alpha \neq 1$, and we would not in general be able to integrate in closed form.)

3. [Optional.] By analogy with our derivation in class of the eigenfunctions of the Laplacian on the cylinder C, derive the eigenfunctions and eigenvalues of the Laplacian on the disk $D = \{(r,\theta) | r < 1\}$ satisfying Dirichlet boundary conditions. Now consider the wave equation on D with Dirichlet boundary conditions:

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u, \quad u|_{\partial D} = 0.$$

Find the set of all possible frequencies f such that the above problem has a solution of the form $e^{2\pi i f t}\Phi(r,\theta)$ for some function $\Phi(r,\theta)$. These are the *natural frequencies* for a circular drumhead: they are the frequencies at which it can oscillate continuously (ignoring losses due to heating in the drumhead and the transmitting of energy from the drumhead to the air to create the sound waves which we actually hear, of course). Any forced motion at another frequency would rapidly die out.

[Sketch.] Since the Laplacian in polar coordinates is the same as the Laplacian in cylindrical coordinates except that it lacks the $\frac{\partial^2}{\partial z^2}$ term, we see the the eigenfunctions for the Laplacian in polar coordinates are simply

$$J_m(\lambda_{mi}\rho)\cos m\phi, \quad J_m(\lambda_{mi}\rho)\sin m\phi,$$

with eigenvalues

$$\lambda = -\lambda_{mi}^2.$$

(It is worthwhile to derive these results by working directly from separation of variables in polar coordinates.) Now suppose that we have a solution u to the above problem which is of the form $u = e^{2\pi i f t}\Phi(r,\theta)$; substituting in, we obtain for Φ the problem

$$-4\pi^2 f^2 \Phi = \nabla^2\Phi, \quad \Phi|_{\partial D} = 0.$$

The left-hand equation here is known as the *Helmholtz equation*, and is easily seen to be simply the eigenvalue problem for the Laplacian on the unit disk. By the foregoing, then, we see that we must have, for some m, i,

$$-4\pi^2 f^2 = -\lambda_{mi}^2,$$
$$f = \pm\frac{1}{2\pi}\lambda_{mi}.$$

[We now have enough background to appreciate at least part of the following question, which arises in the study of inverse problems, and was posed by the mathematician Mark Kac: Can one hear the shape of a drum? More precisely, suppose that for some region D in the plane we are given the set of all possible frequencies f for which the wave equation on D possesses solutions with the single frequency f, i.e., possesses solutions of the form above, $e^{2\pi i f t}\Phi(r,\theta)$. The question then is whether this set of frequencies uniquely determines D. (More generally, one considers a so-called Riemannian manifold and the generalised Laplacian on it.) The answer, as the author saw it put in a course prospectus when he was at Cambridge a long time ago, is No, but Almost Yes. Unfortunately that about exhausts the knowledge of the current author on the subject!]

Appendix: Green's functions in the presence of zero eigenvalues. Let us suppose that the eigenvalue problem

$$\nabla^2 u = \lambda u, \quad \left.\frac{\partial u}{\partial n}\right|_{\partial D} = 0$$

on some region D has a zero eigenvalue, i.e., that there is a nonzero function e which satisfies the above with $\lambda = 0$. We would like to find a Green's function in this case; but the standard formula would involve a division by zero, as noted in the solutions to problem 1 above. To derive an appropriate formula for the solution in this case, we go back to first principles. Suppose that $\{\mathbf{e}_I\}$ is a complete set of eigenfunctions for the above problem with $(\mathbf{e}_I, \mathbf{e}_I) = 1$, with corresponding eigenvalues λ_I. Let $\mathbf{I}_0 = \{I | \lambda_I = 0\}$. Let us consider first the homogeneous problem

$$\nabla^2 u = f, \quad \left.\frac{\partial u}{\partial n}\right|_{\partial D} = 0.$$

If we write $f = \sum_I b_I \mathbf{e}_I$, $u = \sum_I a_I \mathbf{e}_I$, then substituting in gives

$$\sum_I \lambda_I a_I \mathbf{e}_I = \sum_I b_I \mathbf{e}_I.$$

Since $\{e_I\}$ is an orthogonal set of nonzero functions, this gives for $I \notin \mathbf{I}_0$ that $a_I = \frac{1}{\lambda_I} b_I$, which is the same as we had before. If, however, $I \in \mathbf{I}_0$, then this relation gives instead $b_I = 0$; i.e., it becomes a restriction on the functions f for which the problem has a solution, rather than information about the solution. We assume that f is such that $b_I = (f, e_I) = 0$ for all $I \in \mathbf{I}_0$, so that this condition is satisfied. We note also that a_I is undetermined for $I \in \mathbf{I}_0$. Thus we may write

$$u = \sum_{I \notin \mathbf{I}_0} \frac{1}{\lambda_I} b_I \mathbf{e}_I + \sum_{I \in \mathbf{I}_0} a_I \mathbf{e}_I.$$

The second sum can only be determined by auxiliary information (for example, in problem 1 of homework 10 the condition $u(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = 0$ allowed us to determine that part of the sum). Thus here we drop it and consider only the first term. We have, as in our previous derivation of the Green's function, that (assuming as usual that we may interchange integration and summation)

$$u = \sum_{I \notin \mathbf{I}_0} \frac{1}{\lambda_I} \int_D f(\mathbf{x}') \overline{\mathbf{e}_I(\mathbf{x}')}\, d\mathbf{x}' \mathbf{e}_I(\mathbf{x}) = \int_D f(\mathbf{x}') \left[\sum_{I \notin \mathbf{I}_0} \frac{\mathbf{e}_I(\mathbf{x}) \overline{\mathbf{e}_I(\mathbf{x}')}}{\lambda_I} \right] d\mathbf{x}'.$$

This suggests that we should take as the Green's function

$$G(\mathbf{x}, \mathbf{x}') = -\sum_{I \notin \mathbf{I}_0} \frac{\mathbf{e}_I(\mathbf{x}) \overline{\mathbf{e}_I(\mathbf{x}')}}{\lambda_I}.$$

This satisfies a homogeneous Neumann condition by construction. Now if we proceed formally as in our original derivation of a Green's function, we may write

$$\nabla^2_{\mathbf{x}} G(\mathbf{x}, \mathbf{x}') = -\sum_{I \notin \mathbf{I}_0} \mathbf{e}_I(\mathbf{x}) \overline{\mathbf{e}_I(\mathbf{x}')};$$

since for any f satisfying $(f, \mathbf{e}_I) = 0$ when $I \in \mathbf{I}_0$ we have

$$\begin{aligned} f(x) = \sum_I (f, \mathbf{e}_I) e_I &= \sum_{I \notin \mathbf{I}_0} \int_D f(\mathbf{x}') \overline{\mathbf{e}_I(\mathbf{x}')}\, d\mathbf{x}' \mathbf{e}_I(x) \\ &= \int_D f(\mathbf{x}') \left[\sum_{I \notin \mathbf{I}_0} \mathbf{e}_I(\mathbf{x}) \mathbf{e}_I(\mathbf{x}') \right] d\mathbf{x}', \end{aligned}$$

we see that as long as we restrict to functions satisfying the above condition we may write $\nabla^2_{\mathbf{x}} G(\mathbf{x}, \mathbf{x}') = -\delta(\mathbf{x} - \mathbf{x}')$, as before. Examining the proof of the relation used in the solution of problem 1 above in this case, we see that it also holds if we assume that u likewise satisfies $(u, \mathbf{e}_I) = 0$ for $I \in \mathbf{I}_0$. This justifies the solution given above.

Summary:

- We provide some analogies between our work with Fourier transforms and our previous work with orthogonal expansions.
- We then derive the eigenfunctions for the Laplacian on a disk with Dirichlet boundary conditions and use them together with Fourier transforms to study the wave equation on a disk.
- Finally, we derive the solution to the initial value problem for the wave equation on $\mathbf{R}^3$ (and indicate how to derive a similar formula in dimensions 1 and 2). We indicate its qualitative content and sketch an example of its use. In an appendix, we show how to use similar methods to solve the inhomogeneous wave equation.

ANALOGIES BETWEEN FOURIER TRANSFORM METHODS AND ORTHOGONAL EXPANSIONS.
Suppose that D is some bounded region in $\mathbf{R}^m$, and that $\{\mathbf{e}_I\}_{I\in\mathcal{I}}$ is a complete orthonormal set of eigenfunctions for the Laplacian on D with homogeneous Dirichlet boundary conditions, for some set $\mathcal{I}$. Then we know that any 'reasonable' function can be expanded as a series $u = \sum_{I\in\mathcal{I}} u_I \mathbf{e}_I$, where $u_I = (u, \mathbf{e}_I) = \int_D u(\mathbf{x})\overline{\mathbf{e}_I(\mathbf{x})}\, d\mathbf{x}$. Now we may view the coefficients u_I as giving a function from the set of indices $\mathcal{I}$ to the complex numbers $\mathbf{C}$ (for the problems we have dealt with, the coefficients have generally been real; this is because we have used real functions u and real eigenfunctions $\mathbf{e}_I$); we shall write such a function as $\tilde{u} : \mathcal{I} \to \mathbf{C}$, so that $\tilde{u}(I) = (u, \mathbf{e}_I)$. Let us denote the set of all such sequences by $\mathbf{C}^{\mathcal{I}}$ (there is a nice sense in which this set is a Cartesian product of $\mathcal{I}$ copies of $\mathbf{C}$, but it veers off into set theory and we shall not treat it here). Then the foregoing shows that we may define a *transform* $\mathcal{O} : L^1(D) \to \mathbf{C}^{\mathcal{I}}$[1] by

$$\mathcal{O}[u](I) = \tilde{u}(I) = (u, \mathbf{e}_I) = \int_D u(\mathbf{x})\overline{\mathbf{e}_I(\mathbf{x})}\, d\mathbf{x};$$

in other words, $\mathcal{O}[u]$ is the function from $\mathcal{I}$ to $\mathbf{C}$ which, for every $I \in \mathcal{I}$, gives the coefficient $(u, \mathbf{e}_I)$. (If the set $\{\mathbf{e}_I\}_{I\in\mathcal{I}}$ were not assumed to be normalised, then of course we would use $\frac{(u,\mathbf{e}_I)}{(\mathbf{e}_I,\mathbf{e}_I)}$ instead here.) We then have the expansion

$$u(\mathbf{x}) = \sum_{I\in\mathcal{I}} \mathcal{O}[u](I)\mathbf{e}_I(\mathbf{x}).$$

Suppose that we let $\mathbf{O} \subset \mathbf{C}^{\mathcal{I}}$ denote the set of maps $v : \mathcal{I} \to \mathbf{C}$ such that the series

$$\sum_{I\in\mathcal{I}} v(I)\mathbf{e}_I(\mathbf{x})$$

converges in some appropriate sense, and such that this sum is in $L^1(D)$;[2] then we expect that $\mathcal{O}$ actually maps into $\mathbf{O}$ (much as we were able to show that $\mathcal{F}$ actually maps into $C_b(\mathbf{R}^m)$). If we now define the map

$$\mathcal{O}^{-1} : \mathbf{O} \to L^1(D)$$

by

$$\mathcal{O}^{-1}[v](\mathbf{x}) = \sum_{I\in\mathcal{I}} v(I)\mathbf{e}_I(\mathbf{x}),$$

then we see that (as our notation indicates) $\mathcal{O}^{-1}[\mathcal{O}[u]](\mathbf{x}) = u(\mathbf{x})$, $\mathcal{O}[\mathcal{O}^{-1}[v]](I) = v(I)$, i.e., that $\mathcal{O}^{-1}$ is actually an inverse to $\mathcal{O}$.

We may make the following comparison between the foregoing and the Fourier transform:

$$\mathcal{F}[f](\mathbf{k}) = \int_{\mathbf{R}^m} f(\mathbf{x})e^{-2\pi i\mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x} = \hat{f}(\mathbf{k}) \qquad \mathcal{O}[u](I) = \int_D u(\mathbf{x})\overline{\mathbf{e}_I(\mathbf{x})}\, d\mathbf{x} = \tilde{u}(I)$$

$$\mathcal{F}^{-1}[\hat{f}](\mathbf{x}) = \int_{\mathbf{R}^m} \hat{f}(\mathbf{k})e^{2\pi i\mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k} = f(\mathbf{x}) \qquad \mathcal{O}^{-1}[\tilde{u}](I) = \sum_{I\in\mathcal{I}} \tilde{u}(I)\mathbf{e}_I(\mathbf{x}) = u(\mathbf{x})$$

[1]As with the Fourier transform, which, we recall, we showed in the previous set of lecture notes could be viewed as a map $\mathcal{F} : L^1(\mathbf{R}^m) \to C_b(\mathbf{R}^m)$, one can define $\mathcal{O}$ for various different function spaces. We use $L^1(D)$ here for convenience; doing so requires only that the eigenfunctions of the Laplacian are bounded on D, which is true for all of the cases we have studied in this course.

[2]Again, the exact spaces used here are not as important as the general idea.

The transform $\mathcal{O}$ possesses some (though certainly not all) of the properties of the transform $\mathcal{F}$. As an example, we compute the transform of the Laplacian of a function. (As usual, we assume that all relevant transforms exist.) This can be done two ways. The way most closely related to our derivation of a similar property for the Fourier transform is as follows (note that this is the first place we use the fact that the eigenfunctions satisfy homogeneous Dirichlet boundary conditions):

$$\begin{aligned}
\mathcal{O}[\nabla^2 u](I) &= \int_D \nabla^2 u(\mathbf{x})\overline{\mathbf{e}_I(\mathbf{x})}\,d\mathbf{x} = \int_D \nabla\cdot\left(\nabla u(\mathbf{x})\overline{\mathbf{e}_I(\mathbf{x})}\right) - \nabla u(\mathbf{x})\cdot\nabla\overline{\mathbf{e}_I(\mathbf{x})}\,d\mathbf{x} \\
&= \int_{\partial D}\mathbf{n}\cdot\nabla u(\mathbf{x})\overline{\mathbf{e}_I(\mathbf{x})}\,dS - \int_D \nabla\cdot\left(u(\mathbf{x})\nabla\overline{\mathbf{e}_I(\mathbf{x})}\right) - u(\mathbf{x})\nabla^2\overline{\mathbf{e}_I(\mathbf{x})}\,d\mathbf{x} \\
&= -\int_{\partial D} u(\mathbf{x})\mathbf{n}\cdot\nabla\overline{\mathbf{e}_I(\mathbf{x})}\,dS + \int_D u(\mathbf{x})\nabla^2\overline{\mathbf{e}_I(\mathbf{x})}\,d\mathbf{x} = \int_D u(\mathbf{x})\lambda_I\overline{\mathbf{e}_I(\mathbf{x})}\,d\mathbf{x} \\
&= \lambda_I\int_D u(\mathbf{x})\overline{\mathbf{e}_I(\mathbf{x})}\,d\mathbf{x} = \lambda_I\mathcal{O}[u](I);
\end{aligned}$$

here we assume that, since u is a series of functions satisfying homogeneous Dirichlet boundary conditions, it satisfies them itself. This result should be compared to the corresponding result for the Fourier transform:

$$\mathcal{F}[\nabla^2 u](\mathbf{k}) = -4\pi^2|\mathbf{k}|^2\mathcal{F}[u](\mathbf{k}).$$

Another way of deriving the above result for $\mathcal{O}[\nabla^2 u](I)$ which is much closer to our usual methods for manipulating orthogonal expansions (and also more general) is as follows. Writing $\tilde{u}(I) = \mathcal{O}[u](I)$, we have

$$u(\mathbf{x}) = \sum_{I\in\mathcal{I}}\tilde{u}(I)\mathbf{e}_I(\mathbf{x});$$

assuming that we may differentiate term-by-term, we have

$$\nabla^2 u(\mathbf{x}) = \sum_{I\in\mathcal{I}}\tilde{u}(I)\nabla^2\mathbf{e}_I(\mathbf{x}) = \sum_{I\in\mathcal{I}}\lambda_I\tilde{u}(I)\mathbf{e}_I(\mathbf{x}).$$

But this shows that

$$\begin{aligned}
\mathcal{O}[\nabla^2 u](I) = (\nabla^2 u, \mathbf{e}_I) &= \left(\sum_{J\in\mathcal{I}}\lambda_J\tilde{u}(J)\mathbf{e}_J, \mathbf{e}_I\right) \\
&= \lambda_I\tilde{u}(I) = \lambda_I\mathcal{O}[u](I),
\end{aligned}$$

by our usual manipulations with orthogonal expansions. This is our desired result.

The other major property of the Fourier transform, that of turning convolution into multiplication, does not have so happy a fate with $\mathcal{O}$; the details are quite beyond the scope of this course, but we provide an outline in Appendix I at the end for those who are interested. (This Appendix can be skipped entirely, though it does give another perspective on where convolution comes from.)

Given this property, we may proceed to solve the heat equation using $\mathcal{O}$ in a fashion exactly analogous to that by which we solved the heat equation using $\mathcal{F}$. To this end, consider the problem

$$\frac{\partial u}{\partial t} = \nabla^2 u, \quad u|_{t=0} = f, \quad u|_{\partial D} = 0.$$

If we apply $\mathcal{O}$ to the entire problem, we obtain the transformed problem

$$\frac{\partial\tilde{u}(t,I)}{\partial t} = \lambda_I\tilde{u}(t,I), \quad \tilde{u}|_{t=0} = \tilde{f}(I);$$

from this we easily obtain

$$\tilde{u}(t,I) = \tilde{f}(I)e^{\lambda_I t},$$

whence

$$u = \mathcal{O}^{-1}[\tilde{u}](\mathbf{x}) = \sum_{I \in \mathcal{I}} \tilde{f}(I) e^{\lambda_I t} \mathbf{e}_I(\mathbf{x}),$$

where

$$\tilde{f}(I) = (f, \mathbf{e}_I).$$

This is identical to the result we obtained by our usual methods (see, for example, our treatment of the heat equation on the cube in the lecture notes for July 9 – 11).

The point of this is to try to make the Fourier method a little bit more understandable, rather than to suggest that we ought to use this method with orthogonal expansions! (Though we certainly can if we like.)

EIGENFUNCTIONS AND EIGENVALUES FOR THE LAPLACIAN ON A DISK. Let $D = \{(\rho, \phi) | \rho < a\}$, for some positive number a, and consider the problem

$$\nabla^2 u = \lambda u, \quad u|_{\partial D} = 0.$$

Now the Laplacian in polar coordinates can be obtained from the Laplacian in cylindrical coordinates by dropping the final $\frac{\partial^2}{\partial z^2}$; thus this equation becomes

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} = \lambda u.$$

We proceed as usual by looking for separated solutions to this equation. Thus suppose that $u = P(\rho)\Phi(\phi)$; substituting in and dividing through by u then gives

$$\frac{P''}{P} + \frac{1}{\rho}\frac{P'}{P} + \frac{1}{\rho^2}\frac{\Phi''}{\Phi} = \lambda.$$

As usual, since only the term $\frac{\Phi''}{\Phi}$ depends on ϕ, it must be constant; and since ϕ is an angular variable which is only defined up at an additive term of a multiple of 2π, our usual logic shows that this constant must be the negative square of an integer, i.e., that there must be an $m \in \mathbf{Z}$ such that $\frac{\Phi''}{\Phi} = -m^2$. From this we obtain readily the two solutions $\Phi_1(\phi) = \cos m\phi$, $\Phi_2(\phi) = \sin m\phi$. Substituting this back in, we obtain for P

$$\frac{P''}{P} + \frac{1}{\rho}\frac{P'}{P} - \frac{m^2}{\rho^2} = \lambda,$$

or

$$P'' + \frac{1}{\rho}P' + \left(-\lambda - \frac{m^2}{\rho^2}\right) P = 0.$$

This is seen, after scaling by $\sqrt{-\lambda}$, to be simply Bessel's equation; in other words, we must have

$$P(\rho) = J_m(\sqrt{-\lambda}\rho).$$

Somewhat more carefully: if $-\lambda \geq 0$ then we obtain the above formula; if $-\lambda < 0$ then we would obtain $I_m(\sqrt{\lambda}\rho)$. Since we require homogeneous Dirichlet boundary conditions on the boundary, i.e., at $\rho = a$, we must choose J_m and not I_m. This forces $\lambda \leq 0$, say $\lambda = -\mu^2$. Now the boundary condition gives

$$P(a) = J_m(\mu a) = 0,$$

whence we see that $\mu = \frac{\lambda_{mi}}{a}$ for some i, where λ_{mi} denotes as usual the ith positive zero of J_m. Thus we have the eigenfunctions

$$J_m\left(\frac{\lambda_{mi}}{a}\rho\right)\cos m\phi, \quad J_m\left(\frac{\lambda_{mi}}{a}\rho\right)\sin m\phi,$$

with the eigenvalues

$$\lambda = -\frac{\lambda_{mi}^2}{a^2}.$$

This set of eigenfunctions is seen to be complete, since the Bessel function factors are in ρ.

THE WAVE EQUATION ON A DISK. The *wave equation*,

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \nabla^2 u,$$

describes the motion of waves on elastic membranes, in gasses and fluids, and in various other circumstances (at least as long as the quantity u is not large so that nonlinear effects can be neglected). Here c is a parameter called the *wave speed* (we shall see the reason for this terminology later, when we discuss the wave equation on $\mathbf{R}^m$); we shall occasionally set it equal to 1 for convenience – any formula with $c = 1$ can be turned into a formula for general c by multiplying t by c at each occurrence. Now consider the following problem on $(0, +\infty) \times D$:

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u, \quad u|_{\partial D} = 0;$$

this problem could describe the vibrations of a circular drumhead (in that case, u would represent the vertical deflection from the equilibrium plane of the drumhead, so the Dirichlet condition $u|_{\partial D} = 0$ means physically that the edge of the drumhead is fixed and immobile). Here we have specified no initial conditions. If we Fourier transform in t, we obtain, using f as our Fourier variable,

$$-4\pi^2 f^2 \hat{u} = \nabla^2 \hat{u}, \quad \hat{u}|_{\partial D} = 0;$$

from this we see that $-4\pi^2 f^2$ must be an eigenvalue of the Laplacian on D, which means that we must have

$$f = \pm\frac{\lambda_{mi}}{2\pi a}$$

for some m and i; more specifically, for f not of this form we must have $\hat{u}(f, \mathbf{x}) = 0$ for all $\mathbf{x}$. While we shall not pause to give a precise derivation of the following, this means that any solution u must be simply a *sum* (rather than an integral) over frequencies; specifically, since $2\pi i\frac{\lambda_{mi}}{2\pi a}t = \frac{i\lambda_{mi}t}{a}$, we have

$$u(t, \mathbf{x}) = \sum_{m=0}^{\infty}\sum_{i=1}^{\infty} J_m\left(\frac{\lambda_{mi}}{a}\rho\right)\left[e^{\frac{i\lambda_{mi}t}{a}}\left(a_{mi}\cos m\phi + b_{mi}\sin m\phi\right) + e^{-\frac{i\lambda_{mi}t}{a}}\left(c_{mi}\cos m\phi + d_{mi}\sin m\phi\right)\right].$$

Here the coefficients can be complex to make u real.

Let us now consider the slightly different problem

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u, \quad u|_{t=0} = f, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g, \quad u|_{\partial D} = 0.$$

For this problem we shall begin by expanding u in a series in terms of the eigenfunctions found above. We could proceed in the usual fashion; for the sake of illustration, we shall use the transform $\mathcal{O}$ introduced above. Transforming with $\mathcal{O}$, the above problem becomes

$$\frac{\partial^2 \tilde{u}}{\partial t^2} = -\frac{\lambda_{mi}^2}{a^2}\tilde{u}, \quad \tilde{u}|_{t=0} = \tilde{f}, \quad \left.\frac{\partial \tilde{u}}{\partial t}\right|_{t=0} = \tilde{g}.$$

From the equation, we see that the general solution is of the form (writing $I = (m, i, \sigma)$, where $\sigma = 1$ for the eigenfunction with $\cos m\phi$ and $\sigma = -1$ for the eigenfunction with $\sin m\phi$)

$$\tilde{u}(t, I) = a(I)\cos\frac{\lambda_{mi}}{a}t + b(I)\sin\frac{\lambda_{mi}}{a}t;$$

applying the initial conditions gives

$$\tilde{u}|_{t=0} = a(I) = \tilde{f}(I),$$
$$\left.\frac{\partial \tilde{u}}{\partial t}\right|_{t=0} = \frac{\lambda_{mi}}{a}b(I) = \tilde{g}(I),$$
$$b(I) = \frac{a}{\lambda_{mi}}\tilde{g}(I),$$

so that

$$\tilde{u}(t,I) = \tilde{f}(I)\cos\frac{\lambda_{mi}}{a}t + \tilde{g}\frac{a}{\lambda_{mi}}\sin\frac{\lambda_{mi}}{a}t,$$

and the solution is

$$u(t,\mathbf{x}) = \sum_{m=0}^{\infty}\sum_{i=1}^{\infty}\left[\left[\tilde{f}(m,i,1)\cos\frac{\lambda_{mi}}{a}t + \tilde{g}(m,i,1)\frac{a}{\lambda_{mi}}\sin\frac{\lambda_{mi}}{a}t\right]J_m\left(\frac{\lambda_{mi}}{a}\rho\right)\cos m\phi\right.$$
$$\left.+\left[\tilde{f}(m,i,-1)\cos\frac{\lambda_{mi}}{a}t + \tilde{g}(m,i,-1)\frac{a}{\lambda_{mi}}\sin\frac{\lambda_{mi}}{a}t\right]J_m\left(\frac{\lambda_{mi}}{a}\rho\right)\sin m\phi\right].$$

The same result could of course be obtained by our usual methods. We now give a specific example.
EXAMPLE. Solve the following problem on D:

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u, \quad u|_{t=0} = 1, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad u|_{\partial D} = 0.$$

We first determine the transform $\mathcal{O}[1]$:

$$\mathcal{O}[1](m,i,\pm 1) = \frac{2}{(\cos m\phi, \cos m\phi)a^2 J_{m+1}^2(\lambda_{mi})}\int_D J_m\left(\frac{\lambda_{mi}}{a}\rho\right)\begin{cases}\cos m\phi\\ \sin m\phi\end{cases} d\mathbf{x},$$

whence we see that $\mathcal{O}[1](m,i,\pm 1) = 0$ unless $m=0$ and we take the $+1$ in the third slot, and that in that case

$$\begin{aligned}\mathcal{O}[1](0,i,1) &= \frac{1}{\pi a^2 J_1^2(\lambda_{0i})}\int_0^{2\pi}\int_0^a J_0\left(\frac{\lambda_{0i}}{a}\rho\right)\rho\, d\rho\, d\phi \\ &= \frac{2}{a^2 J_1^2(\lambda_{0i})}\frac{a^2}{\lambda_{0i}^2}\left(xJ_0(x)\right)|_0^{\lambda_{mi}} = \frac{2}{\lambda_{0i}J_1(\lambda_{0i})};\end{aligned}$$

we note that the factors of a cancel only because of the value of m involved. Clearly $\mathcal{O}[0]=0$, so substituting back into the general formula above, we have the solution

$$u(t,\mathbf{x}) = \sum_{i=1}^{\infty}\frac{2}{\lambda_{0i}J_1(\lambda_{0i})}\cos\frac{\lambda_{0i}}{a}tJ_0\left(\frac{\lambda_{0i}}{a}\rho\right).$$

This is, of course, what we would expect to obtain had we started by writing out the general series expansion for u and then substituted it into the equation.
THE WAVE EQUATION ON $\mathbf{R}^m$. We now come to the last major topic of the course, namely the treatment of the initial value problem for the wave equation on $\mathbf{R}^m$. Thus we seek solutions to the following problem:

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u, \quad u|_{t=0} = f, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g.$$

(The treatment of the nonhomogeneous version, where there is a term F added to the right-hand side, is beyond the scope of the course proper but will be sketched in Appendix II.) We approach this problem in a fashion analogous to that in which we approached the corresponding version on D. We begin by Fourier transforming:

$$\frac{\partial^2 \hat{u}}{\partial t^2} = -4\pi^2|\mathbf{k}|^2\hat{u}, \quad \hat{u}|_{t=0} = \hat{f}, \quad \left.\frac{\partial \hat{u}}{\partial t}\right|_{t=0} = \hat{g}.$$

Now the first equation above clearly has the general solution

$$\hat{u}(t,\mathbf{k}) = a(\mathbf{k})\cos 2\pi|\mathbf{k}|t + b(\mathbf{k})\sin 2\pi|\mathbf{k}|t,$$

where $a(\mathbf{k})$ and $b(\mathbf{k})$ are two arbitrary functions. Applying the initial conditions, we obtain

$$\begin{aligned} \hat{u}|_{t=0} &= a(\mathbf{k}) = \hat{f}(\mathbf{k}), \\ \left.\frac{\partial \hat{u}}{\partial t}\right|_{t=0} &= 2\pi|\mathbf{k}|b(\mathbf{k}) = \hat{g}(\mathbf{k}), \\ b(\mathbf{k}) = \frac{1}{2\pi|\mathbf{k}|}\hat{g}(\mathbf{k}), \end{aligned}$$

so that

$$\hat{u}(t, \mathbf{k}) = \hat{f}(\mathbf{k}) \cos 2\pi|\mathbf{k}|t + \hat{g}(\mathbf{k})\frac{\sin 2\pi|\mathbf{k}|t}{2\pi|\mathbf{k}|},$$

exactly analogous to the result we obtained above on D. (We note also that the above function is defined for all $\mathbf{k}$, even though $b(\mathbf{k})$ as given above is undefined for $\mathbf{k} = 0$.) We note that the result here is valid for all m; thus the Fourier transform of the solution does not depend in any way on the dimension of the space involved. (This is analogous to the situation for Poisson's equation: if one solves $\nabla^2 u = f$ by Fourier transform, one finds $u = -\frac{1}{4\pi^2|\mathbf{k}|^2} f(\mathbf{k})$, regardless of the dimension.)

We would now like to take the inverse Fourier transform of the above expression. Now the properties of the Fourier transform show that

$$\mathcal{F}^{-1}[\hat{f}\hat{g}](\mathbf{x}) = (f * g)(\mathbf{x})$$

for any appropriate functions f and g; thus if we could recognise the two functions $\cos 2\pi|\mathbf{k}|t$ and $\frac{\sin 2\pi|\mathbf{k}|t}{2\pi|\mathbf{k}|}$ as Fourier transforms, we would be able to write u as a sum of two convolution integrals. We note that the former is the time derivative of the latter, which suggests that we start with the latter function. This is where the dimension of the space comes into play. The main case for us here will be $m = 3$ (and this is the only case we covered systematically in class), but we shall indicate what happens when $m = 1$ or $m = 2$.[3]

Let us denote the inverse transform we seek by $M(t, \mathbf{x})$; then

$$\begin{aligned} M(t, \mathbf{x}) &= \mathcal{F}^{-1}\left[\frac{\sin 2\pi|\mathbf{k}|t}{2\pi|\mathbf{k}|}\right](\mathbf{x}) \\ &= \int_{\mathbf{R}^m} \frac{\sin 2\pi|\mathbf{k}|t}{2\pi|\mathbf{k}|} e^{2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k}. \end{aligned}$$

Before proceeding, we note that this function is real: its conjugate is just

$$\int_{\mathbf{R}^m} \frac{\sin 2\pi|\mathbf{k}|t}{2\pi|\mathbf{k}|} e^{2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k},$$

which can be turned back into the original integral by using the substitution $\mathbf{k}' = -\mathbf{k}$. This will be important below. Now it can be shown that for any $m > 1$ (we shall say more about the case $m = 1$ below), the m-dimensional volume element $d\mathbf{k}$ can be decomposed into the following (for simplicity, we shall write $k = |\mathbf{k}|$ where convenient):

$$d\mathbf{k} = k^{m-1}\, dk\, d\Omega,$$

where $d\Omega$ is an angular element; when $m = 2$ it is simply $d\theta$, while when $m = 3$ it is $\sin\theta d\theta d\phi$ (this is called an element of *solid angle*, in analogy with the element of angle $d\theta$ which one obtains in the case $m = 2$); when

[3]We shall not, however, treat higher values of m since these involve progressively more pathological 'functions': we shall see in a moment that when $m = 3$ we get a Dirac delta function; for $m = 5$ we would get a second derivative of a Dirac delta function, and so on. (Even dimensions turn out to be somewhat more complicated than odd dimensions.) While these derivatives can be defined in a rigorous sense, doing so is beyond the scope of this course.

$m > 3$ it is a similar angular measure in $m-1$ angular variables obtained by parametrising the $m-1$-sphere. (For example, we may parametrise the 4-sphere thus (letting ψ represent the normal polar angle in 3-space):

$$\begin{aligned} w &= \cos\theta \\ z &= \sin\theta\cos\psi \\ x &= \sin\theta\sin\psi\cos\phi \\ y &= \sin\theta\sin\psi\sin\phi, \end{aligned}$$

and for higher dimensions we may proceed by induction.) This general parametrisation is not important, beyond knowing that for all m we can parametrise it in such a way that, for fixed $\mathbf{x}$, we have (writing $r = |\mathbf{x}|$)

$$\mathbf{k}\cdot\mathbf{x} = |\mathbf{k}| r\cos\theta$$

where θ is one of the angles parametrising the $m-1$-sphere, and which runs from 0 to π. (This is clearly true for $m = 2$ in polar coordinates – taking the x axis along $\mathbf{x}$ – and for $m = 3$ in spherical coordinates – take the z axis along $\mathbf{x}$ – and these are the only situations we are really concerned with here.) Thus we may rewrite the above integral as, letting S_1 denote the unit $m-1$-sphere (the unit circle if $m = 2$, the unit sphere if $m = 3$)

$$\int_{S_1}\int_0^\infty \frac{\sin 2\pi kt}{2\pi k} e^{2\pi ikr\cos\theta} k^{m-1}\,dk\,d\Omega.$$

Now as noted above, this integral is always a real number; thus we may replace the complex exponential with its real part, obtaining

$$\int_{S_1}\int_0^\infty \frac{\sin 2\pi kt}{2\pi k}\cos\left(2\pi kr\cos\theta\right) k^{m-1}\,dk\,d\Omega.$$

Now if m is odd (for example, if $m = 3$), the integrand is an even function of k, so this integral equals

$$\frac{1}{2}\int_{S_1}\int_{-\infty}^\infty \frac{\sin 2\pi kt}{2\pi k}\cos\left(2\pi kr\cos\theta\right) k^{m-1}\,dk\,d\Omega = \int_{S_1}\int_{-\infty}^\infty \frac{\sin 2\pi kt}{2\pi k} e^{2\pi ikr\cos\theta} k^{m-1}\,dk\,d\Omega.$$

The point behind all of these manipulations is that the integral over k here is now quite clearly the inverse Fourier transform of the function $k^{m-1}\frac{\sin 2\pi kt}{2\pi k}$ on $\mathbf{R}^1$, evaluated at the point $r\cos\theta$ – in other words, we have reduced a three-dimensional inverse Fourier transform to a one-dimensional one. The factor of k^{m-1} indicates that the inverse transform of this function will be the $m-1$th derivative of the inverse transform of $\frac{\sin 2\pi kt}{2\pi k}$, which we now derive. (This is the reason why the function M becomes increasingly less well-behaved in higher dimensions.)

Directly calculating the inverse Fourier transform of $\frac{\sin 2\pi kt}{2\pi k}$ is not easy, so we shall proceed as we did in class by finding a function whose Fourier transform it is. Let

$$\chi(x) = \chi_{[-t,t]}(x) = \begin{cases} 1, & x \in [-t,t] \\ 0, & x \notin [-t,t] \end{cases};$$

χ is just a rectangular bump function. The Fourier transform of χ is

$$\begin{aligned} \mathcal{F}[\chi](k) &= \int_{-\infty}^\infty \chi(x) e^{-2\pi ikx}\,dx = \int_{-t}^t e^{-2\pi ikx}\,dx = \int_{-t}^t \cos 2\pi kx\,dx \\ &= \left.\frac{\sin 2\pi kx}{2\pi k}\right|_{-t}^t = \frac{\sin 2\pi kt}{\pi k}, \end{aligned}$$

where we have made use of the fact that cos is an odd function and sin an even function. Thus we see that

$$\mathcal{F}^{-1}\left[\frac{\sin 2\pi kt}{2\pi k}\right](x) = \frac{1}{2}\chi(x).$$

It is worth noting that, were we working in dimension $m = 1$, this would be the only inverse Fourier transform we would need, i.e., this would be our function M. We shall not give the details here.

From this we obtain (pretending for the moment that χ is a twice-differentiable function, even though it is not even continuous at $x = \pm t$)

$$\begin{aligned}\mathcal{F}^{-1}\left[k^2\frac{\sin 2\pi kt}{2\pi k}\right](x) &= -\frac{1}{4\pi^2}\mathcal{F}^{-1}\left[-4\pi^2k^2\frac{\sin 2\pi kt}{2\pi k}\right](x)\\ &= -\frac{1}{8\pi^2}\chi''(x),\end{aligned}$$

and we see that our function M is

$$\begin{aligned}M &= -\frac{1}{16\pi^2}\int_0^{2\pi}\int_0^{\pi}\chi''(r\cos\theta)\sin\theta\,d\theta\,d\phi\\ &= -\frac{1}{8\pi}\left[-\frac{1}{r}\chi'(r\cos\theta)\right]\Bigg|_{\theta=0}^{\theta=\pi} = -\frac{1}{8\pi r}\cdot 2\chi'(r) = -\frac{1}{4\pi r}\chi'(r),\end{aligned}$$

where we have used the fact that χ' is odd since χ is even (again, pretending that χ' were a normal function!). We are, now, thus faced with the task of computing $\chi'(r)$, for $r > 0$ (remember that $r = |x|$). Clearly $\chi'(r) = 0$ for $r \neq t$. We claim that in fact $\chi'(r) = -\delta(r-t)$. The simplest way to see this is as follows. Let H denote the *Heaviside function*

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}.$$

Now suppose that H' could be defined in such a way that integration by parts were still valid[4], if f were any function vanishing as $x \to \infty$, we would have

$$\begin{aligned}\int_{-\infty}^{\infty} H'(x)f(x)\,dx &= H(x)f(x)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} H(x)f'(x)\,dx\\ &= -\int_0^{\infty} f'(x)\,dx = -f(x)|_0^{\infty} = f(0),\end{aligned}$$

so that $H'(x)$ does indeed behave as a delta function. Now on $r > 0$, we have $\chi(r) = H(t-r)$, so (proceeding formally) we have $\chi'(r) = -H'(t-r) = -\delta(t-r) = -\delta(r-t)$, as claimed. [Another, perhaps more rigorous, way of seeing this is as follows. Let $\{\phi_n\}$ be the approximate identity given by

$$\phi_n(x) = n\pi^{-\frac{1}{2}}e^{-n^2x^2},$$

and define

$$\Phi_n(x) = \int_0^x \phi_n(u)\,du;$$

then we have, doing a change of variables to $v = nu$,

$$\Phi_n(x) = \int_0^{nx}\phi(v)\,dv,$$

whence it is evident that for $x > 0$ we have $\Phi_n(x) \to \frac{1}{2}$ as $n \to \infty$, while $\Phi_n(x) \to -\frac{1}{2}$ as $n \to -\infty$; in other words, we have for all $x \neq 0$ the limit

$$\lim_{n\to\infty}\Phi_n(x) = H(x) - \frac{1}{2}.$$

[4] This is in fact the way in which differentiation of functions such as H and 'functions' (distributions) such as δ may be defined rigorously: one requires that the normal integration-by-parts formulas hold and proceeds formally.

Thus, assuming that we can interchange differentiation with the limit, we obtain

$$H'(x) = \lim_{n\to\infty} \Phi_n'(x) = \lim_{n\to\infty} \phi_n(x),$$

and this latter limit 'is' just the delta function $\delta(x)$ since $\{\phi_n\}$ is an approximate identity.] Thus, finally, we have for M

$$M(t,\mathbf{x}) = \frac{1}{4\pi|\mathbf{x}|}\delta(|\mathbf{x}|-t) = \frac{1}{4\pi t}\delta(|\mathbf{x}|-t).$$

The inverse transform of $\hat{g}\frac{\sin 2\pi|\mathbf{k}|t}{2\pi|k|}$, which we shall denote $u_2(t,\mathbf{x})$, is thus equal to the convolution integral

$$\frac{1}{4\pi t}\int_{\mathbf{R}^3} g(\mathbf{x}-\mathbf{x}')\delta(|\mathbf{x}'|-t)\,d\mathbf{x}'.$$

Let us now set up a spherical coordinate system in $\mathbf{x}'$; then the above integral becomes

$$\begin{aligned}\frac{1}{4\pi t}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} g(\mathbf{x}-\mathbf{x}')\delta(r'-t)r'^2\,dr'\sin\theta'\,d\theta'\,d\phi' &= \frac{1}{4\pi t}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} g(\mathbf{x}-\mathbf{x}')\delta(r'-t)\,dr't^2\sin\theta'\,d\theta'\,d\phi' \\ &= \frac{1}{4\pi t}\int_{S_t(0)} g(\mathbf{x}-\mathbf{x}')\,dS' = \frac{1}{4\pi t}\int_{S_t(\mathbf{x})} g(\mathbf{x}'')\,dS'',\end{aligned}$$

where in the last equation we have made the substitution $\mathbf{x}'' = \mathbf{x}-\mathbf{x}'$, which translates the sphere $S_t(0)$ to the sphere $S_t(\mathbf{x})$. (Here $S_t(\mathbf{x}) = \{\mathbf{x}'||\mathbf{x}-\mathbf{x}'| = t\}$ is the sphere – not ball! – of radius t centred at $\mathbf{x}$.) The second-to-last equality holds for the following reasons: first of all, the delta function forces the point $\mathbf{x}'$ in $g(\mathbf{x}-\mathbf{x}')$ to lie on the sphere; second, the remaining parts of the volume element, $t^2\sin\theta'\,d\theta'\,d\phi'$, give exactly the surface area element on a sphere of radius t.

This is thus the desired formula for the inverse Fourier transform of the second part of our expression for $\hat{u}$ obtained above.

To work out the first part, we proceed rather formally as follows, assuming that we can interchange $\mathcal{F}^{-1}$ and $\frac{\partial}{\partial t}$:

$$\begin{aligned}\mathcal{F}^{-1}\left[\hat{f}(\mathbf{k})\cos 2\pi|\mathbf{k}|t\right](\mathbf{x}) &= \frac{\partial}{\partial t}\mathcal{F}^{-1}\left[\hat{f}(\mathbf{k})\frac{\sin 2\pi|\mathbf{k}|t}{2\pi|\mathbf{k}|}\right](\mathbf{x}) \\ &= \frac{\partial}{\partial t}\left[\frac{1}{4\pi t}\int_{S_t(\mathbf{x})} f(\mathbf{x}')\,dS'\right].\end{aligned}$$

Thus finally we have the following formula for u:

$$u(t,\mathbf{x}) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi t}\int_{S_t(\mathbf{x})} f(\mathbf{x}')\,dS'\right] + \frac{1}{4\pi t}\int_{S_t(\mathbf{x})} g(\mathbf{x}')\,dS';$$

or, putting back in the speed c,

$$u(t,\mathbf{x}) = \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{1}{4\pi ct}\int_{S_{ct}(\mathbf{x})} f(\mathbf{x}')\,dS'\right] + \frac{1}{4\pi ct}\int_{S_{ct}(\mathbf{x})} g(\mathbf{x}')\,dS'. \tag{1}$$

We note a qualitative result which follows from this: the solution u at a point $\mathbf{x}$ and a time t only depends on the initial data on (or, at any rate, in the case of f, infinitesimally close to) the sphere (not the ball!) of radius ct centred at $\mathbf{x}$ – in other words, on the initial data on the set of points exactly a distance ct from the point $\mathbf{x}$. This means that signals propagate at exactly the speed c. (As mentioned in class – though the derivation does not follow in the way indicated there, since the function k^{m-1} becomes odd and one cannot extend the integral to all of $\mathbf{R}^1$ as done here and suggested there – this property of the wave equation does not hold in two dimensions; and the author has seen it suggested that this is the reason why thunder is usually heard to continue even though the lightning flash (and hence the source of the thunder) is essentially instantaneous: a lightning flash – and hence the intial data for the thunder – is essentially a long straight line, meaning that the source will possess cylindrical symmetry, and the wave will be essentially the same as a two-dimensional wave.)

We now give a concrete example.

EXAMPLE. Solve the following problem on $\mathbf{R}^3$:

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u, \quad u|_{t=0} = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \begin{cases} 1, & |\mathbf{x}| \le 1 \\ 0, & |\mathbf{x}| > 1 \end{cases}.$$

Let $g(\mathbf{x}) = \left.\frac{\partial u}{\partial t}\right|_{t=0}$. By our foregoing work, it suffices to evaluate integrals of the type

$$\int_{S_t(\mathbf{x})} g(\mathbf{x}')\, dS';$$

but a little reflection shows that this is just the area of that part of $S_t(\mathbf{x})$ which lies inside the unit *ball* $B_1(0) = \{\mathbf{x} | |\mathbf{x}| \le 1\}$. This is thus a problem in geometry rather than calculus. We may distinguish four separate cases: (i) $B_t(\mathbf{x}) \subset B_1(0)$; (ii) $B_1(0) \subset B_t(\mathbf{x})$; (iii) $B_1(0) \cap B_t(\mathbf{x}) = \emptyset$; (iv) everything else. For case (i) to hold we must have $|\mathbf{x}| + t \le 1$, for then $|\mathbf{x} - \mathbf{x}'| < t$ implies $|\mathbf{x}'| < t + |\mathbf{x}| < 1$; also, in this case we have clearly

$$\int_{S_t(\mathbf{x})} g(\mathbf{x}')\, dS' = \text{area}(S_t(\mathbf{x})) = 4\pi t^2.$$

For case (ii) to hold we must have $t - |\mathbf{x}| \ge 1$, for then $|\mathbf{x}'| < 1$ implies $|\mathbf{x}' - \mathbf{x}| \le |\mathbf{x}'| + |\mathbf{x}| < 1 + |\mathbf{x}| < t$; and the integral will vanish unless $\mathbf{x} = 0$ and $t = 1$, in the which case it equals 4π. For case (iii) to hold we must have $|\mathbf{x}| - t \ge 1$, for then $|\mathbf{x}'| < 1$ implies $|\mathbf{x}' - \mathbf{x}| \ge |\mathbf{x}| - |\mathbf{x}'| \ge 1 + t - |\mathbf{x}'| > t$; and in this case the integral is also clearly zero. Finally, in case (iv) we have $|\mathbf{x}| + t > 1$, $|t - |\mathbf{x}|| < 1$, and we see geometrically (try drawing a picture of the situation in two dimensions!) that the intersection of $S_t(\mathbf{x})$ with $B_1(0)$ is a spherical cap with central half-angle θ satisfying

$$1 = |\mathbf{x}|^2 + t^2 - 2t|\mathbf{x}|\cos\theta,$$

i.e., $\cos\theta = \frac{|\mathbf{x}|^2 + t^2 - 1}{2t|\mathbf{x}|}$. The area of such a spherical cap is given by

$$\begin{aligned}\int_0^{2\pi}\int_0^{\theta} t^2 \sin\theta'\, d\theta'\, d\phi' &= 2\pi t^2 \int_{\cos\theta}^{1} dx = 2\pi t^2 (1 - \cos\theta) \\ &= 2\pi t^2 \frac{2t|\mathbf{x}| - |\mathbf{x}|^2 - t^2 + 1}{2t|\mathbf{x}|} = \frac{\pi t}{|\mathbf{x}|}\left(1 - (t - |\mathbf{x}|)^2\right).\end{aligned}$$

We thus see that the second part u_2 of the solution u depends only on $\mathbf{x}$ (which makes sense, since the original problem was spherically symmetric), and that we have in particular (remembering the overall factor of $\frac{1}{4\pi t}$)

$$\begin{aligned} u_2(t, \mathbf{x}) &= \frac{1}{4\pi t}\begin{cases} 4\pi t^2, & |\mathbf{x}| + t \le 1 \\ \frac{\pi t}{|\mathbf{x}|}\left(1 - (t - |\mathbf{x}|)^2\right), & |\mathbf{x}| + t > 1,\ |t - |\mathbf{x}|| < 1 \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} t, & |\mathbf{x}| + t \le 1 \\ \frac{1 - (t - |\mathbf{x}|)^2}{4|\mathbf{x}|}, & |\mathbf{x}| + t > 1,\ |t - |\mathbf{x}|| < 1 \\ 0, & \text{otherwise} \end{cases}. \end{aligned}$$

Since in this case $f = 0$, the first part of the solution will vanish and the above formula for u_2 gives in fact the full solution u. Let us consider what it means qualitatively. Let us fix some observation point $\mathbf{x}$ and consider $u(t, \mathbf{x})$ as a function of t only. We identify two cases: (i) $|\mathbf{x}| \le 1$; (ii) $|\mathbf{x}| > 1$. In case (i), we see that at time $t = 0$ we have $u = 0$, while for $t \le 1 - |\mathbf{x}|$ we have $u(t, \mathbf{x}) = t$ by the above formula. Now suppose that $t > 1 - |\mathbf{x}|$, but that we still have $|t - |\mathbf{x}|| < 1$: this means that $-1 + |\mathbf{x}| < t < 1 + |\mathbf{x}|$, but the first inequality is trivial since $-1+|\mathbf{x}|<0$, so only the second inequality is meaningful, and we see that overall we have $|t-1|<|\mathbf{x}|$. In this case we have $u(t,\mathbf{x})=\frac{1-(t-|\mathbf{x}|)^2}{4|\mathbf{x}|}$, which is a segment of a parabola going from

$$u(1-|\mathbf{x}|,\mathbf{x})=\frac{1-(1-2|\mathbf{x}|)^2}{4|\mathbf{x}|}=\frac{4|\mathbf{x}|-4|\mathbf{x}|^2}{4|\mathbf{x}|}=1-|\mathbf{x}|$$

to

$$u(1+|\mathbf{x}|,\mathbf{x})=\frac{1-1}{4|\mathbf{x}|}=0.$$

Finally, if $|x|+t>1$ and $|t-|\mathbf{x}||\geq 1$, which in this case means (as indicated above) that $t>1+|\mathbf{x}|$, then we have $u(t,\mathbf{x})=0$. Thus we have finally

$$u(t,\mathbf{x})=\begin{cases} t, & 0\leq t\leq 1-|\mathbf{x}| \\ \frac{1-(t-|\mathbf{x}|)^2}{4|\mathbf{x}|}, & 1-|\mathbf{x}|\leq t\leq 1+|\mathbf{x}| \\ 0, & t\geq 1+|\mathbf{x}| \end{cases};$$

note that these three functions agree on the endpoints (except in the special case $\mathbf{x}=0$), so that the resulting function u is continuous in time. This means that $u(t,\mathbf{x})$ first grows linearly, then drops of quadratically to zero, and finally stays at zero for all future time.

Now suppose that $|\mathbf{x}|>1$; in this case, the first case for u_2 above never happens, so we are only concerned with the cases $|t-|\mathbf{x}||<1$ and $|t-|\mathbf{x}||\geq 1$. The first case gives $-1+|\mathbf{x}|<t<1+|\mathbf{x}|$, while the second case (naturally) gives everything else; thus we have simply

$$u(t,\mathbf{x})=\begin{cases} \frac{1-(t-|\mathbf{x}|)^2}{4|\mathbf{x}|}, & -1+|\mathbf{x}|<t<1+|\mathbf{x}| \\ 0, & \text{otherwise} \end{cases}.$$

In this case, u is zero up to time $-1+|\mathbf{x}|$ (this is the minimum time it takes for a signal to pass from the unit ball to the point $\mathbf{x}$); it then exhibits a quadratic increase and decrease, before dropping to zero at time $1+|\mathbf{x}|$ (which is the maximum time it takes for a signal to pass from the unit ball to the point $\mathbf{x}$), after which it remains zero for all time. In other words, then, at points $\mathbf{x}$ outside the unit ball, the solution is a quadratic pulse of width 2 whose height is inversely proportional to the distance $|\mathbf{x}|$ of the point from the origin.

We may use our work in this example to quickly do one more example, as follows.

EXAMPLE. Solve the following problem on $\mathbf{R}^3$:

$$\frac{\partial^2 u}{\partial t^2}=\nabla^2 u,\quad u|_{t=0}=\begin{cases} 1, & |\mathbf{x}|\leq 1 \\ 0, & |\mathbf{x}|>1 \end{cases},\quad \left.\frac{\partial u}{\partial t}\right|_{t=0}=0.$$

In this case only the first term in the solution for u remains, and we have by equation (1)

$$u(t,\mathbf{x})=\begin{cases} 1, & |\mathbf{x}|+t<1 \\ \frac{|\mathbf{x}|-t}{2|\mathbf{x}|}, & |\mathbf{x}|+t>1,\ |t-|\mathbf{x}||<1 \\ 0, & |\mathbf{x}|+t>1,\ |t-|\mathbf{x}||>1 \end{cases},$$

where we have dropped the boundary points since the function $u_2(t,\mathbf{x})$ derived above is not in general differentiable there. If we proceed with the same type of analysis that we performed in the previous example, we see that for a fixed $\mathbf{x}$ with $|\mathbf{x}|<1$, we have

$$u(t,\mathbf{x})=\begin{cases} 1, & 0\leq t<1-|\mathbf{x}| \\ \frac{|\mathbf{x}|-t}{2|\mathbf{x}|}, & 1-|\mathbf{x}|<t<1+|\mathbf{x}| \\ 0, & t>1+|\mathbf{x}| \end{cases};$$

we note that this function is not continuous. Qualitatively, at a point inside the unit ball u is uniformly equal to 1 until the time $1-|\mathbf{x}|$, which is the least amount of time required for a signal to pass from outside the unit ball to the point $\mathbf{x}$; after that it jumps discontinuously to the value $1 - \frac{1}{2|\mathbf{x}|}$, before continually decreasing up to time $t = 1 + |\mathbf{x}|$, at which point it jumps again from the value $-\frac{1}{2|\mathbf{x}|}$ to 0, where it stays for all time.

Similarly, for a fixed $\mathbf{x}$ with $|\mathbf{x}| > 1$, we have

$$u(t, \mathbf{x}) = \begin{cases} \frac{|\mathbf{x}|-t}{2|\mathbf{x}|}, & -1 + |\mathbf{x}| < t < 1 + |\mathbf{x}| \\ 0, & |t - |\mathbf{x}|| > 1 \end{cases},$$

which is not continuous either. This is a general feature of solutions to the wave equation with discontinuous initial data: whereas the heat equation smooths out initial discontinuities, the wave equation propagates them. Qualitatively, in this case we see that u is initially zero, and stays zero until time $-1+|\mathbf{x}|$, which is the minimum amount of time required for a signal from inside the unit ball to reach the point $\mathbf{x}$; then it jumps discontinuously to the value $\frac{1}{2|\mathbf{x}|}$ before decreasing linearly to the value $-\frac{1}{2|\mathbf{x}|}$ at time $t = 1 + |\mathbf{x}|$ (which, similarly, is the maximum amount of time for a signal from inside the unit ball to reach $\mathbf{x}$), whereupon it jumps discontinuously back to 0. Thus we have again a single pulse, but the front and back edges are now discontinuous jumps, unlike the previous example.

These two examples end the examinable material for this course. (The last result done in class on August 8, about solutions to Laplace's equation, will be added to the notes on Green's functions.) The following appendices are not examinable (though some of the formulas in Appendix I may shed light on why we define convolution the way we do). The author thanks you for your patience, and hopes that you have gained something from your studies through this course. He would be happy to receive feedback on these notes at ncarruth@math.toronto.edu.

APPENDIX I. We would like to know what becomes of convolution under $\mathcal{O}$. To do this, we first consider in more detail exactly how the Fourier transform turns convolution into multiplication. Suppose that f and g are two suitable functions such that all needed Fourier transforms exist and can be inverted. Then we have

$$\begin{aligned} \mathcal{F}[f * g](\mathbf{k}) &= \int_{\mathbf{R}^m} \left[\int_{\mathbf{R}^m} f(\mathbf{x} - \mathbf{x}')g(\mathbf{x}')\, d\mathbf{x}' \right] e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x} \\ &= \int_{\mathbf{R}^m \times \mathbf{R}^m} f(\mathbf{x} - \mathbf{x}')g(\mathbf{x}')e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x}'\, d\mathbf{x} \\ &= \int_{\mathbf{R}^m \times \mathbf{R}^m} f(\mathbf{x} - \mathbf{x}')e^{-2\pi i \mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}g(\mathbf{x}')e^{-2\pi i \mathbf{k}\cdot\mathbf{x}'}\, d\mathbf{x}'\, d\mathbf{x} \\ &, \end{aligned}$$

from which the result follows after the change of variables $\mathbf{u} = \mathbf{x} - \mathbf{x}'$, $\mathbf{v} = \mathbf{x}'$. We note that the crucial property above was that the expansion functions (the analogoues of the eigenfunctions e_I) satisfied the property

$$e^{-2\pi i \mathbf{k}\cdot\mathbf{x}} = e^{-2\pi i \mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}e^{-2\pi i \mathbf{k}\cdot\mathbf{x}'};$$

mathematically, if we set for convenience $\mathbf{e}_\mathbf{k}(\mathbf{x}) = e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}$, then the $\mathbf{e}_\mathbf{k}$ are so-called *homomorphisms* from the Abelian group $\mathbf{R}^m$ (under vector addition) to the group of complex numbers of unit modulus $\{z \in \mathbf{C} | |z| = 1\}$ – in other words, they take addition of vectors to multiplication of complex numbers:

$$\mathbf{e}_\mathbf{k}(\mathbf{x} + \mathbf{y}) = \mathbf{e}_\mathbf{k}(\mathbf{x})\mathbf{e}_\mathbf{k}(\mathbf{y}).$$

Now on a general region D, it does not make sense to ask whether the eigenfunctions $\mathbf{e}_I$ satisfy a similar property, since if $\mathbf{x}$, $\mathbf{y} \in D$ there is no reason at all to expect that $\mathbf{x} + \mathbf{y} \in D$.[5] Thus there does not appear to be any way to generalise this property of $\mathcal{F}$ to $\mathcal{O}$.

[5]One could, however, ask whether there were not a more general group structure on D. The mathematical field of *harmonic analysis* studies the extension of the transforms here to situations where the domains of the functions are topological groups. These groups are not, however, in general, open subsets of $\mathbf{R}^m$.

With some reflection, though, we note that $\mathbf{e}_{\mathbf{k}}(\mathbf{x})$ is a homomorphism in $\mathbf{k}$ as well as in $\mathbf{x}$ (this is actually a rather trivial observation, since $\mathbf{k}$ and $\mathbf{x}$ appear in $\mathbf{e}_{\mathbf{k}}(\mathbf{x})$ interchangeably, i.e., $\mathbf{e}_{\mathbf{k}}(\mathbf{x}) = \mathbf{e}_{\mathbf{x}}(\mathbf{k})$):

$$\mathbf{e}_{\mathbf{k}+\mathbf{l}}(\mathbf{x}) = \mathbf{e}_{\mathbf{k}}(\mathbf{x})\mathbf{e}_{\mathbf{l}}(\mathbf{x}).$$

From this we can show that the inverse Fourier transform also maps convolutions to products: suppose that we have two Fourier representations

$$f(\mathbf{x}) = \int_{\mathbf{R}^m} \hat{f}(\mathbf{k})e^{-2\pi i\mathbf{k}\cdot\mathbf{x}}\,d\mathbf{k}, \quad g(\mathbf{x}) = \int_{\mathbf{R}^m} \hat{g}(\mathbf{k}')e^{-2\pi i\mathbf{k}'\cdot\mathbf{x}}\,d\mathbf{k}';$$

then we may write their product as

$$\begin{aligned}
f(\mathbf{x})g(\mathbf{x}) &= \int_{\mathbf{R}^m\times\mathbf{R}^m} \hat{f}(\mathbf{k})\hat{g}(\mathbf{k}')e^{-2\pi i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}}\,d\mathbf{k}\,d\mathbf{k}' \\
&= \int_{\mathbf{R}^m\times\mathbf{R}^m} \hat{f}(\mathbf{k}''-\mathbf{k}')\hat{g}(\mathbf{k}')e^{-2\pi i\mathbf{k}''\cdot\mathbf{x}}\,d\mathbf{k}''\,d\mathbf{k}' = \int_{\mathbf{R}^m}\left[\int_{\mathbf{R}^m} \hat{f}(\mathbf{k}''-\mathbf{k}')\hat{g}(\mathbf{k}')\,d\mathbf{k}'\right]e^{-2\pi i\mathbf{k}''\cdot\mathbf{x}}\,d\mathbf{k}'' \\
&= \mathcal{F}^{-1}[f*g](\mathbf{x}),
\end{aligned}$$

where as before we have performed the change of variables $\mathbf{k}'' = \mathbf{k}+\mathbf{k}'$. (We note that the same kind of procedure could be used with the forward Fourier transform $\mathcal{F}$.) Now for the $\mathbf{e}_I$ the prospects of generalising this result are brighter since, for the index sets we have studied, if $I,\ J \in \mathcal{I}$, then in fact we also have $I+J\in\mathcal{I}$. This suggests that, while $\mathcal{O}$ might not turn convolutions into products, perhaps $\mathcal{O}^{-1}$ turns (some generalised form of) convolutions into products. We investigate this in more detail. Suppose that we have two expansions

$$u = \sum_{I\in\mathcal{I}}\tilde{u}(I)\mathbf{e}_I, \quad v = \sum_{I\in\mathcal{I}}\tilde{v}(I)\mathbf{e}_I;$$

then we may write, as before,

$$uv = \sum_{I,J\in\mathcal{I}}\tilde{u}(I)\tilde{v}(J)\mathbf{e}_I\mathbf{e}_J.$$

In general, though, there is now no clear way to proceed, since we do not know anything about the $\mathbf{e}_I$. Suppose that we still had the result $\mathbf{e}_I\mathbf{e}_J = \mathbf{e}_{I+J}$ (none of the sets of eigenfunctions we have dealt with actually satisfy this property); then the above sum would become

$$uv = \sum_{I,J\in\mathcal{I}}\tilde{u}(I)\tilde{v}(J)\mathbf{e}_{I+J} = \sum_{K\in\mathcal{I}}\sum_{J\in\mathcal{I}}\tilde{u}(K-J)\tilde{v}(J)\mathbf{e}_K,$$

from which we see that

$$\mathcal{O}[uv](I) = \sum_{J\in\mathcal{I}}\tilde{u}(I-J)\tilde{v}(J).$$

In general, the best we can hope for is some sort of expansion

$$\mathbf{e}_I\mathbf{e}_J = \sum_{K\in\mathcal{I}}\pi_{IJK}\mathbf{e}_K;$$

such an expansion surely exists, assuming anyway that the eigenfunctions $\mathbf{e}_I$ are not too pathological, and allows us to write

$$uv = \sum_{I,J,K\in\mathcal{I}}\tilde{u}(I)\tilde{v}(J)\pi_{IJK}\mathbf{e}_K,$$

where

$$\pi_{IJK} = (\mathbf{e}_I\mathbf{e}_J,\mathbf{e}_K),$$

meaning that

$$\mathcal{O}[uv](K) = \sum_{I,J\in\mathcal{I}} \tilde{u}(I)\pi_{IJK}\tilde{v}(J).$$

This is probably the closest we can come to generalising the property of mapping convolutions into products enjoyed by the Fourier transform. If π_{IJK} is zero for most values of the parameters IJK, then this result may still be useful; if not, it is probably just a curiousity.

We give an example.

EXAMPLE. Let us consider the simple case of the eigenfunctions of the Laplacian on the unit square with Dirichlet boundary conditions. We have not considered this case directly but a quick review of our derivation of the eigenfunctions of the Laplacian on the unit cube shows that the eigenfunctions are $\mathbf{e}_I = \sin \ell\pi x \sin m\pi y$, where $I = (\ell, m)$, $\ell, m \in \mathbf{Z}$, $\ell, m > 0$. Thus in this case, letting $I = (\ell, m)$, $J = (\ell', m')$, and $K = (\ell'', m'')$, we have

$$\pi_{IJK} = \int_Q \sin \ell\pi x \sin m\pi y \sin \ell'\pi x \sin m'\pi y \sin \ell''\pi x \sin m''\pi y\, dx\, dy.$$

Now

$$\begin{aligned}\int_0^1 \sin \ell\pi x \sin \ell'\pi x \sin \ell''\pi x\, dx &= \frac{1}{2}\int_0^1 [\cos(\ell-\ell')\pi x - \cos(\ell+\ell')\pi x] \sin \ell''\pi x\, dx\\ &= \frac{1}{4}\int_0^1 \sin(\ell''+\ell-\ell')\pi x - \sin(\ell''-\ell+\ell')\pi x\\ &\qquad - \sin(\ell''+\ell+\ell')\pi x + \sin(\ell''-\ell-\ell')\pi x\, dx,\end{aligned}$$

which we shall not evaluate explicitly but only determine when it is zero. Clearly, $\int_0^1 \sin n\pi x = \frac{1}{n\pi}(1-(-1)^n)$ is zero exactly when n is even; thus the above integral will be zero unless at least one of the quantities

$$\ell''+\ell-\ell', \quad \ell''-\ell+\ell', \quad \ell''+\ell+\ell', \quad \ell''-\ell-\ell'$$

is odd; but the first two are odd together, as are the last two, and thus the integral will vanish unless at least one of

$$\ell''+\ell-\ell', \quad \ell''-\ell-\ell'$$

is odd. But these are also seen to be odd together, so we find at last that the integral will vanish unless

$$\ell''-\ell-\ell'$$

is odd. Since analogous results hold for the corresponding y integrals, we see that π_{IJK} will be zero unless the quantity

$$K-(I+J)$$

is odd (meaning that both of its components are odd). While this is not nearly as nice as requiring it to vanish, it does tell us that π_{IJK} vanishes for a sizeable number of indices IJK.

Similar triple products can (I believe) be worked out for the Legendre polynomials and the Legendre functions, and probably Bessel functions as well. If anyone is interested in knowing more about this particular topic, please let me know and I can provide more references.

APPENDIX II. SOLUTIONS TO THE NONHOMOGENEOUS WAVE EQUATION. We sketch a solution to the nonhomogeneous wave equation on $\mathbf{R}^3$. Thus consider the problem

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u + F, \quad u|_{t=0} = f, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g.$$

Fourier transforming as usual, we have

$$\frac{\partial^2 \hat{u}}{\partial t^2} = -4\pi^2|\mathbf{k}|^2\hat{u} + \hat{F}, \quad \hat{u}|_{t=0} = \hat{f}, \quad \left.\frac{\partial \hat{u}}{\partial t}\right|_{t=0} = \hat{g}.$$

Thus we must now solve an equation of the form

$$y'' + \alpha^2 y = h \tag{2}$$

where $\alpha > 0$ and h is some given function. We may do this by the method of *variation of parameters* (also called *variation of constants*). (See [1], sections 3.4 and 3.6 (especially Theorem 3.6.4 and accompanying discussion) for a treatment of this method in a general setting.) The general solution to the corresponding homogeneous equation

$$y'' + \alpha^2 y = 0$$

is

$$y = a\cos\alpha x + b\sin\alpha x,$$

where $a = y(0)$ and $b = \frac{y'(0)}{\alpha}$. The method of variation of parameters starts by looking for solutions to equation (2) of the form

$$y = a(x)\cos\alpha x + b(x)\sin\alpha x.$$

Differentiating once, we obtain

$$y' = a'\cos\alpha x + b'\sin\alpha x + \alpha\left(-a(x)\sin\alpha x + b(x)\cos\alpha x\right).$$

We require the sum of the first two terms to vanish; then differentiating again, we obtain

$$y'' = \alpha\left(-a'\sin\alpha x + b'\cos\alpha x\right) - \alpha^2\left(a(x)\cos\alpha x + b(x)\sin\alpha x\right),$$

from which we see easily that

$$y'' + \alpha y = h = \alpha\left(-a'\sin\alpha x + b'\cos\alpha x\right).$$

Combining this with the requirement

$$a'\cos\alpha x + b'\sin\alpha x = 0,$$

we see that we now have the system

$$\begin{aligned} \cos\alpha x a' + \sin\alpha x b' &= 0 \\ -\alpha\sin\alpha x a' + \alpha\cos\alpha x b' &= h. \end{aligned}$$

Now the determinant of the coefficient matrix is just the Wronskian of the two solutions:

$$W = \begin{vmatrix} \cos\alpha x & \sin\alpha x \\ -\alpha\sin\alpha x & \alpha\cos\alpha x \end{vmatrix} = \alpha,$$

so that as long as we assume $\alpha \neq 0$ we may solve the above system; in fact, we have (using our formula for the inverse of a two by two matrix)

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \frac{1}{\alpha}\begin{pmatrix} \alpha\cos\alpha x & -\sin\alpha x \\ \alpha\sin\alpha x & \cos\alpha x \end{pmatrix}\begin{pmatrix} 0 \\ h \end{pmatrix} = \begin{pmatrix} -h\frac{\sin\alpha x}{\alpha} \\ h\frac{\cos\alpha x}{\alpha} \end{pmatrix}.$$

From this we have

$$\begin{aligned} a &= y(0) - \frac{1}{\alpha}\int_0^x h(u)\sin\alpha u\, du \\ b &= \frac{1}{\alpha}y'(0) + \frac{1}{\alpha}\int_0^x h(u)\cos\alpha u\, du, \end{aligned}$$

so that

$$\begin{aligned} y &= y(0)\cos\alpha x + y'(0)\frac{\sin\alpha x}{\alpha} + \frac{1}{\alpha}\int_0^x h(u)\sin\alpha x\cos\alpha u - \sin\alpha u\cos\alpha x\, du \\ &= y(0)\cos\alpha x + y'(0)\frac{\sin\alpha x}{\alpha} + \int_0^x h(u)\frac{\sin\alpha(x-u)}{\alpha}\, du. \end{aligned}$$

We now return to our original problem:

$$\frac{\partial^2 \hat{u}}{\partial t^2} = -4\pi^2|\mathbf{k}|^2\hat{u} + \hat{F}, \quad \hat{u}|_{t=0} = \hat{f}, \quad \left.\frac{\partial \hat{u}}{\partial t}\right|_{t=0} = \hat{g}.$$

The above formula gives

$$\hat{u}(t,\mathbf{k}) = \hat{f}\cos 2\pi|\mathbf{k}|t + \hat{g}\frac{\sin 2\pi|\mathbf{k}|t}{2\pi|\mathbf{k}|} + \int_0^t \hat{F}(s,\mathbf{k})\frac{\sin 2\pi|\mathbf{k}|(t-s)}{2\pi|\mathbf{k}|}\,ds.$$

The first two terms are of course the same as those we obtained for the homogeneous equation above. We see that we may invert this formula in much the same way as we did the formula for the solution to the homogeneous problem previously. Specifically, we obtain

$$u(t,\mathbf{x}) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi t}\int_{S_t(\mathbf{x})} f(\mathbf{x}')\,d\mathbf{x}'\right] + \frac{1}{4\pi t}\int_{S_t(\mathbf{x})} g(\mathbf{x}')\,d\mathbf{x}' + \int_0^t \frac{1}{4\pi(t-s)}\int_{S_{t-s}(\mathbf{x})} F(s,\mathbf{x}')\,dS'\,ds.$$

Let us investigate the final term here, which is the only new thing. We see that the contribution which it gives to $u(\mathbf{x})$ is equal to the integral over all times from 0 to t of a quantity which at time s is (proportional to) the integral over the sphere of radius $t-s$ centred at $\mathbf{x}$ – in other words, the integral over the surface from which a signal will take exactly the time $t-s$ remaining to reach the point $\mathbf{x}$. More succinctly, the contribution F makes to u at the point $\mathbf{x}$ and time t is the integral over the set of all points (through all of space-time, not just space) $(s,\mathbf{x}')$ satisfying $|\mathbf{x}-\mathbf{x}'| = t-s$, i.e., the set of all points just able to send a signal to $\mathbf{x}$ by time t.

We may write the above result more simply as follows. First, let us do a change of variables and write $u = t-s$, $\mathbf{x}'' = \mathbf{x}' - \mathbf{x}$; then the last integral above becomes

$$\int_0^t \int_{S_u(\mathbf{x})} \frac{1}{4\pi u} F(t-u,\mathbf{x}')\,dS'\,du = \int_0^t \int_{S_u(0)} \frac{1}{4\pi u} F(t-u,\mathbf{x}''+\mathbf{x})\,dS''\,du;$$

if we now introduce spherical coordinates (r'',θ'',ϕ'') for $\mathbf{x}''$, we may write this integral as (noting that $dS'' = u^2 \sin\theta''\,d\theta''\,d\phi''$ since it is the full surface-area element for the sphere of radius u)

$$\begin{aligned}\int_0^t \int_0^{2\pi}\int_0^{\pi} \frac{1}{4\pi u}F(t-u,\mathbf{x}''+\mathbf{x})\sin\theta''\,d\theta''\,d\phi''\,u^2\,du &= \int_{B_t(0)} \frac{F(t-r'',\mathbf{x}''+\mathbf{x})}{4\pi r''}\,dV \\ &= \int_{B_t(\mathbf{x})} \frac{F(t-|\mathbf{x}-\mathbf{x}'|,\mathbf{x}')}{4\pi|\mathbf{x}-\mathbf{x}'|}\,dV,\end{aligned}$$

where we have changed back to $\mathbf{x}' = \mathbf{x}''+\mathbf{x}$ in the last line, and noted that $r'' = |\mathbf{x}''| = |\mathbf{x}-\mathbf{x}'|$. This expression is related to the so-called *retarded potential* which is used in studying electromagnetic radiation. We recognise the quantity $\frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|}$ as being (up to a sign) the Green's function for the Laplacian on $\mathbf{R}^3$; what is different here is that we are integrating it against a function $F(t-|\mathbf{x}-\mathbf{x}'|,\mathbf{x}')$ instead of a function of $\mathbf{x}'$ alone. In other words, roughly speaking, the effect of the source F on the solution u is obtained by integrating against the ordinary Green's function for the Laplacian, but using the *retarded* source function $F(t-|\mathbf{x}-\mathbf{x}'|,\mathbf{x}')$ at times which are such that a signal from the point of integration $\mathbf{x}'$ can just reach the observation point $\mathbf{x}$ by the observation time t.

We note that the above method of variation of parameters can be used with only slight modifications to solve the nonhomogeneous wave equation on a bounded region, in a manner analogous to our solution to the wave equation on a disk given above.

REFERENCES

Coddington, E. A., and Levinson, N. Theory of Ordinary Differential Equations. New York: McGraw-Hill Book Company, Inc., 1955.

Additional solutions to Laplace's equation

Laplace's equation $\nabla^2 u = 0$ has the following general series expansions as its solutions when solved in the indicated regions and with the indicated boundary conditions:

Region and boundary conditions, and dates for notes	Series expansion, related complete orthogonal set, and inner product
$\{(\rho,\phi,z)\vert\rho\leq a, 0\leq z\leq b\}$ $u\vert_{z=0}=u\vert_{z=b}=0$ July 2 – 4	$\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} I_m\left(\frac{n\pi}{b}\rho\right)(a_{nm}\cos m\phi + b_{nm}\sin m\phi)\sin\frac{n\pi}{b}z$ $\{\cos m\phi\sin\frac{n\pi}{b}z, \sin m\phi\sin\frac{n\pi}{b}z \vert n,m\in\mathbf{Z}, n\geq 1, m\geq 0\}$ $(f(\phi,z),g(\phi,z)) = \int_0^{2\pi}\int_0^b f(\phi,z)\overline{g(\phi,z)}\,dz\,d\phi$
$\{(x,y,z)\vert 0\leq x,y,z\leq 1\}$ $u\vert_{x=0}=u\vert_{x=1}=$ $u\vert_{y=0}=u\vert_{y=1}=0$ July 9 – 11	$\sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty}\sin\ell\pi x\sin m\pi y\left(a_{\ell m}\cosh\sqrt{\ell^2+m^2}\pi z + b_{\ell m}\sinh\sqrt{\ell^2+m^2}\pi z\right)$ $\{\sin\ell\pi x\sin m\pi y\vert\ell,m\in\mathbf{Z},\ell,m\geq 1\}, (f(x,y),g(x,y)) = \int_0^1\int_0^1 f(x,y)\overline{g(x,y)}\,dx\,dy$

We may interchange x, y, and z in the last example to obtain additional solutions on the cube.

In cases where more than one set of boundary conditions is inhomogeneous, we may express the solution as a sum of two or three separate ones, each of which satisfies a problem with one set of inhomogeneous boundary conditions. See notes of July 2 – 4, pp. 3 – 6 for an example.

Eigenfunctions and eigenvalues for the Laplacian: $\nabla^2 u = \lambda u$

Region and boundary conditions, and dates for notes	Eigenfunctions, eigenvalues, and parameter ranges
$Q=\{(x,y,z)\vert 0\leq x,y,z\leq 1\},\ u\vert_{\partial Q}=0$ July 9 – 11	$\sin\ell\pi x\sin m\pi y\sin n\pi z,\ -\pi^2\left(\ell^2+m^2+n^2\right),\ \ell,m,n\in\mathbf{Z},\ \ell,m,n\geq 1$
$Q=\{(x,y,z)\vert 0\leq x,y,z\leq 1\},\ \left.\frac{\partial u}{\partial n}\right\vert_{\partial Q}=0$ [Homeworks 10 and 11]	$\cos\ell\pi x\cos m\pi y\cos n\pi z,\ -\pi^2\left(\ell^2+m^2+n^2\right),\ \ell,m,n\in\mathbf{Z},\ \ell,m,n\geq 0$
$C=\{(\rho,\phi,z)\vert\rho\leq 1, 0\leq z\leq 1\},\ u\vert_{\partial C}=0$ July 9 – 11, 16 – 18	$J_m(\lambda_{mi}\rho)\sin n\pi z\begin{cases}\cos m\phi\\ \sin m\phi\end{cases},\ -\lambda_{mi}^2-n^2\pi^2,$ $m,n,i\in\mathbf{Z},\ m\geq 0,\ n,i\geq 1,\ \lambda_{mi}$ the ith positive zero of $J_m(x)$
$B=\{(r,\theta,\phi)\vert r<1\},\ u\vert_{\partial B}=0$ July 16 – 18	$j_\ell(\kappa_{\ell i}r)P_{\ell m}(\cos\theta)\begin{cases}\cos m\phi\\ \sin m\phi\end{cases},\ -\kappa_{\ell i}^2,\ \ell,m,i\in\mathbf{Z},\ \ell\geq 0,\ 0\leq m\leq\ell,\ i\geq 1,$ $\kappa_{\ell i}=\lambda_{\ell+\frac{1}{2},i}$ the ith positive zero of $j_\ell(x)$
$D=\{(\rho,\phi)\vert\rho<a\},\ u\vert_{\partial D}=0$ August 6 – 8	$J_m\left(\frac{\lambda_{mi}}{a}\rho\right)\begin{cases}\cos m\phi\\ \sin m\phi\end{cases},\ -\frac{1}{a^2}\lambda_{mi}^2,\ m,i\in\mathbf{Z},m\geq 0,i\geq 1,$ λ_{mi} the ith positive zero of $J_m(x)$

The inner product used is $(f,x)=\int_X f(\mathbf{x})\overline{g(\mathbf{x})}\,d\mathbf{x}$, where X is the region and $d\mathbf{x}$ is the volume or area element.

All of the above sets are complete and orthogonal with respect to their respective inner product.

For general concepts relating to eigenfunctions and eigenvalues, see notes of July 2 – 4.

Additional special functions: equations and properties

Modified Bessel functions. These are solutions $I_m(x)$, $m\in\mathbf{Z}$, $m\geq 0$ to the equation

$$\frac{d^2I}{dx^2}+\frac{1}{x}\frac{dI}{dx}-\left(1+\frac{m^2}{x^2}\right)I=0$$

(compare the equation satisfied by Bessel functions $J_m(x)$). They are exponential rather than oscillatory in nature and hence do not form an orthogonal basis. They satisfy many similar identities to the unmodified Bessel functions but we do not need these identities in this course. *(continued)*

Spherical Bessel functions. These are solutions $j_\ell(x)$, $\ell \in \mathbf{Z}$, $\ell \geq 0$ to the equation

$$\frac{d^2 j}{dx^2} + \frac{2}{x}\frac{dj}{dx} + \left(1 - \frac{\ell(\ell+1)}{x^2}\right) j = 0$$

and can be expressed as $j_\ell(x) = \sqrt{\frac{\pi}{2x}} J_{\ell+\frac{1}{2}}(x)$. They can be expressed in terms of elementary functions (though we don't use that here). If $\kappa_{\ell i}$ denotes the ith positive zero of $j_\ell(x)$, then for each ℓ the set $\{j_\ell(\kappa_{\ell i} r)\}_{i=1}^\infty$ forms a complete orthogonal set on $[0,1]$ with respect to the inner product

$$(f(r), g(r)) = \int_0^1 f(r)\overline{g(r)}\, r^2\, dr.$$

Their normalisation with respect to this inner product is

$$(j_\ell(\kappa_{\ell i} r), j_\ell(\kappa_{\ell i} r)) = \frac{1}{2} j_{\ell+1}^2(\kappa_{\ell i}).$$

The j_ℓ satisfy many identities similar to those satisfied by the ordinary Bessel functions, but everything we shall need to calculate can be obtained by reducing to the ordinary Bessel functions so we do not give them.

Poisson's equation on a bounded domain. Let X denote one of Q, C, and B. The problem on X

$$\nabla^2 u = f, \quad u|_{\partial X} = 0$$

can be solved by expanding $f = \sum_I a_I \mathbf{e}_I$, where $a_I = \frac{(f, \mathbf{e}_I)}{(\mathbf{e}_I, \mathbf{e}_I)}$, and $u = \sum_I b_I \mathbf{e}_I$; $\nabla^2 u = f$ then gives

$$\lambda_I^2 b_I = a_I.$$

Here $\mathbf{e}_I$ is the eigenfunction of the Laplacian satisfying

$$\nabla^2 \mathbf{e}_I = \lambda_I \mathbf{e}_I, \quad \mathbf{e}_I|_{\partial X} = 0.$$

See the notes of July 9 – 11 and 16 – 18 for examples. The more general problem

$$\nabla^2 u = f, \quad u|_{\partial X} = g$$

may be solved as the sum $u = u_1 + u_2$ of the two problems

$$\nabla^2 u_1 = f, \quad u_1|_{\partial X} = 0, \qquad \nabla^2 u_2 = 0, \quad u_2|_{\partial X} = g.$$

See the notes of July 16 – 18 for examples of this type of problem. The related problem

$$\nabla^2 u = f, \quad \frac{\partial u}{\partial n}\Big|_{\partial X} = 0 \qquad [\tfrac{\partial}{\partial n} \text{ the outward normal derivative}]$$

may be solved in the same way, using the eigenfunctions satisfying

$$\nabla^2 \mathbf{e}_I = \lambda_I \mathbf{e}_I, \quad \frac{\partial \mathbf{e}_I}{\partial n}\Big|_\partial X = 0,$$

except when one or more of the eigenvalues vanish: in that case f must be orthogonal to all corresponding eigenfunctions, and additional conditions must be imposed on u to get a unique solution. See the Appendix to the solutions for Homework 11, and the notes for July 2 – 4. The inhomogeneous problem may then be treated as above.

Green's functions for Poisson's equation. Suppose that $G(\mathbf{x}, \mathbf{x}')$ is a function satisfying

$$\nabla_{\mathbf{x}}^2 G(\mathbf{x}, \mathbf{x}') = -\delta(\mathbf{x} - \mathbf{x}')$$

where δ is the Dirac delta function (see the next page for a review of this function). Then for u sufficiently differentiable on a domain D we have

$$u(\mathbf{x}) = -\int_D G(\mathbf{x}, \mathbf{x}')\nabla_{\mathbf{x}'}^2 u(\mathbf{x}')\, d\mathbf{x}' + \int_{\partial D} G(\mathbf{x}, \mathbf{x}')\frac{\partial u}{\partial n'} - u(\mathbf{x}')\frac{\partial G}{\partial n'}\, dS'.$$

We may use Green's functions satisfying certain boundary conditions to solve boundary-value problems.

$$G(\mathbf{x}, \mathbf{x}')|_{\mathbf{x} \in \partial D} = 0: \qquad u = -\int_D G(\mathbf{x}, \mathbf{x}') f(\mathbf{x}')\, d\mathbf{x}' - \int_{\partial D} \frac{\partial G}{\partial n'} g(\mathbf{x}')\, dS' \text{ solves } \nabla^2 u = f,\ u|_{\partial D} = g$$

$$\frac{\partial G}{\partial n}\bigg|_{\mathbf{x} \in \partial D} = 0: \qquad u = -\int_D G(\mathbf{x}, \mathbf{x}') f(\mathbf{x}')\, d\mathbf{x}' + \int_{\partial D} G(\mathbf{x}, \mathbf{x}') g(\mathbf{x}')\, dS' \text{ solves } \nabla^2 u = f,\ \frac{\partial u}{\partial n}\bigg|_{\partial D} = g$$

On $\mathbf{R}^3$, the solution vanishing at infinity to

$$\nabla_{\mathbf{x}}^2 G(\mathbf{x}, \mathbf{x}') = -\delta(\mathbf{x} - \mathbf{x}') \quad \text{is} \quad G(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|}.$$

Thus on $\mathbf{R}^3$ the solution vanishing at infinity to Poisson's equation

$$\nabla^2 u = f \quad \text{is} \quad u(\mathbf{x}) = -\int_{\mathbf{R}^3} G(\mathbf{x}, \mathbf{x}') f(\mathbf{x}')\, d\mathbf{x}'.$$

Fourier transforms. These are covered in the notes for July 23 – 25, July 30 – August 1, and August 6 – August 8. If $f(\mathbf{x})$ is a function on $\mathbf{R}^m$ which satisfies $\int_{\mathbf{R}^m} |f(\mathbf{x})|\,d\mathbf{x} < \infty$, then we say that f *is in* L^1 and define its *Fourier transform*

$$\hat{f}(\mathbf{k}) = \mathcal{F}[f(\mathbf{x})](\mathbf{k}) = \int_{\mathbf{R}^m} f(\mathbf{x})e^{-2\pi i\mathbf{k}\cdot\mathbf{x}}\,d\mathbf{x}.$$

If f is continuous and bounded and such that $\hat{f}(\mathbf{k})$ is in L^1, then we have the *Fourier inversion theorem*

$$f(\mathbf{x}) = \mathcal{F}^{-1}[\hat{f}(\mathbf{k})](\mathbf{x}) = \int_{\mathbf{R}^m} \hat{f}(\mathbf{k})e^{2\pi i\mathbf{k}\cdot\mathbf{x}}\,d\mathbf{k}.$$

This may be shewn by making use of so-called *approximate identities*, which are sequences $\{\phi_n(\mathbf{x})\}$ of functions in L^1 satisfying

$$\textstyle\int_{\mathbf{R}^m}\phi_n(\mathbf{x})\,d\mathbf{x} = 1, \quad \int_{\mathbf{R}^m}\phi_n(\mathbf{x})f(\mathbf{x})\,d\mathbf{x} \to f(0) \text{ as } n\to\infty$$

for all suitable (e.g., continuous and bounded) functions $f(\mathbf{x})$. If ϕ is any individual function in L^1 satisfying $\int_{\mathbf{R}^m}\phi(\mathbf{x})\,d\mathbf{x} = 1$, then the sequence $\{n^m\phi(n\mathbf{x})\}$ is an approximate identity.

If $f(\mathbf{x})$ and $g(\mathbf{x})$ are two functions in L^1 on $\mathbf{R}^m$, we define their *convolution* $f * g$ by

$$(f*g)(\mathbf{x}) = \int_{\mathbf{R}^m} f(\mathbf{x}-\mathbf{x}')g(\mathbf{x}')\,d\mathbf{x}'.$$

The Fourier transform maps convolution to multiplication in the following sense:

$$\mathcal{F}[(f*g)(\mathbf{x})](\mathbf{k}) = \hat{f}(\mathbf{k})\hat{g}(\mathbf{k}), \quad \mathcal{F}^{-1}[f(\mathbf{k})g(\mathbf{k})](\mathbf{x}) = (\mathcal{F}^{-1}[f]*\mathcal{F}^{-1}[g])(\mathbf{x}).$$

The Fourier transform possesses the following properties (see notes for July 23 – 25, p. 15):

$$\mathcal{F}[af+bg](\mathbf{k}) = a\mathcal{F}[f](\mathbf{k}) + b\mathcal{F}[g](\mathbf{k}), \quad \mathcal{F}[\partial_j f](\mathbf{k}) = 2\pi i k_j\mathcal{F}[f](\mathbf{k}), \quad \mathcal{F}[2\pi i x_j f](\mathbf{k}) = -\frac{\partial}{\partial k_j}\mathcal{F}[f](\mathbf{k})$$

$$\mathcal{F}[f(\mathbf{x}-\boldsymbol{\alpha})](\mathbf{k}) = e^{-2\pi i\mathbf{k}\cdot\boldsymbol{\alpha}}\hat{f}(\mathbf{k}), \quad \mathcal{F}[e^{2\pi i\boldsymbol{\alpha}\cdot\mathbf{x}}f(\mathbf{x})](\mathbf{k}) = \mathcal{F}[f](\mathbf{k}-\boldsymbol{\alpha}).$$

The Fourier transform of a Gaussian is

$$\mathcal{F}\left[e^{-a|\mathbf{x}|^2}\right](\mathbf{k}) = \left(\frac{\pi}{a}\right)^{\frac{m}{2}} e^{-\frac{\pi^2|\mathbf{k}|^2}{a}}, \quad \mathcal{F}^{-1}\left[e^{-a|\mathbf{k}|^2}\right](\mathbf{x}) = \left(\frac{\pi}{a}\right)^{\frac{m}{2}} e^{-\frac{\pi^2|\mathbf{x}|^2}{a}}.$$

Heat equation: bounded domains. Let X denote one of Q, C, and B. The problem on $(0,+\infty)\times X$

$$\frac{\partial u}{\partial t} = \nabla^2 u, \quad u|_{t=0} = f, \quad u|_{\partial X} = 0$$

can be solved by expanding $f = \sum_I a_I\mathbf{e}_I$, where $a_I = \frac{(f,\mathbf{e}_I)}{(\mathbf{e}_I,\mathbf{e}_I)}$, and $u = \sum_I b_I(t)\mathbf{e}_I$; the equation and initial condition then give

$$b_I'(t) = \lambda_I b_I, \quad b_I(0) = a_I, \quad \text{whence} \quad b_I(t) = a_I e^{\lambda_I t}.$$

Here $\mathbf{e}_I$ and λ_I denote the appropriate eigenfunctions and eigenvalues, as in our discussion of the Poisson equation. See the notes of July 2 – 4, 9 – 11, and 16 – 18 for details and examples. The more general problem

$$\tfrac{\partial u}{\partial t} = \nabla^2 u, \quad u|_{t=0} = f, \quad u|_{\partial X} = g,$$

where g is a function of $\mathbf{x}$ alone, can be solved as the sum $u = u_1 + u_2$ of the two problems

$$\nabla^2 u_1 = 0, \quad u_1|_{\partial X} = g, \quad \tfrac{\partial u_2}{\partial t} = \nabla^2 u_2, \quad u_2|_{t=0} = f - g, \quad u_2|_{\partial X} = 0.$$

See the notes of July 16 – 18, pp. 7 – 8, for discussion and an example.

Heat equation and generalisations on $\mathbf{R}^m$. The problem on $(0,+\infty)\times\mathbf{R}^m$

$$\frac{\partial u}{\partial t} = \nabla^2 u, \quad u|_{t=0} = f, \quad \lim_{|\mathbf{x}|\to\infty} u(\mathbf{x}) = 0$$

can be solved using Fourier transforms, obtaining $\frac{\partial\hat{u}}{\partial t} = -4\pi^2|\mathbf{k}|^2\hat{u}$, $\hat{u}|_{t=0} = \hat{f}$, whence $\hat{u} = \hat{f}e^{-4\pi^2 t|\mathbf{k}|^2}$, and

$$u(t,\mathbf{x}) = (K_t * f)(\mathbf{x}), \quad \text{where the \textit{heat kernel} } K_t(\mathbf{x}) = \tfrac{1}{(4\pi t)^{\frac{m}{2}}}e^{-\frac{|\mathbf{x}|^2}{4t}}.$$

Note that the heat kernel is an approximate identity in the limit $t\to 0^+$. The more general problem on $(0,+\infty)\times\mathbf{R}^m$

$$\tfrac{\partial u}{\partial t} = \nabla^2 u + g(\mathbf{x},t), \quad u|_{t=0} = f$$

has solution

$$u(t,\mathbf{x}) = K_t(\mathbf{x}) * f(\mathbf{x}) + \textstyle\int_0^t K_{t-s}(\mathbf{x}) * g(s,\mathbf{x})\,ds.$$

In practice it may be simpler to solve both of these problems by working directly with Fourier transforms. More general equations such as $\frac{\partial u}{\partial t} = \nabla^2 u + \mathbf{n}\cdot\nabla u$ can be solved in this way. See notes for July 30 – August 1 and Homework 12, and the practice problems for week 12 and the final.

Wave equation: bounded domains. Again, let X denote one of Q, C, and B. The problem on $(0, +\infty) \times X$

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u, \quad u|_{t=0} = f, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g, \quad u|_{\partial X} = 0$$

can be solved by expanding $f = \sum_I a_I \mathbf{e}_I$, $g = \sum_I b_I \mathbf{e}_I$, where $a_I = \frac{(f, \mathbf{e}_I)}{(\mathbf{e}_I, \mathbf{e}_I)}$ and $b_I = \frac{(g, \mathbf{e}_I)}{(\mathbf{e}_I, \mathbf{e}_I)}$, and also $u = \sum_I c_I(t) \mathbf{e}_I$; the equation and initial conditions then give

$$c_I''(t) = \lambda_I c_I(t), \quad c_I(0) = a_I, \quad c_I'(0) = b_I.$$

This is a simple second-order constant-coefficient ordinary differential equation and can be solved easily. Here $\mathbf{e}_I$ and λ_I denote the appropriate eigenfunctions and eigenvalues, as above. The *frequencies* are $\sqrt{-\lambda_I}$. See the notes for August 6 – August 8 for an example on the disk.
Wave equation on $\mathbf{R}^3$. The problem on $(0, +\infty) \times \mathbf{R}^3$

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u, \quad u|_{t=0} = f, \left.\frac{\partial u}{\partial t}\right|_{t=0} = g$$

can be solved using Fourier transforms, obtaining $\frac{\partial^2 \hat{u}}{\partial t^2} = -4\pi^2 |\mathbf{k}|^2 \hat{u}$, $\hat{u}|_{t=0} = \hat{f}$, $\left.\frac{\partial \hat{u}}{\partial t}\right|_{t=0} = \hat{g}$. Ultimately,

$$u(t, \mathbf{x}) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi t}\int_{S_t(\mathbf{x})} f(\mathbf{x}')\, dS'\right] + \frac{1}{4\pi t}\int_{S_t(\mathbf{x})} g(\mathbf{x}')\, dS',$$

where $S_t(\mathbf{x})$ is the sphere of radius t centred at $\mathbf{x}$.

More general equations on $\mathbf{R}^m$. The problem on $(0, +\infty) \times \mathbf{R}^m$

$$\frac{\partial u}{\partial t} = \nabla^2 u + \mathbf{n} \cdot \nabla u + bu, \quad u|_{t=0} = f$$

can be solved by taking Fourier transforms, obtaining

$$\frac{\partial \hat{u}}{\partial t} = -4\pi^2 |\mathbf{k}|^2 \hat{u} + 2\pi i \mathbf{n} \cdot \mathbf{k} \hat{u} + b\hat{u}, \quad \hat{u}|_{t=0} = \hat{f},$$

whence

$$\hat{u} = e^{-4\pi^2 t|\mathbf{k}|^2 + 2\pi i t \mathbf{n} \cdot \mathbf{k} + bt} \hat{f},$$

which can be inverted using properties of the Fourier transform to obtain u.

Last (Family) Name: Carruth

First (Given) Name: Nathan

Student Number: [purposely omitted]

University of Toronto, Faculty of Arts and Science

APM346 H1Y, August 2019 Final Examinations

Instructor: Nathan Carruth

Duration: 3 hours. **No aids allowed.**

Please read the following instructions:

- Please fill out the front of this exam booklet, but do not begin writing the actual exam until the announcements are over and the Exam Facilitator has started the exam.
- No aids of any form are allowed on this exam. Possessing an aid during this exam may result in your being charged with an academic offence.
- All cell phones, smart watches, electronic devices, toasters, etc., must be turned off and placed in your bag under your desk. All study materials must also be placed in your bag under your desk. Having such items on your person after the exam has started may be an academic offence.
- [Your instructor does not recommend carrying toasters in your pockets anyway.]
- When you are done with your exam, please raise your hand and wait for someone to come and collect it. Do not collect your bag and jacket while still in possession of the exam paper.
- If you are feeling ill and unable to finish your exam, please let an Exam Facilitator know this prior to leaving the exam hall so it can be properly noted.
- In the event of a fire alarm, do not check your cell phone when escorted outside.

Special instructions:

- You *must* use the definition of the Fourier transform given in class. Use of a different definition (including that given in the textbook) may result in lost marks.
- Use of an incorrect orthogonal set on a problem may result in a very low score for the entire problem. Please check the sets you use. Sets which were derived in class, in the notes, or in the homework solutions on the course webpage may be used without derivation. Other sets, if needed, may be stated without derivation, but then no partial credit will be given for a partially correct set.
- This exam has eight questions, for a total of 125 marks. The weighting is indicated on each question. Note that weighting may not directly correspond either to difficulty or to amount of writing required, and that the ordering of the problems may not be the best order in which to write the exam. You must show all of your work for credit.
- You may use the back sides of the pages, as well as the last four pages, to continue your solutions, as long as this is clearly indicated.

- Unless otherwise stated, you must write out the full form of the final answer for full marks.

1. [8 marks] Solve the following boundary-value problem on the unit cube $Q = \{(x, y, z) | x, y, z \in [0, 1]\}$:

$$\nabla^2 u = 0, \quad u|_{\partial Q} = \begin{cases} 1, & z = 1 \\ 0, & \text{otherwise} \end{cases}.$$

We have from class that the general solution to $\nabla^2 u = 0$ on Q with $u|_{x=0} = u|_{x=1} = u|_{y=0} = u|_{y=1} = 0$ is

$$u = \sum_{\ell=1}^{\infty} \sum_{m=1}^{\infty} \sin \ell\pi x \sin m\pi y \left(a_{\ell m} \sinh \pi\sqrt{\ell^2 + m^2} z + b_{\ell m} \cosh \pi\sqrt{\ell^2 + m^2} z \right). \qquad \text{[2 marks]}$$

The boundary conditions then give

$$u|_{z=0} = 0 = \sum_{\ell,m=1}^{\infty} \sin \ell\pi x \sin m\pi y \, (b_{\ell m}) \,, \text{[1 mark]} \qquad \text{so} \qquad b_{\ell m} = 0 \text{ [1 mark]}$$

$$u|_{z=1} = 1 = \sum_{\ell,m=1}^{\infty} \sin \ell\pi x \sin m\pi y \left(a_{\ell m} \sinh \pi\sqrt{\ell^2 + m^2} \right) \qquad \text{[1 mark]}$$

$$\begin{aligned} a_{\ell m} \sinh \pi\sqrt{\ell^2 + m^2} &= 4 \int_0^1 \int_0^1 \sin \ell\pi x \sin m\pi y \, dy \, dx = 4 \left(\int_0^1 \sin \ell\pi x \, dx \right) \left(\int_0^1 \sin m\pi y \, dy \right) \text{ [1 mark]} \\ &= r \left(-\frac{1}{\ell\pi} \cos \ell\pi x \Big|_0^1 \right) \left(-\frac{1}{m\pi} \cos m\pi y \Big|_0^1 \right) \\ &= \frac{4}{\pi^2 \ell m} \left(1 - (-1)^\ell\right) \left(1 - (-1)^m\right), \text{ [1 mark]} \end{aligned}$$

so the solution is

$$u = \sum_{\ell=1, \ell \text{ odd}}^{\infty} \sum_{m=1, m \text{ odd}}^{\infty} \frac{16}{\pi^2 \ell m \sinh \pi\sqrt{\ell^2 + m^2}} \sin \ell\pi x \sin m\pi y \sinh \pi\sqrt{\ell^2 + m^2} z. \qquad \text{[1 mark]}$$

Notes. 1 mark was given if the form for the expansion was not quite correct. Writing out a sum over only ℓ and m odd (as done here) was not required. Taking the initial value of ℓ and m to be 0 instead of 1 should typically result in a deduction of 0.5 marks, since in this case the final expression is meaningless.

2. [22 marks] Solve the following boundary-value problem on the spherical shell $\{(r,\theta,\phi)|1<r<2\}$:

$$\nabla^2 u = 0, \quad u|_{r=1} = \begin{cases} 0, & 0 \le \theta < \frac{\pi}{2} \\ \sin 2\phi, & \frac{\pi}{2} < \theta \le \pi \end{cases}, \quad u|_{r=2} = \begin{cases} \sin 2\phi, & 0 \le \theta < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < \theta \le \pi \end{cases}.$$

Recall *Legendre's equation:* $(1-x^2)P_\ell'' - 2xP_\ell' + \ell(\ell+1)P_\ell = 0$. [Can you see a certain $P_{\ell m}$ hiding here?] The following identities may be useful: $P_{\ell+1}' - xP_\ell' = (\ell+1)P_\ell$, $(2\ell+1)P_\ell = P_{\ell+1}' - P_{\ell-1}'$. [Hint: the algebra is probably easiest if you write everything in terms of derivatives of P_n for various n *before* integrating.] Your answer may include $P_n(0)$ for values of n for which this is nonzero. You may also use the normalisation integral for $P_{\ell m}$: $\int_{-1}^1 P_{\ell m}^2(x)\,dx = \frac{(\ell+m)!}{(\ell-m)!}\frac{2}{2\ell+1}$.

We have the general solution

$$u(r,\theta,\phi) = \sum_{\ell=0}^\infty \sum_{m=0}^\ell P_{\ell m}(\cos\theta)\left[\cos m\phi\left(\alpha_{\ell m} r^\ell + \beta_{\ell m} r^{-(\ell+1)}\right) + \sin m\phi\left(\gamma_{\ell m} r^\ell + \delta_{\ell m} r^{-(\ell+1)}\right)\right]. \quad \text{[1 mark]}$$

The first boundary condition [1 mark] then gives

$$\alpha_{\ell m} + \beta_{\ell m} = 0, \qquad \text{all } \ell, m \qquad \text{[1 mark]}$$
$$\gamma_{\ell m} + \delta_{\ell m} = 0, \qquad m \neq 2 \qquad \text{[0.5 marks]}$$
$$\sum_{\ell=2}^\infty P_{\ell 2}(\cos\theta)\left(\gamma_{\ell 2} + \delta_{\ell 2}\right) = \begin{cases} 0, & 0 \le \theta < \frac{\pi}{2} \\ 1, & \frac{\pi}{2} < \theta \le \pi \end{cases} \qquad \text{[1 mark]}$$

Similarly, the second boundary condition [1 mark] gives

$$2^\ell \alpha_{\ell m} + 2^{-(\ell+1)}\beta_{\ell m} = 0, \qquad \text{all } \ell, m \qquad \text{[1 mark]}$$
$$2^\ell \gamma_{\ell m} + 2^{-(\ell+1)}\delta_{\ell m} = 0, \qquad m \neq 2 \qquad \text{[0.5 marks]}$$
$$\sum_{\ell=2}^\infty P_{\ell 2}(\cos\theta)\left(2^\ell\gamma_{\ell 2} + 2^{-(\ell+1)}\delta_{\ell 2}\right) = \begin{cases} 1, & 0 \le \theta < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < \theta \le \pi \end{cases} \qquad \text{[1 mark]}$$

Since the matrix

$$\begin{pmatrix} 1 & 1 \\ 2^\ell & 2^{-(\ell+1)} \end{pmatrix} \qquad \text{has inverse} \qquad \frac{1}{2^{-(\ell+1)} - 2^\ell}\begin{pmatrix} 2^{-(\ell+1)} & -1 \\ -2^\ell & 1 \end{pmatrix},$$

we see that $\alpha_{\ell m} = \beta_{\ell m} = 0$ for all ℓ, m [1 mark], while $\gamma_{\ell m} = \delta_{\ell m} = 0$ for all $m \neq 2$ [1 mark]. We now need to expand the two functions appearing in the remaining two conditions. To do this, we note that

$$\begin{aligned} P_{\ell 2}(x) &= (1-x^2)P_\ell'' \text{[0.5 marks]} = 2xP_\ell' - \ell(\ell+1)P_\ell \text{[0.5 marks]} = 2\left[P_{\ell+1}' - (\ell+1)P_\ell\right] - \ell(\ell+1)P_\ell \\ &= 2P_{\ell+1}' - (\ell+2)(\ell+1)P_\ell = 2P_{\ell+1}' - \frac{(\ell+2)(\ell+1)}{2\ell+1}\left(P_{\ell+1}' - P_{\ell-1}'\right) \\ &= \frac{4\ell+2-(\ell^2+3\ell+2)}{2\ell+1}P_{\ell+1}' + \frac{(\ell+2)(\ell+1)}{2\ell+1}P_{\ell-1}' = -\frac{\ell(\ell-1)}{2\ell+1}P_{\ell+1}' + \frac{(\ell+2)(\ell+1)}{2\ell+1}P_{\ell-1}', \end{aligned}$$

[3.5 marks]

so

$$\int P_{\ell 2}(x)\,dx = -\frac{\ell(\ell-1)}{2\ell+1}P_{\ell+1} + \frac{(\ell+2)(\ell+1)}{2\ell+1}P_{\ell-1} + C.$$

Thus (making the change of variables $x = \cos\theta$, as usual)

$$\begin{aligned}\left[\frac{(\ell+2)!}{(\ell-2)!}\frac{2}{2\ell+1}\right] \text{[0.5 marks]} \left(2^\ell\gamma_{\ell 2} + 2^{-(\ell+1)}\delta_{\ell 2}\right) &= -\frac{\ell(\ell-1)}{2\ell+1}P_{\ell+1} + \frac{(\ell+2)(\ell+1)}{2\ell+1}P_{\ell-1}\bigg|_0^1 \text{ [0.5 marks]}\\ &= \frac{\ell^2+3\ell+2-\ell^2+\ell}{2\ell+1} + \frac{\ell(\ell-1)}{2\ell+1}P_{\ell+1}(0) - \frac{(\ell+2)(\ell+1)}{2\ell+1}P_{\ell-1}(0)\\ &= 2 + \frac{\ell(\ell-1)}{2\ell+1}P_{\ell+1}(0) - \frac{(\ell+2)(\ell+1)}{2\ell+1}P_{\ell-1}(0), \quad \text{[1 mark]}\end{aligned}$$

while since $P'_{\ell+1}$, $P'_{\ell-1}$ are even or odd as ℓ is [1 mark],

$$\left[\frac{(\ell+2)!}{(\ell-2)!}\frac{2}{2\ell+1}\right](\gamma_{\ell 2} + \delta_{\ell 2}) = (-1)^\ell\left[2 + \frac{\ell(\ell-1)}{2\ell+1}P_{\ell+1}(0) - \frac{(\ell+2)(\ell+1)}{2\ell+1}P_{\ell-1}(0)\right].$$

[0.5 marks] [0.5 marks]

Thus finally

$$\begin{aligned}\begin{pmatrix}\gamma_{\ell 2}\\ \delta_{\ell 2}\end{pmatrix} &= \frac{(\ell-2)!}{(\ell+2)!}\frac{2\ell+1}{2}\frac{1}{2^{-(\ell+1)}-2^\ell}\begin{pmatrix}2^{-(\ell+1)} & -1\\ -2^\ell & 1\end{pmatrix}\begin{pmatrix}(-1)^\ell\\ 1\end{pmatrix}\\ &\qquad\cdot\left[2 + \frac{\ell(\ell-1)}{2\ell+1}P_{\ell+1}(0) - \frac{(\ell+2)(\ell+1)}{2\ell+1}P_{\ell-1}(0)\right]\\ &= \frac{(-1)^\ell}{2^{-(\ell+1)}-2^\ell}\left[\frac{2\ell+1}{(\ell+2)(\ell+1)\ell(\ell-1)} + \frac{1}{2(\ell+2)(\ell+1)}P_{\ell+1}(0) - \frac{1}{2\ell(\ell-1)}P_{\ell-1}(0)\right]\\ &\qquad\cdot\begin{pmatrix}2^{-(\ell+1)} - (-1)^\ell\\ -2^\ell + (-1)^\ell\end{pmatrix}\end{aligned}$$

[3.5 marks]

and we have the final answer

$$\begin{aligned}u = \sum_{\ell=2}^{\infty} P_{\ell 2}(\cos\theta)\sin 2\phi &\frac{2\ell+1}{(\ell+2)(\ell+1)\ell(\ell-1)\left(2^{-(\ell+1)}-2^\ell\right)}\\ &\cdot\left[r^\ell\left((-1)^\ell 2^{-(\ell+1)} - 1\right) - r^{-(\ell+1)}\left((-1)^\ell 2^\ell - 1\right)\right]\\ + \sum_{\ell=2,\ell\,\text{odd}}^{\infty} P_{\ell 2}(\cos\theta)\sin 2\phi &\frac{1}{2^{-(\ell+1)}-2^\ell}\left[\frac{1}{2(\ell+2)(\ell+1)}P_{\ell+1}(0) - \frac{1}{2\ell(\ell-1)}P_{\ell-1}(0)\right]\\ &\cdot\left[r^\ell\left((-1)^\ell 2^{-(\ell+1)} - 1\right) - r^{-(\ell+1)}\left((-1)^\ell 2^\ell - 1\right)\right]. \quad \text{[0.5 marks]}\end{aligned}$$

Notes. One can also use the alternative (less general) form for the solution

$$u(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell} P_{\ell m}(\cos\theta)\left(a_{\ell m}\cos m\phi + b_{\ell m}\sin m\phi\right)\left(c_{\ell m}r^\ell + d_{\ell m}r^{-(\ell+1)}\right).$$

However, in either case it is necessary to solve systems for all of the coordinates; and concluding too quickly that (for example) $a_{\ell m} = 0$ for all ℓ and m led to lost marks. (This is analogous to problem 3 on the midterm.) Additionally, the identity $(2\ell+1)P_\ell = P'_{\ell+1} - P'_{\ell-1}$ only applies to P_ℓ, *not* to the $P_{\ell 2}$ with which we need to work here: attempting to solve the problem that way probably led to little credit being given.

Beyond the foregoing, most lost marks on this problem were probably due to algebraic errors or simply not finishing.

The alert reader will note that the marks above add up to 22.5, not 22. This was an inadvertant slip on the part of the instructor which was felt not to be serious enough to attempt to correct once it was discovered. Thus this problem had effectively 0.5 bonus marks attached to it.

3. [9 marks] Solve the following boundary-value problem on the cylinder $\{(\rho,\phi,z)|\rho<1,\ 0<z<2\}$:

$$\nabla^2 u = 0, \quad u|_{z=0} = u|_{z=2} = 0, \quad u|_{\rho=1} = z\cos 2\phi.$$

We have the general expansion

$$u(\rho,\phi,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} I_m\left(\frac{n\pi}{2}\rho\right)\left[\cos m\phi\beta_{nm}\sin\frac{n\pi}{2}z + \sin m\phi\delta_{nm}\sin\frac{n\pi}{2}z\right]; \qquad \text{[1.5 marks]}$$

applying the boundary condition [1 mark] gives

$$\begin{aligned} \delta_{nm} &= 0 \qquad \text{for all}\, n,m && \text{[1 mark]}\\ \beta_{nm} &= 0 \qquad \text{for all}\, m\neq 2 && \text{[0.5 marks]}\\ \sum_{n=1}^{\infty} I_2\left(\frac{n\pi}{2}\right)\left(\beta_{n2}\sin\frac{n\pi}{2}z\right) &= z; && \text{[1 mark]} \end{aligned}$$

thus (since $\int_0^2 \sin^2\frac{n\pi}{2}z\,dz = 1$ [0.5 mark])

$$\begin{aligned} \beta_{n2}I_2\left(\frac{n\pi}{2}\right) &= \int_0^2 z\sin\frac{n\pi}{2}z\,dz \text{ [0.5 marks]} = \left[-\frac{2}{n\pi}z\cos\frac{n\pi}{2}z\Big|_0^2 + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}z\Big|_0^2\right] \text{ [1 mark]}\\ &= \frac{4}{n\pi}(-1)^{n+1}, \text{ [1 mark]} \end{aligned}$$

so $\beta_{n2} = \frac{4(-1)^{n+1}}{n\pi I_2\left(\frac{n\pi}{2}\right)}$ [0.5 marks], and the solution is

$$u(\rho,\phi,z) = \sum_{n=1}^{\infty} I_2\left(\frac{n\pi}{2}\rho\right)\cos 2\phi(-1)^{n+1}\frac{4}{n\pi I_2\left(\frac{n\pi}{2}\right)}\sin\frac{n\pi}{2}z. \qquad \text{[0.5 marks]}$$

NOTES. Probably the single most common mistake on this problem was forgetting the factor of $\frac{1}{2}$ in the z separation constant, i.e., using $n\pi$ instead of $\frac{n\pi}{2}$ in the foregoing. This fails to give a correct answer since $\{\sin n\pi z\}$ is not a complete set on the interval $[0,2]$. This generally resulted in the deduction of 0.5 marks. As with problem 1, beginning the sum for n at 0 instead of 1 should generally result in a deduction of 0.5 marks.

4. [12 marks] Suppose that $n \in \mathbf{Z}$, $n > 0$. Solve the following problem on $(0, +\infty) \times \mathbf{R}^3$, using Fourier transforms:

$$\frac{\partial u}{\partial t} = \nabla^2 u + (4\pi t)^{-\frac{3}{2}} e^{-\frac{x^2}{4t}}, \quad u|_{t=0} = \left(\frac{\pi}{n^2}\right)^{-\frac{3}{2}} e^{-n^2|\mathbf{x}|^2}.$$

Find the limit of the solution as $n \to \infty$. What does the initial data behave like in this limit?

We have, upon Fourier transforming in space,

$$\begin{aligned}\frac{\partial \hat{u}}{\partial t} &= -4\pi^2|\mathbf{k}|^2\hat{u} \text{ [1 mark]} + (4\pi t)^{-\frac{3}{2}}\left(\frac{\pi}{\frac{1}{4t}}\right)^{\frac{3}{2}} e^{-4\pi^2|\mathbf{k}|^2 t} \\ &= -4\pi^2|\mathbf{k}|^2\hat{u} + e^{-4\pi^2|\mathbf{k}|^2 t} \text{ [1 mark]}\end{aligned} \qquad \begin{aligned}\hat{u}|_{t=0} &= \left(\frac{\pi}{n^2}\right)^{-\frac{3}{2}}\left(\frac{\pi}{n^2}\right)^{\frac{3}{2}} e^{-\frac{\pi^2|\mathbf{k}|^2}{n^2}} \\ &= e^{-\frac{\pi^2|\mathbf{k}|^2}{n^2}} \text{ [1 mark]}\end{aligned}$$

whence, using the integrating factor $e^{4\pi^2|\mathbf{k}|^2 t}$ [1 mark],

$$\begin{aligned}\frac{\partial}{\partial t}\left(e^{4\pi^2|\mathbf{k}|^2 t}\hat{u}\right) &= 1, \\ \hat{u} &= [\hat{u}(0) \text{ [1 mark]} + t \text{ [1 mark]}]\, e^{-4\pi^2|\mathbf{k}|^2 t} \\ &= te^{-4\pi^2|\mathbf{k}|^2 t} + e^{-|\mathbf{k}|^2\pi^2\left(4t+\frac{1}{n^2}\right)}, \text{ [0.5 marks]}\end{aligned}$$

whence we obtain upon inverse transforming

$$\begin{aligned}u &= t\left(\frac{\pi}{4\pi^2 t}\right)^{\frac{3}{2}} e^{-\frac{|\mathbf{x}|^2}{4t}} \text{ [1 mark]} + \left(\frac{\pi}{\pi^2\left(4t+\frac{1}{n^2}\right)}\right)^{\frac{3}{2}} e^{-\frac{|\mathbf{x}|^2}{4t+\frac{1}{n^2}}} \text{ [1 mark]} \\ &= \frac{1}{8\pi^{\frac{3}{2}} t^{\frac{1}{2}}} e^{-\frac{|\mathbf{x}|^2}{4t}} \text{ [0.5 marks]} + \frac{1}{\left(\pi\left(4t+\frac{1}{n^2}\right)\right)^{\frac{3}{2}}} e^{-\frac{|\mathbf{x}|^2}{4t+\frac{1}{n^2}}} \text{ [1 mark]}.\end{aligned}$$

In the limit as $n \to \infty$, the second term becomes simply $\frac{1}{(4\pi t)^{\frac{3}{2}}} e^{-\frac{|\mathbf{x}|^2}{4t}}$, and the whole solution is

$$u = \frac{1}{(4\pi t)^{\frac{3}{2}}}(1+t)e^{-\frac{|\mathbf{x}|^2}{4t}}. \qquad \text{[1 mark]}$$

Since

$$\int_{\mathbf{R}^3} \pi^{-\frac{3}{2}} e^{-|\mathbf{x}|^2}\, d\mathbf{x} = \pi^{-\frac{3}{2}}\left(\frac{\pi}{1}\right)^{\frac{3}{2}} = 1,$$

and

$$\left(\frac{\pi}{n^2}\right)^{-\frac{3}{2}} e^{-n^2|\mathbf{x}|^2} = n^3\left[\pi^{-\frac{3}{2}} e^{-|n\mathbf{x}|^2}\right],$$

we see that the initial data is an approximate identity and behaves like the delta function $\delta(\mathbf{x})$ in the limit $n \to \infty$.[1 mark]

NOTES. Probably the most common mistake here was incorrectly taking the forward or inverse Fourier transform of a Gaussian. I think almost nobody correctly found the indicated limit of the initial data (many people said it was zero, which is true only for $\mathbf{x} \neq 0$).

5. (a) [19 marks] Solve the following problem on $(0,+\infty)\times B$, where B is the unit ball $\{(r,\theta,\phi)|r<1\}$:

$$\frac{\partial u}{\partial t}=\nabla^2 u,\quad u|_{t=0}=r^2\sin^2\theta\sin 2\phi,\quad u|_{\partial B}=0.$$

[If you wish to use quantities like $\kappa_{\ell n}$, you must define them explicitly.] Find the limit of the solution as $t\to+\infty$.

(b) [4 marks] Suppose that the condition $u|_{\partial B}=0$ were replaced by the condition $u|_{\partial B}=\cos\theta$. Explain how you would solve the problem in this case (you need not actually calculate anything). What would you expect the limit of the solution to be in this case as $t\to+\infty$? [You need not give an explicit formula, but your answer must be a definite function, not just a description in words.]

(a) The Laplacian on B with Dirichlet boundary conditions has eigenfunctions

$$j_\ell\left(\kappa_{\ell n}r\right)P_{\ell m}(\cos\theta)\begin{cases}\cos m\phi\\ \sin m\phi\end{cases}\qquad \text{[1 mark]}$$

(where $\kappa_{\ell n}$, $n=1,2,\ldots$, is the nth positive root of j_ℓ [0.5 marks]) with corresponding eigenvalues $\lambda_{\ell nm}=-\kappa_{\ell n}^2$ [1 mark]. Suppose that we expand u in this basis as

$$u=\sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}\sum_{n=1}^{\infty}j_\ell\left(\kappa_{\ell n}r\right)P_{\ell m}(\cos\theta)\left(a_{\ell nm}\cos m\phi+b_{\ell nm}\sin m\phi\right).\qquad \text{[1 mark]}$$

Then substituting into the equation gives

$$\sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}\sum_{n=1}^{\infty}j_\ell\left(\kappa_{\ell n}r\right)P_{\ell m}(\cos\theta)\left(a'_{\ell nm}\cos m\phi+b'_{\ell nm}\sin m\phi\right)\qquad \text{[1 mark]}$$

$$=\sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}\sum_{n=1}^{\infty}-\kappa_{\ell n}^2 j_\ell\left(\kappa_{\ell n}r\right)P_{\ell m}(\cos\theta)\left(a_{\ell nm}\cos m\phi+b_{\ell nm}\sin m\phi\right),$$

[1 mark]

so

$$a'_{\ell nm}=-\kappa_{\ell n}^2 a_{\ell nm},\ \text{[1 mark]}\qquad b'_{\ell nm}=-\kappa_{\ell n}^2 b_{\ell nm},\ \text{[1 mark]}$$

and

$$a_{\ell nm}(t)=a_{\ell nm}(0)e^{-\kappa_{\ell n}^2 t},\ \text{[1 mark]}\qquad b_{\ell nm}(t)=b_{\ell nm}(0)e^{-\kappa_{\ell n}^2 t}.\ \text{[1.5 marks]}$$

The initial values can be obtained from $u|_{t=0}$:

$$\sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}\sum_{n=1}^{\infty}j_\ell\left(\kappa_{\ell n}r\right)P_{\ell m}(\cos\theta)\left(a_{\ell nm}(0)\cos m\phi+b_{\ell nm}(0)\sin m\phi\right)=r^2\sin^2\theta\sin 2\phi.\qquad \text{[1 mark]}$$

Since $P_{22}(\cos\theta) = 3\sin^2\theta$, we see that $b_{\ell nm}(0) = 0$ unless $\ell = m = 2$ [0.5 marks], and $a_{\ell nm} = 0$ for all ℓ, n, m [1 mark]; finally

$$\sum_{n=1}^{\infty} b_{2n2}(0) j_2(\kappa_{2n} r) = \frac{r^2}{3}, \qquad \text{[1 mark]}$$

$$\begin{aligned} b_{2n2}(0) &= \frac{2}{j_3^2(\kappa_{2n})}\text{[1 mark]} \int_0^1 \frac{r^4}{3} j_2(\kappa_{2n} r)\, dr \text{[1 mark]} = \frac{2}{3j_3^2(\kappa_{2n})}\sqrt{\frac{\pi}{2\kappa_{2n}}} \int_0^1 r^{\frac{7}{2}} J_{\frac{5}{2}}\left(\lambda_{\frac{5}{2},n} r\right) \text{ [0.5 marks] } dr \\ &= \frac{2}{3j_3^2(\kappa_{2n})}\sqrt{\frac{\pi}{2}} \frac{1}{\kappa_{2n}^{\frac{3}{2}}} J_{\frac{7}{2}}(\kappa_{2n}) \text{ [1 mark]} = \frac{2}{3j_3^2(\kappa_{2n})\kappa_{2n}} j_3(\kappa_{2n}) \\ &= \frac{2}{3j_3(\kappa_{2n})\kappa_{2n}}, \text{ [0.5 marks]} \end{aligned}$$

and the final solution is

$$u = \sum_{n=1}^{\infty} j_2(\kappa_{2n} r) P_{22}(\cos\theta) \sin 2\phi \frac{2}{3j_3(\kappa_{2n})\kappa_{2n}} e^{-\kappa_{2n}^2 t}. \qquad \text{[0.5 marks]}$$

Since $\kappa_{2n} > 0$ for all n, we see that $u \to 0$ as $t \to +\infty$. [1 mark]
(b) In this case we would first solve the problem on B

$$\nabla^2 U_1 = 0, \quad U_1|_{\partial B} = \cos\theta, \qquad \text{[1 mark]}$$

and then solve on $(0, +\infty) \times B$

$$\frac{\partial u_2}{\partial t} = \nabla^2 u_2, \quad u_2|_{t=0} = r^2 \sin^2\theta \sin 2\phi - U_1, \quad u_2|_{\partial B} = 0; \qquad \text{[1 mark]}$$

the full solution would be $u = U_1 + u_2$ [1 mark]. We expect $\lim_{t\to+\infty} u = U_1$ [1 mark] in this case.

Notes. Probably the biggest single reason for deducted marks in (a) was not deriving the equations satisfied by the coefficients, but rather assuming the solutions from the outset. For (b), the single biggest quantitative error was probably taking $u_2|_{t=0} = r^2\sin^2\theta\sin 2\phi - \cos\theta$, or even dropping the subtracted term altogether.

Starting the n sum at 0 instead of 1 should not result in lost marks (since n is just a counter, which can just as well be started at 0 as at 1, though in class we always started it at 1).

The curious asymmetry in marking the expressions for $a_{\ell nm}(t)$ and $b_{\ell nm}(t)$ was not intended to create any asymmetry in practice, in that if only one appeared, it would be given the higher mark. (I probably had some reason in mind when I wrote 1 mark for a and 1.5 marks for b, but I have long since forgotten what it was.)

6. [24 marks] Solve the following problem on the unit disk $D = \{(\rho, \phi) | \rho < 1\}$:

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u, \quad u|_{\partial D} = 0, \quad u|_{t=0} = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \rho^2 \sin 2\phi.$$

[As in problem 5, if you wish to use quantities like λ_{mi}, you must define them explicitly.] What is the lowest frequency occurring? [A symbolic answer is sufficient.]

In this case we have the eigenfunctions $J_m(\lambda_{mi}\rho)\begin{cases}\cos m\phi \\ \sin m\phi\end{cases}$ [1 mark] (where λ_{mi} is the ith positive root of $J_{\ell m}(x)$ [0.5 marks]) with eigenvalues $-\lambda_{mi}^2$ [1 mark]. Expanding u as

$$u = \sum_{m=0}^{\infty}\sum_{i=1}^{\infty} J_m(\lambda_{mi}\rho)(a_{mi}\cos m\phi + b_{mi}\sin m\phi), \qquad \text{[1 mark]}$$

we have, upon substituting into the equation,

$$\sum_{n=0}^{\infty}\sum_{i=1}^{\infty} J_m(\lambda_{mi}\rho)(a''_{mi}\cos m\phi + b''_{mi}\sin m\phi) \text{ [1 mark]}$$

$$= \sum_{m=0}^{\infty}\sum_{i=1}^{\infty} J_m(\lambda_{mi}\rho)\left(-\lambda_{mi}^2\right)(a_{mi}\cos m\phi + b_{mi}\sin m\phi), \text{ [1 mark]}$$

so that the a_{mi} and b_{mi} satisfy

$$a''_{mi} = -\lambda_{mi}^2 a_{mi}, \text{ [1 mark]} \qquad b''_{mi} = -\lambda_{mi}^2 b_m i, \text{ [1 mark]}$$

so

$$a_{mi}(t) = \alpha_{mi}\cos\lambda_{mi}t + \beta_{mi}\sin\lambda_{mi}t, \qquad b_{mi}(t) = \gamma_{mi}\cos\lambda_{mi}t + \delta_{mi}\sin\lambda_{mi}t. \qquad \text{[1 mark]}$$

Now we see that

$$a_{mi}(0) = \alpha_{mi}, \quad a'_{mi}(0) = \lambda_{mi}\beta_{mi}, \quad b_{mi}(0) = \gamma_{mi}, \quad b'_{mi}(0) = \lambda_{mi}\delta_{mi};$$

and these initial values can be determined from the initial conditions for u:

$$0 = u|_{t=0} = \sum_{m=0}^{\infty}\sum_{i=1}^{\infty} J_m(\lambda_{mi}\rho)(a_{mi}(0)\cos m\phi + b_{mi}(0)\sin m\phi) \text{ [1 mark]}$$

so $\alpha_{mi} = \gamma_{mi} = 0$ for all m, i [1 mark];

$$\rho^2\sin 2\phi = u_t|_{t=0} = \sum_{m=0}^{\infty}\sum_{i=1}^{\infty} J_m(\lambda_{mi}\rho)(a'_{mi}(0)\cos m\phi + b'_{mi}(0)\sin m\phi), \text{ [1 mark]}$$

so $a'_{mi}(0) = 0$ for all m, i [1 mark], which gives $\beta_{mi} = 0$ and $a_{mi}(t) = 0$ for all t, all m, i [2 marks], while $b'_{mi}(0) = 0$ for all $m \neq 2$ [0.5 marks] , which gives $\delta_{mi} = 0$, hence $b_{mi}(t) = 0$ for all t [0.5 marks], for $m \neq 2$ [1 mark]; finally,

$$\rho^2 = \sum_{i=1}^{\infty} J_2(\lambda_{2i}\rho)\, b'_{2i}(0), \qquad \text{[1 mark]}$$

so

$$b_{2i}'(0) = \frac{2}{J_3^2(\lambda_{2i})} \int_0^1 \rho^3 J_2(\lambda_{2i}\rho)\, d\rho = \frac{2}{J_3^2(\lambda_{2i})} \frac{1}{\lambda_{2i}} J_3(\lambda_{2i}) = \frac{2}{\lambda_{2i} J_3(\lambda_{2i})}, \qquad \text{[3.5 marks]}$$

whence

$$\delta_{2i} = \frac{2}{\lambda_{2i}^2 J_3(\lambda_{2i})} \qquad \text{[1 mark]}$$

and we have finally for u

$$u(t,\rho,\phi) = \sum_{i=1}^{\infty} \frac{2}{\lambda_{2i}^2 J_3(\lambda_{2i})} J_2(\lambda_{2i}\rho) \sin 2\phi \sin \lambda_{2i} t. \qquad \text{[1 mark]}$$

The lowest frequency is thus $\frac{\lambda_{21}}{2\pi}$. [1 mark]

NOTES. As with problem 5, probably the biggest reason for lost marks was starting directly with the solutions for the coefficients rather than deriving them as here. For the last part of the question, an answer λ_{21} was also acceptable (missing the factor of 2π did not result in lost marks): while technically only $\frac{\lambda_{21}}{2\pi}$ is the frequency, λ_{21} is the so-called *angular frequency*, and since we didn't spend much time on this point in class I didn't see a point in deducting marks for missing the 2π.

As with problem 5, starting the i sum at 0 should not result in lost marks.

7. [18 marks] Solve the following problem on the unit cube Q (defined in problem 1):

$$\nabla^2 u = \sin 4\pi x \sin 2\pi y \cos \pi z, \qquad \left.\frac{\partial u}{\partial n}\right|_{\partial Q} = 0, \qquad u(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = 0.$$

(Here $\frac{\partial}{\partial n}$ denotes the derivative in the normal direction to the surface ∂Q.)

We have the eigenfunctions $\cos \ell\pi x \cos m\pi y \cos n\pi z$ [2 marks], with eigenvalues $-\pi^2\left(\ell^2+m^2+n^2\right)$ [1 mark]. Expanding u as

$$u(x,y,z) = \sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{\ell mn} \cos \ell\pi x \cos m\pi y \cos n\pi z, \qquad \text{[1 mark]}$$

we see that the equation gives

$$\sum_{\ell,m,n=0}^{\infty} -\pi^2\left(\ell^2+m^2+n^2\right) a_{\ell mn} \cos \ell\pi x \cos m\pi y \cos n\pi z = \sin 4\pi x \sin 2\pi y \cos \pi z, \qquad \text{[1 mark]}$$

whence we see that

$$-\pi^2\left(\ell^2+m^2+n^2\right) a_{\ell mn} = n_\ell n_m n_n \int_Q \sin 4\pi x \sin 2\pi y \cos \pi z \cos \ell\pi x \cos m\pi y \cos n\pi z \, dV, \tag{1}$$

where $n_\ell = \begin{cases} 2, & \ell \neq 0 \\ 1, & \ell = 0 \end{cases}$ is the appropriate normalisation constant. Now we see that the integral above vanishes for $n \neq 1$, while

$$\begin{aligned}
\int_0^1 \sin 2k\pi x \cos \ell\pi x \, dx\text{[0.5 marks]} &= \frac{1}{2}\int_0^1 \sin\left[(2k\pi+\ell\pi)x\right] + \sin\left[(2k\pi-\ell\pi)x\right] \, dx\text{[1 mark]} \\
&\left[= 0 \text{ if } \ell = 2k\right]\text{[1 mark]} \\
&= -\frac{1}{2}\left[\frac{1}{(2k+\ell)\pi}\cos\left[(2k+\ell)\pi x\right]\Big|_0^1 + \frac{1}{(2k-\ell)\pi}\cos\left[(2k-\ell)\pi x\right]\Big|_0^1\right], \\
&\qquad\qquad \text{[0.5 marks]} \quad \ell \neq 2k \\
&= \frac{1}{2\pi}\left(1-(-1)^\ell\right)\left(\frac{1}{2k+\ell}+\frac{1}{2k-\ell}\right) = \frac{2k}{\pi}\left(1-(-1)^\ell\right)\frac{1}{4k^2-\ell^2}, \\
&\qquad\qquad \text{[1 mark]}
\end{aligned}$$

so for $(\ell, m, n) \neq (0,0,0)$ we have

$$\begin{aligned}
a_{\ell mn} &= \begin{cases} 0, & n \neq 1, \text{ or } m = 2, \text{[0.5 marks] or } \ell = 4 \text{ [0.5 marks]} \\ n_\ell n_m \frac{8}{\pi^2}\left(1-(-1)^\ell\right)\left(1-(-1)^m\right)\frac{1}{16-\ell^2} \\ \quad \cdot \frac{1}{4-m^2}\text{[1 mark]} \cdot \frac{-1}{\pi^2(\ell^2+m^2+1)}, & \text{otherwise} \end{cases} \\
&= \begin{cases} 0\text{[0.5 marks]}, & n \neq 1, \text{ or } m \text{ or } \ell \text{ even[0.5 marks]} \\ -\frac{128}{\pi^2}\frac{1}{16-\ell^2}\frac{1}{4-m^2}\text{[0.5 marks]}\frac{1}{\pi^2(\ell^2+m^2+1)}\text{[1 mark]}, & \text{otherwise.} \end{cases}
\end{aligned}$$

Now if $\ell = m = n = 0$, then the integral in (1) is zero, as is the left-hand side. Thus this equation is consistent but tells us nothing about a_{000} [1 mark]. However, since our series for u only has nonzero coefficients for ℓ, m, n all odd, and

$$\cos\frac{\ell\pi}{2}\cos\frac{m\pi}{2}\cos\frac{n\pi}{2} = 0 \qquad \text{[1 mark]}$$

in such a case, the final condition gives $a_{000} = 0$ [1 mark]. Thus

$$u = \sum_{\ell,m=1,\ell,m\text{ odd}}^{\infty} -\frac{128}{\pi^2}\frac{1}{16-\ell^2}\frac{1}{4-m^2}\frac{1}{\pi^2\left(\ell^2+m^2+1\right)}\cos\ell\pi x\cos m\pi y\cos\pi z. \qquad \text{[1.5 marks]}$$

NOTES. Again, some marks were lost by simply assuming the general form of the solution to Poisson's equation rather than deriving it as here (though this is less of an issue than with 5 and especially 6). Marks were also lost for being insufficiently careful with the term a_{000}.

8. [9 marks] Solve the following problem on $\mathbf{R}^3$ (here x is the first coordinate of $\mathbf{x} = (x, y, z)$):

$$\frac{\partial u}{\partial t} = \nabla^2 u + \frac{\partial u}{\partial x}, \qquad u|_{t=0} = e^{-|\mathbf{x}|^2}.$$

We have the Fourier transform:

$$\frac{\partial \hat{u}}{\partial t} = -4\pi^2|\mathbf{k}|^2\hat{u}\text{[1 mark]} + 2\pi i k_1 \hat{u}\text{[1 mark]}, \quad \hat{u}|_{t=0} = \pi^{\frac{3}{2}} e^{-\pi^2|\mathbf{k}|^2}.\text{[1 mark]}$$

The equation gives

$$\begin{aligned}\frac{\partial}{\partial t}\left(e^{\left(4\pi^2|\mathbf{k}|^2 - 2\pi i k_1\right)t}\hat{u}\text{[1 mark]}\right) &= 0,\\ \hat{u} &= \hat{u}(0)e^{-\left(4\pi^2|\mathbf{k}|^2 - 2\pi i k_1\right)t}\\ &= \pi^{\frac{3}{2}} e^{-\pi^2|\mathbf{k}|^2(4t+1)} e^{2\pi i k_1 t}.\text{[1 mark]}\end{aligned}$$

Since

$$\mathcal{F}^{-1}\left[\pi^{\frac{3}{2}} e^{-\pi^2|\mathbf{k}|^2(4t+1)}\right] = \pi^{\frac{3}{2}}\left(\frac{1}{\pi(4t+1)}\right)^{\frac{3}{2}} e^{-\frac{|\mathbf{x}|^2}{4t+1}} = \frac{1}{(4t+1)^{\frac{3}{2}}} e^{-\frac{|\mathbf{x}|^2}{4t+1}}, \qquad \text{[2 marks]}$$

we see by properties of Fourier transforms that

$$u = \frac{1}{(4t+1)^{\frac{3}{2}}} e^{-\frac{1}{4t+1}\left(y^2+z^2+(x+t)^2\right)}. \qquad \text{[2 marks]}$$

Notes. Again, probably the biggest issue with this problem was the mishandling of the relevant Fourier transforms. Another issue which came up was failure to use the property

$$\mathcal{F}[f(\mathbf{x} - \mathbf{x}_0)](\mathbf{k}) = e^{-2\pi i \mathbf{k}\cdot\mathbf{x}_0}\hat{f}(\mathbf{k}).$$

Some solutions wrote effectively $\mathcal{F}\left[\frac{\partial u}{\partial x}\right] = \frac{\partial \hat{u}}{\partial x}$, probably by analogy with a similar (though correct) formula for $\frac{\partial u}{\partial t}$: unfortunately this formula is not only wrong in actuality but meaningless even in principle, since $\hat{u}$ is a function of $\mathbf{k}$ and t and hence does not depend on x. The point behind the analogous formula for $\frac{\partial u}{\partial t}$ is that we are taking a function of $(t, \mathbf{x})$ and transforming *only in* $\mathbf{x}$, meaning that t is essentially a parameter with respect to which we can differentiate either before or after transforming (assuming, as always, that our functions are sufficiently well-behaved that we are allowed to take the derivative inside the integral representing the Fourier transform). x, however, is one of the variables with respect to which we are transforming; i.e., it will be one of the variables over which we integrate, and hence it does not appear in the transformed function and it makes no sense to speak of the derivative of the transform with respect to it. More explicitly:

$$\begin{aligned}\mathcal{F}\left[\frac{\partial u}{\partial t}\right] &= \int_{\mathbf{R}^3}\frac{\partial u}{\partial t}e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\,d\mathbf{x} = \frac{\partial}{\partial t}\int_{\mathbf{R}^3} u e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\,d\mathbf{x}\\ &= \frac{\partial}{\partial t}\mathcal{F}[u] = \frac{\partial \hat{u}}{\partial t},\end{aligned}$$

while attempting to do the same thing with $\frac{\partial u}{\partial x}$ leads to

$$\mathcal{F}\left[\frac{\partial u}{\partial x}\right] = \int_{\mathbf{R}^3}\frac{\partial u}{\partial x}e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}\,d\mathbf{x},$$

and now there is no way to take the derivative outside of the integral since the integral over $\mathbf{x}$ includes an integral over x: one needs instead to do an integration by parts, which leads to the formula

$$\mathcal{F}\left[\frac{\partial u}{\partial x}\right] = 2\pi i k_1 \hat{u}$$

used here, as derived in class. (Here k_1 represents the component of $\mathbf{k}$ corresponding to x.)

Scratch paper

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Scratch paper

– End of exam booklet –

We give a brief review of complex numbers, and some results which we shall need in this course.

Definition. A *complex number* is a number of the form $a+ib$, where $a,b \in$ and i satisfies $i^2=-1$. If $a+ib$ and $c+id$ are two complex numbers, we define their sum, difference, product, and quotient as follows:

$$\begin{aligned}
(a+ib)+(c+id) &= (a+c)+i(b+d) \\
(a+ib)-(c+id) &= (a-c)+i(b-d) \\
(a+ib)\cdot(c+id) &= (ac-bd)+i(bc+ad) \\
\frac{1}{c+id} &= \frac{c}{c^2+d^2}+i\frac{-d}{c^2+d^2}=\frac{c-id}{c^2+d^2}, \qquad c^2+d^2\neq 0 \\
\frac{a+ib}{c+id} &= (a+ib)\cdot\frac{1}{c+id}=\frac{(ac+bd)+i(bc-ad)}{c^2+d^2}, \qquad c^2+d^2\neq 0.
\end{aligned}$$

If $z=a+ib$ is a complex number, we call a its *real part* and b its *imaginary part*, and write $a=\operatorname{Re} z$, $b=\operatorname{Im} z$. The *conjugate* of z is the number $\overline{z}=a-ib$. The quantity $\sqrt{z\overline{z}}=\sqrt{a^2+b^2}$ is called the *norm* of z, and is denoted $|z|$; note that it is equal to the norm of the vector $a+b$ in ^{2}.[1] (We note in passing that $c^2+d^2=0$ if and only if both c and d are zero; thus requiring $c^2+d^2\neq 0$ is equivalent to saying that at least one of c and d is nonzero.)

Commentary. Essentially, the above definitions say that complex numbers obey all of the usual rules of algebra, supplemented by the condition $i^2=-1$. It turns out to be convenient to require them to also behave in a natural way with respect to the operations of the calculus, as in the following.[2]

Definition. Let $f:[a,b]\to$, and suppose that $f_1=\operatorname{Re} f$ and $f_2=\operatorname{Im} f$ are differentiable. Then we define

$$f'(t)=f_1'(t)+if_2'(t)$$

(compare to the definition of a tangent vector to a plane parametric curve). If f_1 and f_2 are integrable, then we define

$$\int_a^b f(t)dt=\int_a^b f_1(t)dt+i\int_a^b f_2(t)dt.$$

Definition. A sequence $\{z_n\}$ of complex numbers is said to *converge* to the complex number z if the sequence $\{|z_n-z|\}$ of real numbers converges to 0. (It can be shewn that this is equivalent to saying that $\operatorname{Re} z_n$ converges to $\operatorname{Re} z$ and $\operatorname{Im} z_n$ converges to $\operatorname{Im} z$.) Convergence of a series as convergence of its partial sums is defined as for real series.

Commentary. Suppose that $\sum\limits_{n=0}^{\infty} a_n x^n$ is a power series with radius of convergence $R>0$ (we include the case $R=\infty$). If $z=a+ib$ is such that $|z|<R$, then it can be shewn that $\sum\limits_{n=0}^{\infty} a_n z^n$ converges also. This allows us to extend functions with convergent power series representations (such as e^x, $\sin x$, $\cos x$, etc.) to the complex plane. In particular, if we define e^z in this way for $z\in$, then it can be shewn that (exercise)

$$e^{i\theta}=\cos\theta+i\sin\theta.$$

[1]There are, in fact, some deep connections here, and various parts of two-dimensional calculus have some analogue in complex analysis. Three-dimensional vector calculus is related to a still higher kind of number, called a *quaternion*. Quaternions are very interesting and useful for some purposes (and have nice connections to the concept of *spin* in quantum mechanics) but we do not need them here (and unfortunately shall probably not have occasion to use them anywhere in this course).

[2]Note that we are developing here only a very small part of the theory of complex analysis – in particular, all of our functions are functions of a real variable, and we differentiate only with respect to real variables. As those of you who have had complex analysis are aware, the true power and depth of complex analysis only comes out when one considers derivatives with respect to a *complex* variable.

We review some concepts and methodology from linear algebra.

Definition. Let V and W be two vector spaces[1]. A map $T : V \to W$ is called a *linear transformation* if it satisfies $T(\alpha v + \beta w) = \alpha T(v) + \beta T(w)$ for all $v, w \in V$ and all scalars α, β.

Definition. Let V be a vector space, and let $S \subset V$. We say that S *spans* V if for all $v \in V$ there are $w_1, \ldots, w_n \in S$ and scalars $\alpha_1, \ldots, \alpha_n$ such that $v = \alpha_1 w_1 + \cdots + \alpha_n w_n$. We say that S is *linearly independent* if for any $w_1, \ldots, w_n \in S$ the equation $\alpha_1 w_1 + \cdots + \alpha_n w_n = 0$ has only $\alpha_1 = \cdots = \alpha_n = 0$ as a solution. If S both spans V and is linearly independent then it is called a *basis* for V. In this case, the number of elements of S is called the *dimension* of V. It could be finite or infinite[2].

Example. If $V = {}^n$ or $V = {}^n$, then

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

is a basis for V.

Definition. Let V, W be vector spaces with bases $B = \{v_1, \ldots, v_n\}$, $D = \{w_1, \ldots, w_m\}$, and let $T : V \to W$ be a linear transformation. Then the *basis representation of T with respect to B and D*, $[T]_B^D$, is defined as follows. For each $v_k \in V$, $T(v_k) \in W$ can be expressed in a unique way as a linear combination of elements of D, say

$$T(v_k) = a_{1k} w_1 + \cdots + a_{mk} w_m.$$

We define

$$[T]_B^D = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

In linear algebra courses, one learns about the properties of these matrices, and how to transform from one basis to another, but we do not need all this at the moment.

Definition. Let V be a vector space, and let $T : V \to V$. If $v \in V$, $v \neq 0$, is such that $T(v) = \lambda v$ for some scalar λ, then v is said to be an *eigenvector* of T with *eigenvalue* λ. If there is a basis $B = \{v_1, \ldots, v_n\}$ of V, each element of which is an eigenvector of T, then T is said to be *diagonalisable*.

In this case, it is not hard to see that $[T]_B^B$ is a diagonal matrix, with the kth element being the eigenvalue corresponding to v_k.

[1] I am not going to give the formal definition of a vector space here. Roughly, a vector space is a collection of objects (which can be vectors in ^{n} but can also be other things, such as functions) which can be added and multiplied by scalars (real or complex numbers) in such a way that vector addition and scalar multiplication interact as one would expect. Those of you who have never seen abstract vector spaces can think of ^{n} or ^{n} for the time being.

[2] For the benefit of those who know a little set theory, we note that in the case of an infinite-dimensional vector space V, by 'the number of elements of S' we mean the cardinality of S. Much of the numerology of finite-dimensional linear algebra can be carried over to the infinite-dimensional case in this way. However, in this case an (algebraic) basis as defined here is not particularly useful and one prefers to use something like an orthogonal basis, as we shall see later, where one is able (essentially) to represent elements of V as infinite linear combinations of elements of S.

Definition. Let V be a complex vector space. An *inner product* on V is a map $(\cdot,\cdot) : V \times V$ to satisfying the following properties:

1. $(av + bw, u) = a(v, u) + b(w, u)$ for all $v, w, u \in V$ and all $a, b \in$;
2. $(v, u) = \overline{(u, v)}$ for all $v, u \in V$;
3. $(v, v) \geq 0$ for all $v \in V$, and $(v, v) = 0$ if and only if $v = 0$.

The first and second properties imply that $(\cdot,\cdot)$ is *conjugate linear* in the second argument, i.e., $(v, aw + bu) = \overline{a}(v, w) + \overline{b}(v, u)$. This is sometimes combined with property 1 above to say that $(\cdot,\cdot)$ is a *sesquilinear*[3] map. (It would be *bilinear*, i.e., linear in each argument separately, if it weren't for the conjugate on the a and b.)

The text has an introduction to inner products in section 0.3, and we shall go over similar material from a slightly different perspective in class.

[3]While I have never checked this, 'sesqui' apparently means 'one-and-a-half', as in sesquicentennial, or 150th anniversary.

We give a brief review of integration of piecewise-defined functions. Suppose that we have a function $F : [a, b] \to$ (or would work just as well), suppose that $a_0 = a < a_1 < \cdots < a_n = b$ is some finite sequence of numbers in $[a, b]$, let $F_k : [a_{k-1}, a_k] \to$ $(k = 1, 2, \ldots, n)$ be continuous[1], and suppose that for each $k = 1, 2, \ldots, n$ we have

$$F(x) = F_k(x) \quad \text{for all}\, x \in [a_{k-1}, a_k],$$

in other words, that we have the piecewise definition

$$F(x) = \begin{cases} F_1(x), & x \in [a, a_1] \\ F_2(x), & x \in [a_1, a_2] \\ & \vdots \\ F_n(x), & x \in [a_{n-1}, b] \end{cases}$$

Then it can be shewn that F is integrable on $[a, b]$, and

$$\int_a^b F(x)\, dx = \sum_{k=1}^{n} \int_{a_{k-1}}^{a_k} F_k(x)\, dx.$$

If some F_k is zero, then its integral over its domain $[a_{k-1}, a_k]$ will also be zero, and hence it will not contribute to the sum and may be dropped.

The foregoing applies in particular when we are computing the coefficients in the expansion of a function (say $f(x)$) on $[a, b]$ in terms of a complete orthogonal set of functions on $[a, b]$; see, for example, the solution to problem 2 on homework 3. In particular, since coefficients in such an expansion do not depend on x, and must therefore take into account the function values over the entire interval, in cases where f has a piecewise definition it will be necessary to use a formula like that above to calculate inner products involving f.

[1] 'Integrable' would work just as well here.

We review the definition and an elementary property of the Wronskian. We recall that the notation $f^{(n)}(x)$ denotes the nth derivative of the function f.

DEFINITION. Let $f_1, \ldots, f_n : (a, b) \to$, $a, b \in \cup\{-\infty, +\infty\}$, and suppose that the first $n-1$ derivatives of all n functions exist on (a, b). Then the Wronskian of $f_1, \ldots, f_n$ is the function $W : (a, b) \to$ defined by

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \ldots & f_n(x) \\ f_1'(x) & f_2'(x) & \ldots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \ldots & f_n^{(n-1)}(x) \end{vmatrix}.$$

EXAMPLES.

(a) Let $(a, b) =$, $f_1(x) = x$, $f_2(x) = x^2$. Then the Wronskian of f_1 and f_2 is given by

$$\begin{aligned} W(x) &= \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} \\ &= 2x^2 - x^2 = x^2. \end{aligned}$$

(b) Let $(a, b) =$, $f_1(x) = e^x$, $f_2(x) = e^{-x}$. Then the Wronskian of f_1 and f_2 is

$$\begin{aligned} W(x) &= \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} \\ &= -1 - 1 = -2. \end{aligned}$$

The importance of the Wronskian can be seen from the following proposition.

PROPOSITION. Suppose that the functions $f_1, f_2, \ldots, f_n : (a, b) \to$ possess derivatives of up to order $n-1$ and are linearly dependent on (a, b). Then their Wronskian is zero everywhere on (a, b).

Proof. Since the functions $f_1, f_2, \ldots, f_n$ are linearly dependent on (a, b), there must exist constants $c_1, c_2, \ldots, c_n$ such that for all $x \in (a, b)$ we have

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0.$$

Since $c_1, c_2, \ldots, c_n$ are all constants, we may differentiate this equation k times to obtain

$$c_1 f_1^{(k)}(x) + c_2 f_2^{(k)}(x) + \cdots + c_n f_n^{(k)}(x) = 0,$$

where $k = 1, \ldots, n-1$. Thus we see that for each $x \in (a, b)$, the vectors

$$\begin{pmatrix} f_1(x) \\ f_1'(x) \\ \vdots \\ f_1^{(n-1)}(x) \end{pmatrix}, \quad \begin{pmatrix} f_2(x) \\ f_2'(x) \\ \vdots \\ f_2^{(n-1)}(x) \end{pmatrix}, \quad \ldots \quad \begin{pmatrix} f_n(x) \\ f_n'(x) \\ \vdots \\ f_n^{(n-1)}(x) \end{pmatrix}$$

are linearly dependent. Thus the matrix

$$D = \begin{pmatrix} f_1(x) & f_2(x) & \ldots & f_n(x) \\ f_1'(x) & f_2'(x) & \ldots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \ldots & f_n^{(n-1)}(x) \end{pmatrix}$$

is not full-rank, so its determinant $|D|$ must be zero. But $|D|$ is exactly the Wronskian of $f_1, f_2, \ldots, f_n$, so this completes the proof of the proposition. QED.

From this it follows that if the Wronskian is not identically zero on (a, b), then $f_1, f_2, \ldots, f_n$ must be linearly independent on (a, b).

EXAMPLES.

(c) From examples (a) and (b) above, we see that x and x^2 are linearly independent on , as are e^x and e^{-x}.